

# ESE 5310: Digital Signal Processing

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Lecture 18: Apr 4, 2024

Discrete Fourier Transform



# Today

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- ❑ Discrete Fourier Series
- ❑ Discrete Fourier Transform (DFT)
- ❑ DFT Properties
  - Circular Shift
  - Circular Convolution

# Discrete Fourier Series

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# Reminder: Eigenvalue (DTFT)

□  $x[n] = e^{j\omega n}$

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} x[n-k]h[k] \\ &= \sum_{k=-\infty}^{\infty} e^{j\omega(n-k)}h[k] \\ &= e^{j\omega n} \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k} \\ &= H(e^{j\omega})e^{j\omega n} \end{aligned}$$

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$

- Describes the change in amplitude and phase of signal at frequency  $\omega$
- Frequency response
- Complex value
  - Re and Im
  - Mag and Phase



# DTFT

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DTFT:

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n} d\omega$$



# DTFT

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- If  $x[n]$  is periodic, then finite set of frequencies in the finite range
  - Can represent with discrete Fourier series



# Discrete Fourier Series

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## □ Definition:

- Consider N-periodic signal:

$$\tilde{x}[n + N] = \tilde{x}[n] \quad \forall n$$

- Frequency-domain also periodic in N:

$$\tilde{X}[k + N] = \tilde{X}[k] \quad \forall k$$

- “~” indicates periodic signal/spectrum

# Discrete Fourier Series

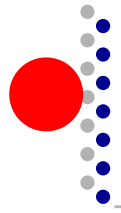
□ Define:

$$W_N \triangleq e^{-j2\pi/N}$$

□ DFS:

$$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] W_N^{-kn}$$

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn}$$



# Discrete Fourier Series

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$$W_N \triangleq e^{-j2\pi/N}$$

## □ Properties of $W_N$ :

- $W_N^0 = W_N^N = W_N^{2N} = \dots = 1$
- $W_N^{k+r} = W_N^k W_N^r$  and,  $W_N^{k+N} = W_N^k$



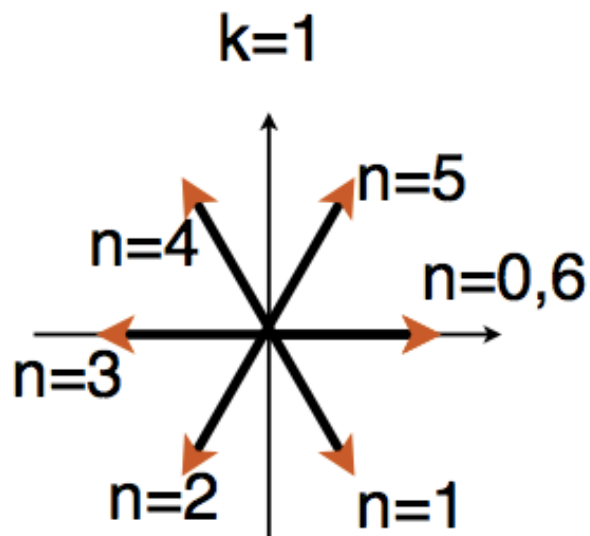
# Discrete Fourier Series

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## □ Example: $W_N^{kn}$ ( $N=6$ )



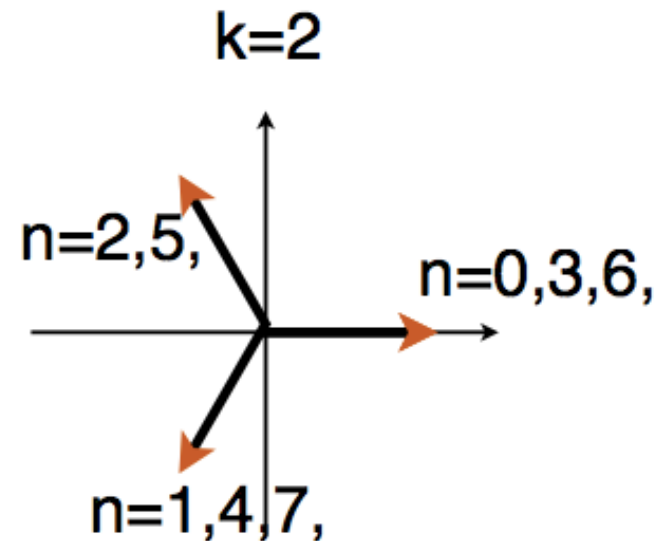
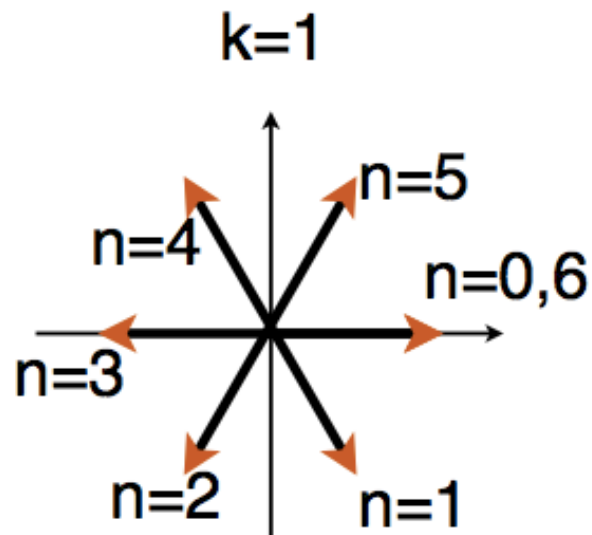
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## □ Example: $W_N^{kn}$ ( $N=6$ )



# Discrete Fourier Transform

- By convention, work with one period:

$$x[n] \triangleq \begin{cases} \tilde{x}[n] & 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases}$$
$$X[k] \triangleq \begin{cases} \tilde{X}[k] & 0 \leq k \leq N-1 \\ 0 & \text{otherwise} \end{cases}$$

Same, but different!

# Discrete Fourier Transform

## □ The DFT

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn} \quad \text{Inverse DFT, synthesis}$$

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} \quad \text{DFT, analysis}$$

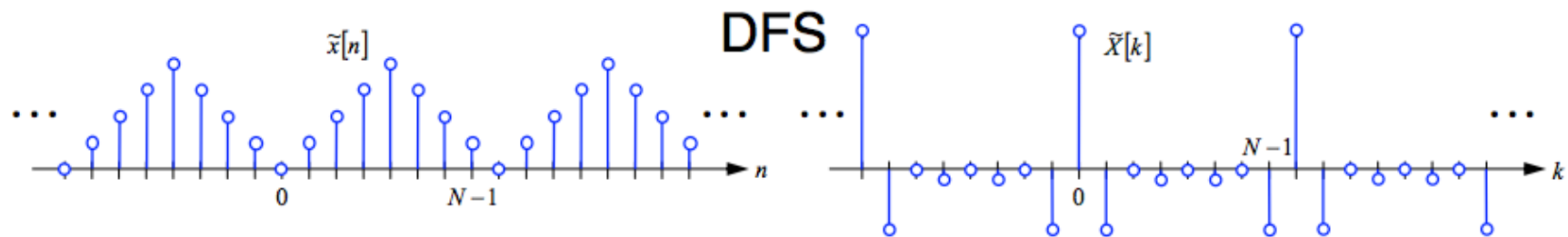
## □ It is understood that,

$$x[n] = 0 \quad \text{outside } 0 \leq n \leq N - 1$$

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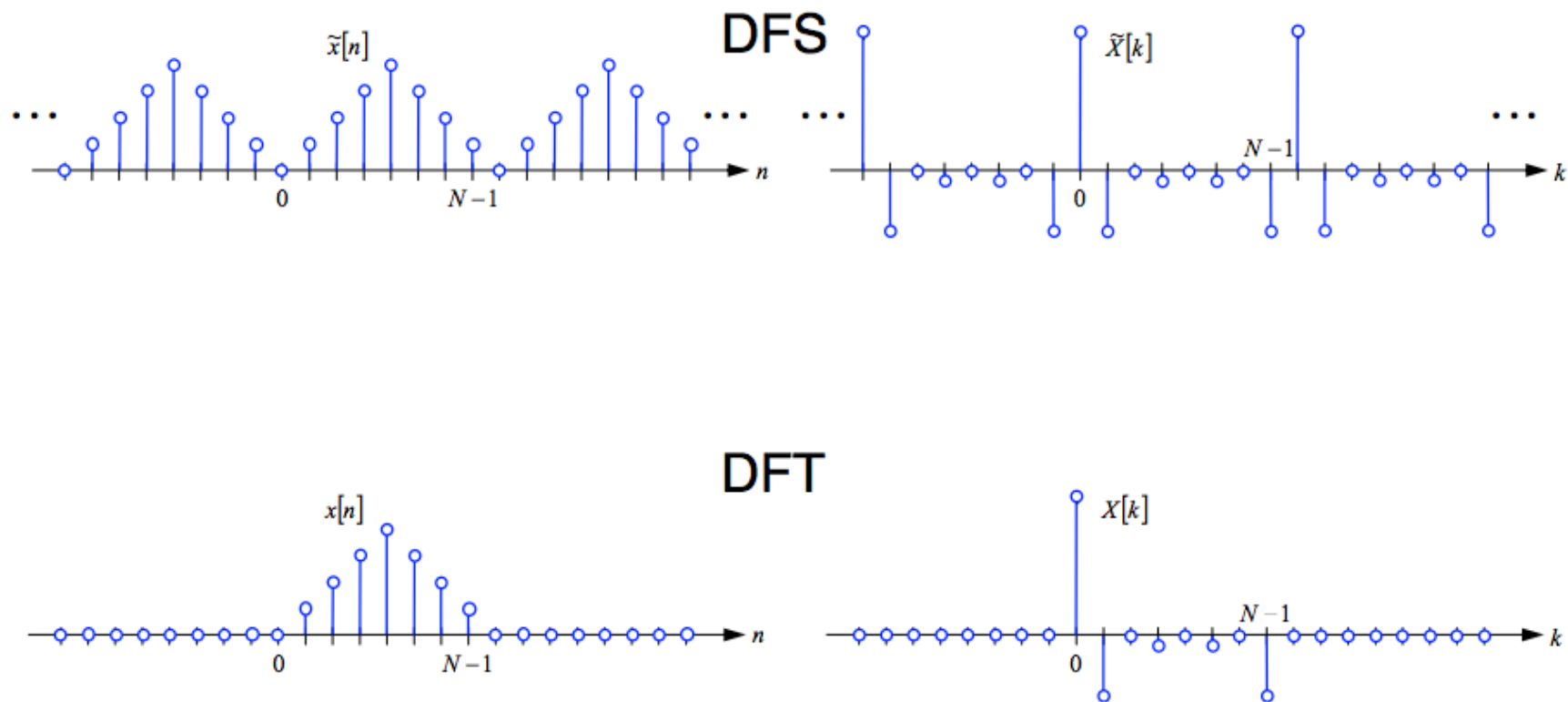


# DFS vs. DFT





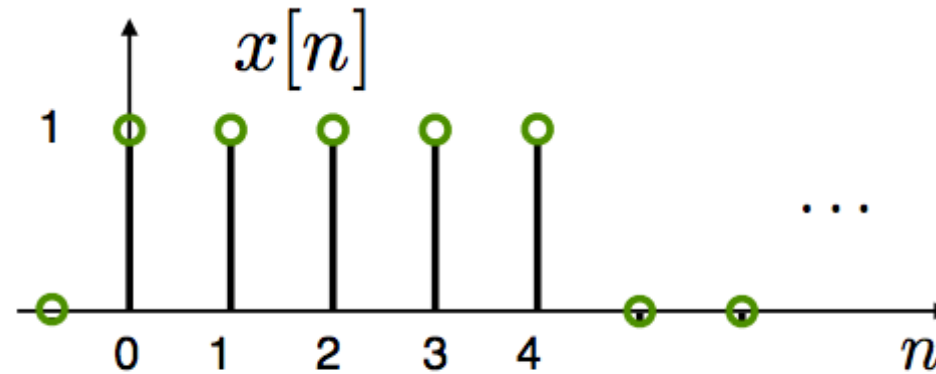
# DFS vs. DFT





# Example

$$W_N \triangleq e^{-j2\pi/N}$$



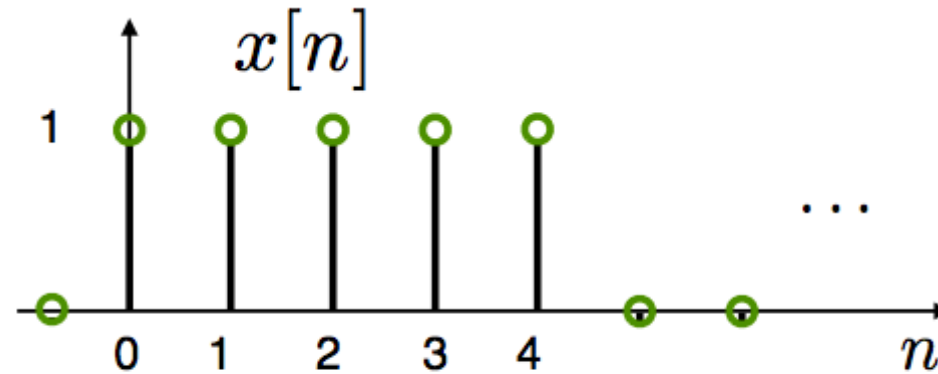
Take  $N=5$

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$$



# Example

$$W_N \triangleq e^{-j2\pi/N}$$



Take  $N=5$

$$X[k] = \begin{cases} \sum_{n=0}^4 W_5^{nk} & k = 0, 1, 2, 3, 4 \\ 0 & \text{otherwise} \end{cases}$$
$$= 5\delta[k]$$

“5-point DFT”

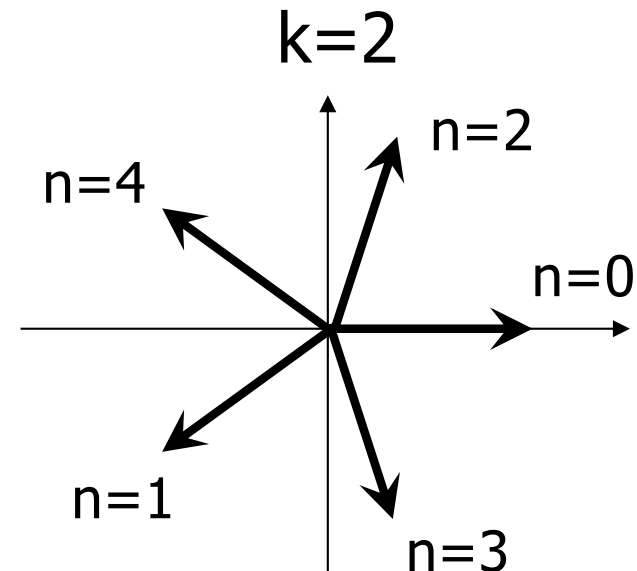
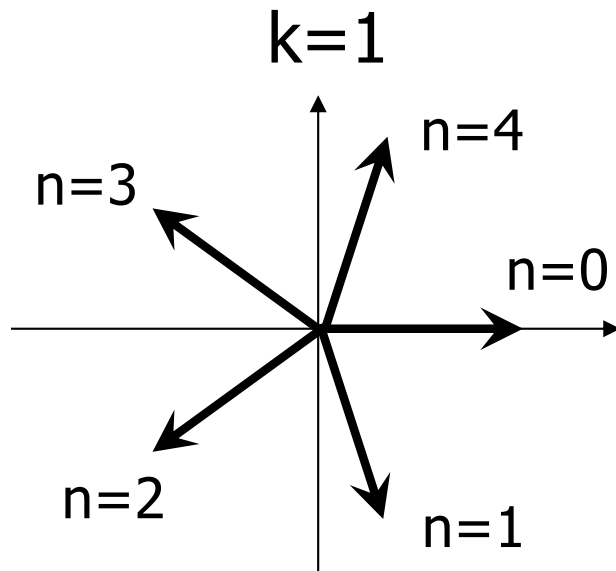
# Discrete Fourier Series

$$W_N \triangleq e^{-j2\pi/N}$$

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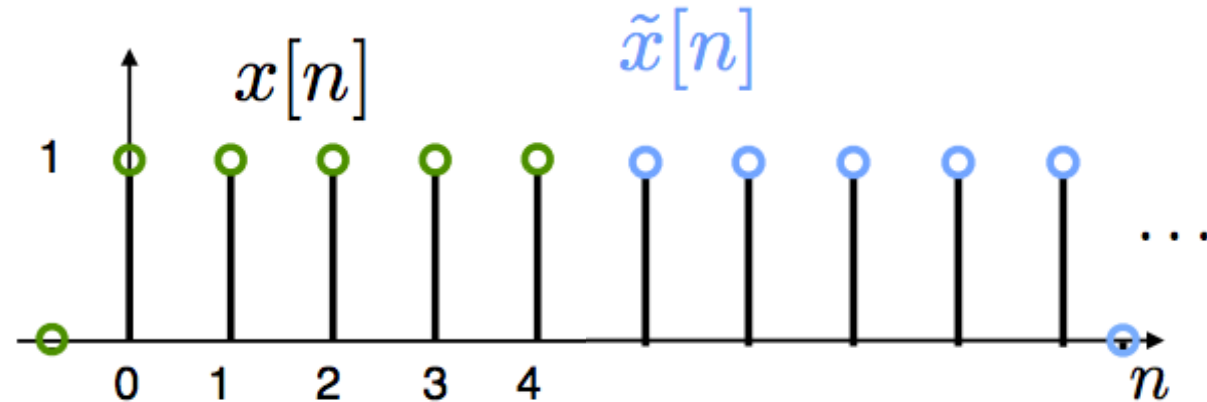
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# Example

$$W_N \triangleq e^{-j2\pi/N}$$



Take  $N=5$

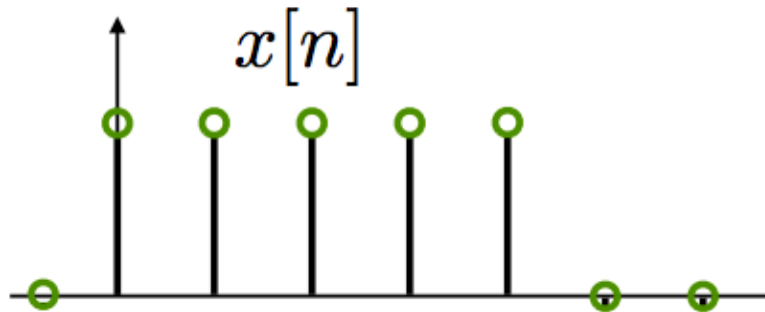
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“5-point DFT”

# Example

$$W_N \triangleq e^{-j2\pi/N}$$

- Q: What if we take  $N=10$ ?
- A:  $X[k] = \tilde{X}[k]$  where  $\tilde{x}[n]$  is a period-10 seq.



# Example

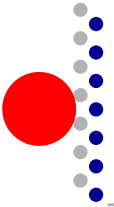
$$W_N \triangleq e^{-j2\pi/N}$$

- Q: What if we take  $N=10$ ?
- A:  $X[k] = \tilde{X}[k]$  where  $\tilde{x}[n]$  is a period-10 seq.



$$X[k] = \begin{cases} \sum_{n=0}^9 W_{10}^{nk} & k = 0, 1, 2, \dots, 9 \\ 0 & \text{otherwise} \end{cases}$$

“10-point DFT”



# Example

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□ Now, sum from  $n=0$  to 9

$$X[k] = \sum_{n=0}^9 x[n] W_{10}^{nk}$$

# Example

- Now, sum from  $n=0$  to 9

$$\begin{aligned} X[k] &= \sum_{n=0}^9 x[n] W_{10}^{nk} \\ &= \sum_{n=0}^4 W_{10}^{nk} \\ &= e^{-j \frac{4\pi}{10} k} \frac{\sin(\frac{\pi}{2} k)}{\sin(\frac{\pi}{10} k)} \end{aligned}$$



# DFT vs. DTFT

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□ For finite sequences of length  $N$ :

■ The  $N$ -point DFT of  $x[n]$  is:

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi/N)nk} \quad 0 \leq k \leq N-1$$

■ The DTFT of  $x[n]$  is:

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n] e^{-j\omega n} \quad -\infty < \omega < \infty$$



# DFT vs. DTFT

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- The DFT are samples of the DTFT at  $N$  equally spaced frequencies

$$X[k] = X(e^{j\omega})|_{\omega=k\frac{2\pi}{N}} \quad 0 \leq k \leq N - 1$$



# DFT vs DTFT

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□ Back to example

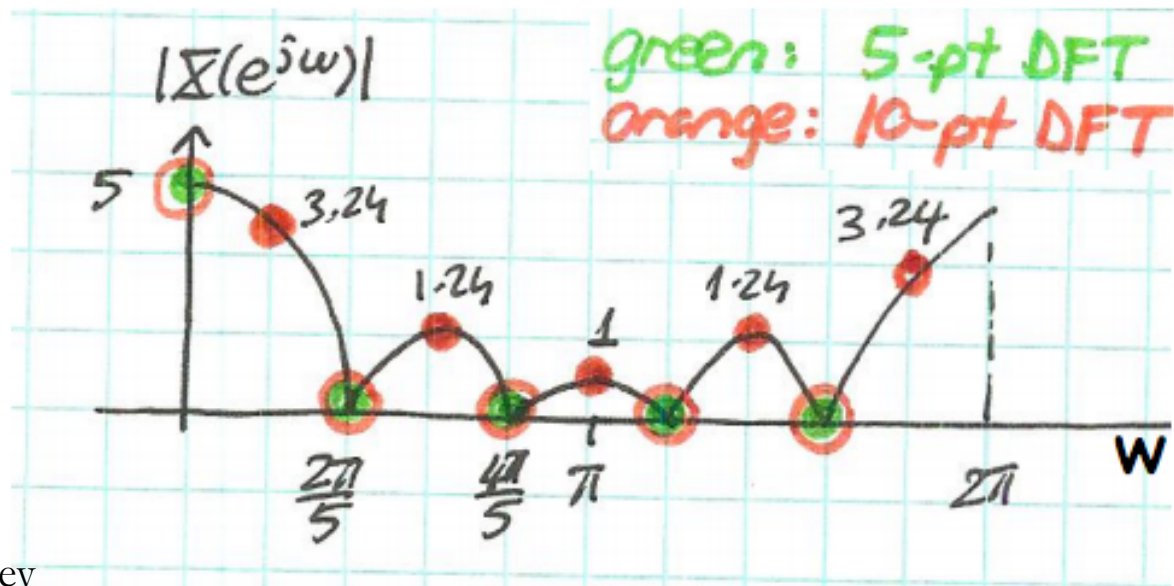
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“10-point DFT”

# DFT vs DTFT

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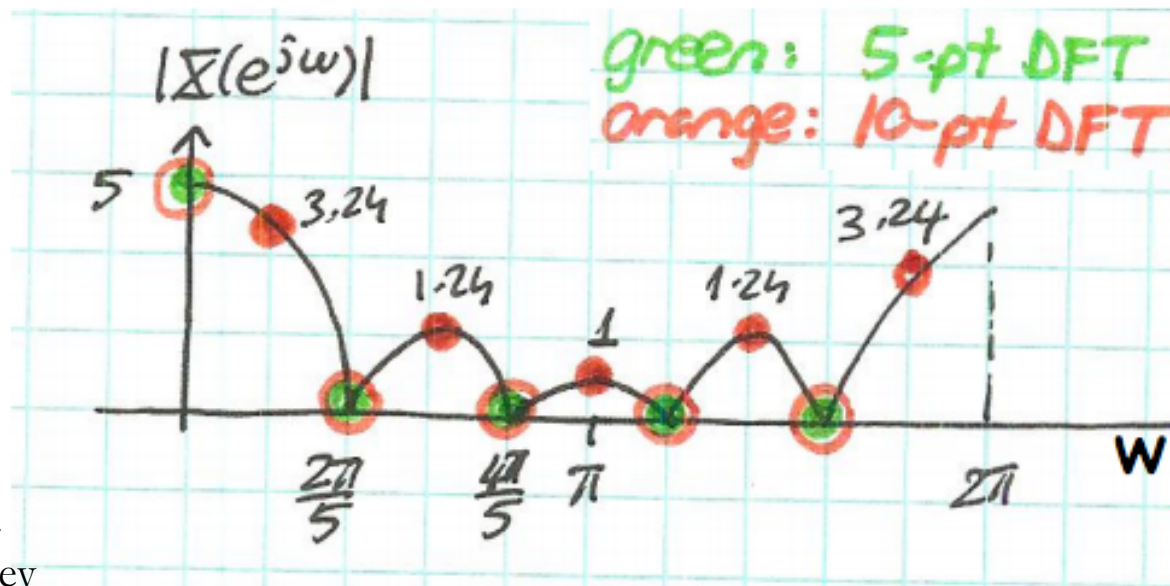


# DFT vs DTFT

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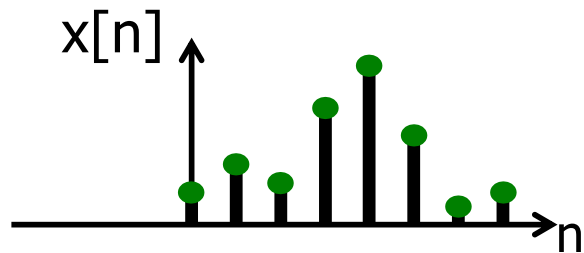
Use `fftshift`  
to center  
around dc





# DFT Intuition

Time

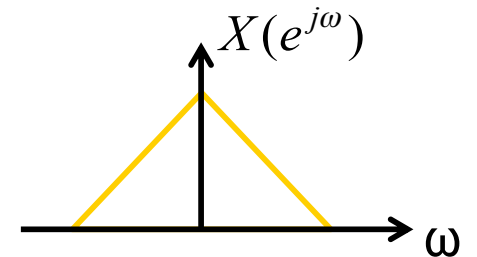


Transform

DTFT

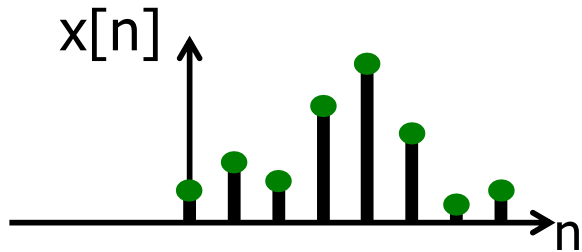
$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

Frequency



# DFT Intuition

Time

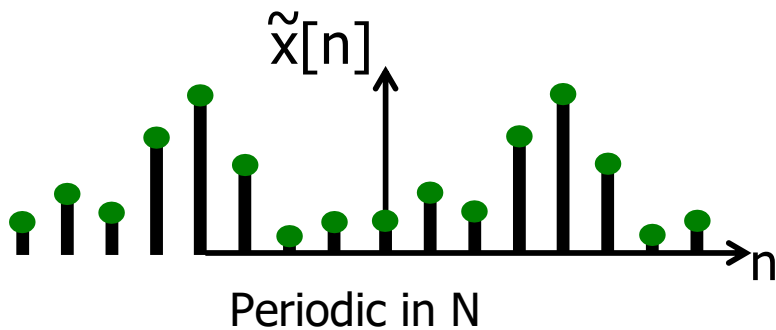
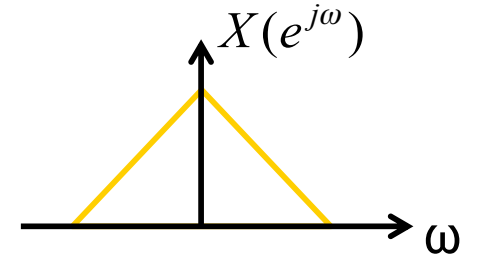


Transform

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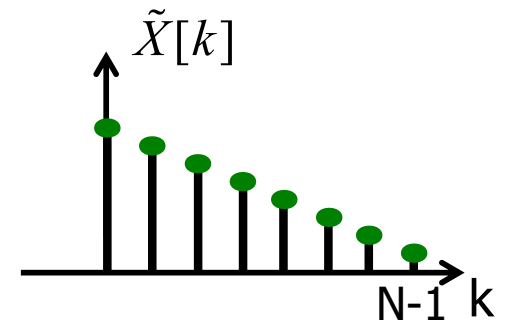
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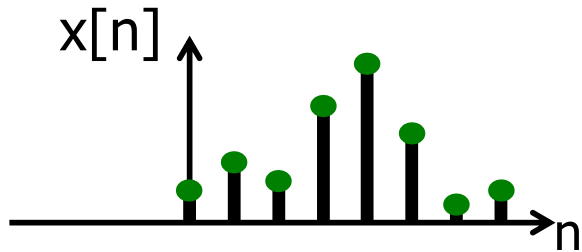
DFS

$$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] W_N^{-kn}$$



# DFT Intuition

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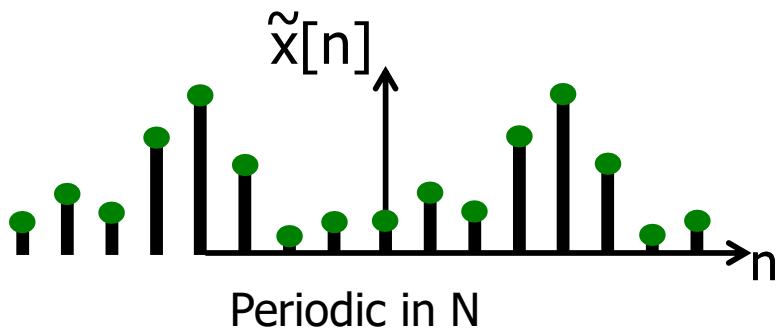
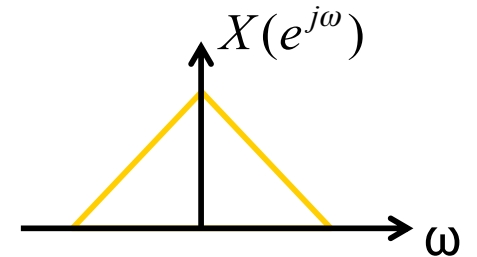


Transform

DTFT

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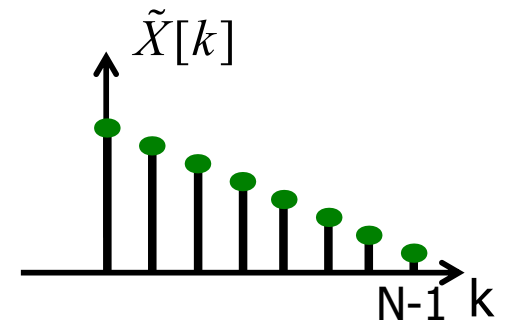
Frequency



DFS

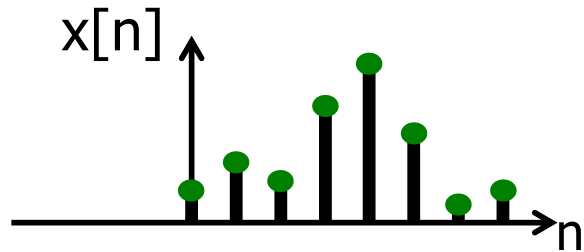
$$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] W_N^{-kn}$$

$$W_N = e^{-j\frac{2\pi}{N}}$$



# DFT Intuition

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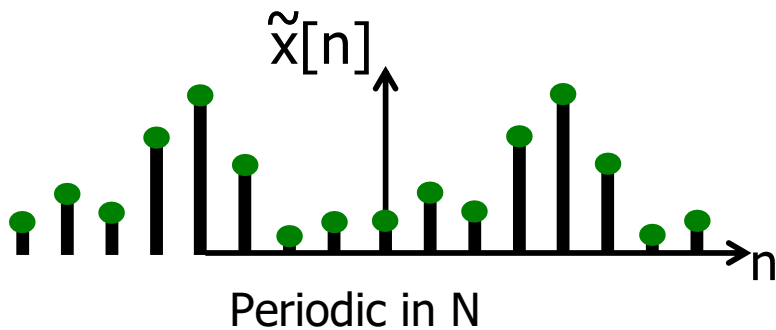
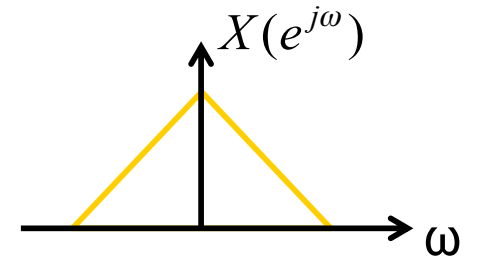


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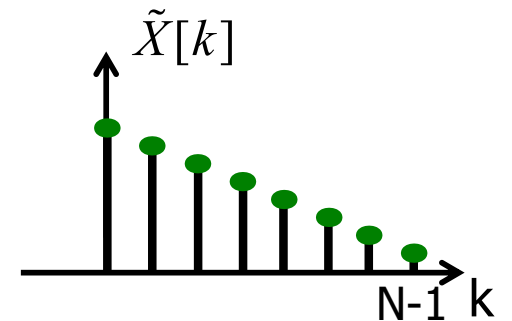
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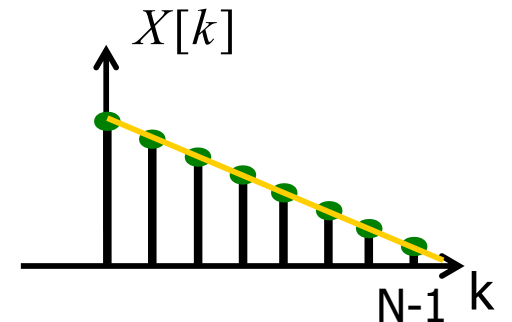
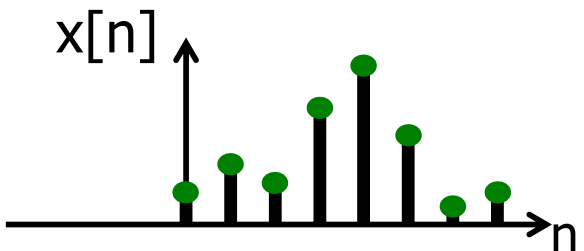
DFS

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DFT

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}$$





# DFT and Inverse DFT

---

- Use the DFT to compute the inverse DFT. How?

$$x[n] = \mathcal{DFT}^{-1}\{X[k]\}$$



# DFT and Inverse DFT

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- Use the DFT to compute the inverse DFT. How?

$$N \cdot x^*[n] = N \left( \mathcal{DFT}^{-1} \{X[k]\} \right)^*$$

# Reminder: Discrete Fourier Transform

## □ The DFT

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn} \quad \text{Inverse DFT, synthesis}$$

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} \quad \text{DFT, analysis}$$

## □ It is understood that,

$$x[n] = 0 \quad \text{outside } 0 \leq n \leq N - 1$$

$$X[k] = 0 \quad \text{outside } 0 \leq k \leq N - 1$$

# DFT and Inverse DFT

- Use the DFT to compute the inverse DFT. How?

$$\begin{aligned} N \cdot x^*[n] &= N \left( \mathcal{DFT}^{-1} \{X[k]\} \right)^* \\ &= N \left( \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn} \right)^* \\ &= \sum_{k=0}^{N-1} X^*[k] W_N^{kn} \\ &= \mathcal{DFT} \{X^*[k]\}. \end{aligned}$$



# DFT and Inverse DFT

---

□ So

$$\mathcal{DFT} \{X^*[k]\} = N (\mathcal{DFT}^{-1} \{X[k]\})^*$$

# DFT and Inverse DFT

□ So

$$\mathcal{DFT} \{X^*[k]\} = N (\mathcal{DFT}^{-1} \{X[k]\})^*$$

$$\mathcal{DFT}^{-1} \{X[k]\} = \left(\frac{1}{N}\right) (\mathcal{DFT} \{X^*[k]\})^*$$

□ Implement IDFT by:

- Take complex conjugate
- Take DFT
- Multiply by 1/N
- Take complex conjugate



# DFT as Matrix Operator

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- Can write compactly as

$$\begin{aligned}\mathbf{X} &= \mathbf{W}_N \mathbf{x} \\ \mathbf{x} &= \frac{1}{N} \mathbf{W}_N^* \mathbf{X}\end{aligned}$$



# DFT as Matrix Operator

---

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$$

DFT:

$$\begin{pmatrix} X[0] \\ \vdots \\ X[k] \\ \vdots \\ X[N-1] \end{pmatrix} = \begin{pmatrix} W_N^{00} & \dots & W_N^{0n} & \dots & W_N^{0(N-1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ W_N^{k0} & \dots & W_N^{kn} & \dots & W_N^{k(N-1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ W_N^{(N-1)0} & \dots & W_N^{(N-1)n} & \dots & W_N^{(N-1)(N-1)} \end{pmatrix} \begin{pmatrix} x[0] \\ \vdots \\ x[n] \\ \vdots \\ x[N-1] \end{pmatrix}$$

# DFT as Matrix Operator

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**IDFT:**

$$\begin{pmatrix} x[0] \\ \vdots \\ x[n] \\ \vdots \\ x[N-1] \end{pmatrix} = \frac{1}{N} \begin{pmatrix} W_N^{-00} & \dots & W_N^{-0k} & \dots & W_N^{-0(N-1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ W_N^{-n0} & \dots & W_N^{-nk} & \dots & W_N^{-n(N-1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ W_N^{-(N-1)0} & \dots & W_N^{-(N-1)k} & \dots & W_N^{-(N-1)(N-1)} \end{pmatrix} \begin{pmatrix} X[0] \\ \vdots \\ X[k] \\ \vdots \\ X[N-1] \end{pmatrix}$$

# DFT as Matrix Operator

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DFT:

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IDFT:

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# Properties of the DFT

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- ❑ Properties of DFT inherited from DFS
- ❑ Linearity

$$\alpha_1 x_1[n] + \alpha_2 x_2[n] \leftrightarrow \alpha_1 X_1[k] + \alpha_2 X_2[k]$$



# Properties of the DFT

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- Properties of DFT inherited from DFS
- Linearity

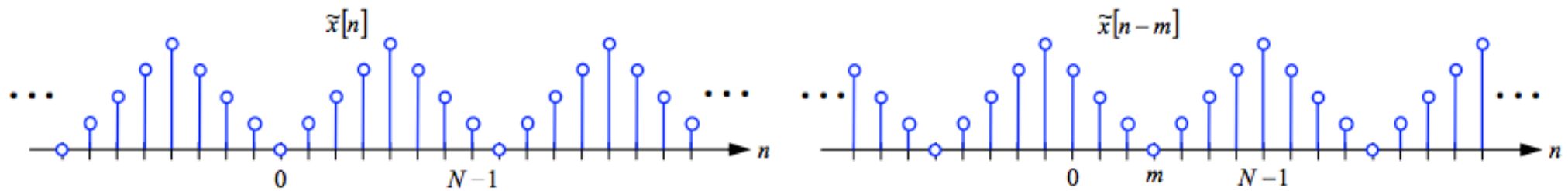
$$\alpha_1 x_1[n] + \alpha_2 x_2[n] \leftrightarrow \alpha_1 X_1[k] + \alpha_2 X_2[k]$$

- Circular Time Shift

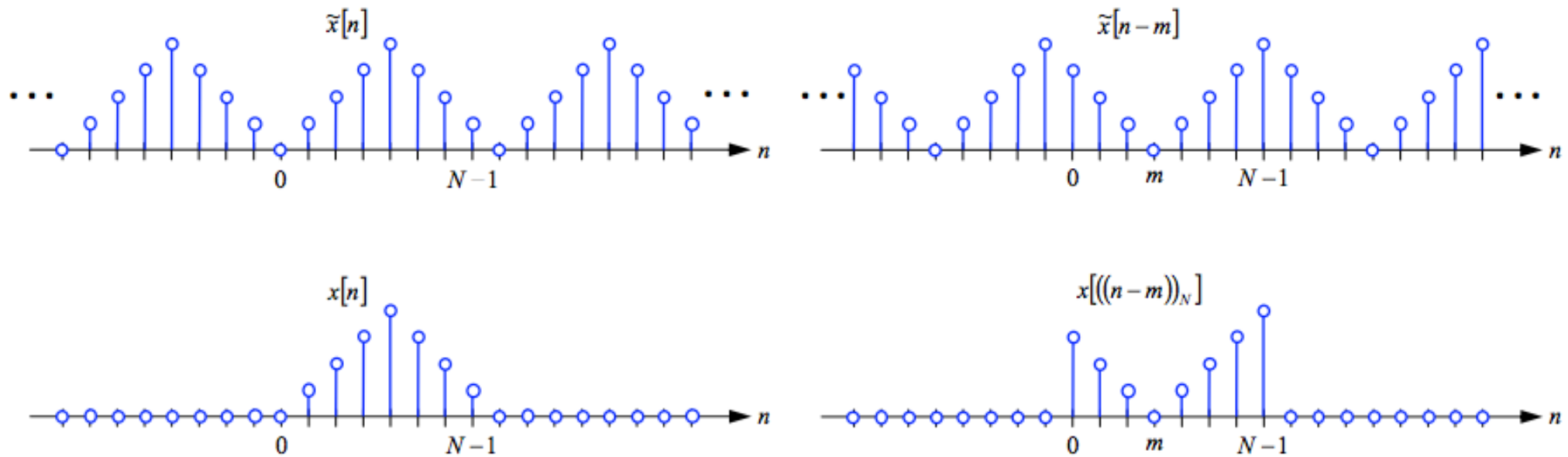
$$x[((n - m))_N] \leftrightarrow X[k]e^{-j(2\pi/N)km} = X[k]W_N^{km}$$



# Circular Shift



# Circular Shift





# Properties of DFT

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- Circular frequency shift

$$x[n]e^{j(2\pi/N)nl} = x[n]W_N^{-nl} \leftrightarrow X[((k-l))_N]$$

- Complex Conjugation

$$x^*[n] \leftrightarrow X^*[((-k))_N]$$

- Conjugate Symmetry for Real Signals

$$x[n] = x^*[n] \leftrightarrow X[k] = X^*[((-k))_N]$$

# Properties of the DFS/DFT

| Discrete Fourier Series |                                                    |                                                              | Discrete Fourier Transform |                                         |                                                    |
|-------------------------|----------------------------------------------------|--------------------------------------------------------------|----------------------------|-----------------------------------------|----------------------------------------------------|
| Property                | $N$ -periodic sequence                             | $N$ -periodic DFS                                            | Property                   | $N$ -point sequence                     | $N$ -point DFT                                     |
|                         | $\tilde{x}[n]$<br>$\tilde{x}_1[n], \tilde{x}_2[n]$ | $\tilde{X}[k]$<br>$\tilde{X}_1[k], \tilde{X}_2[k]$           |                            | $x[n]$<br>$x_1[n], x_2[n]$              | $X[k]$<br>$X_1[k], X_2[k]$                         |
| Linearity               | $a\tilde{x}_1[n] + b\tilde{x}_2[n]$                | $a\tilde{X}_1[k] + b\tilde{X}_2[k]$                          | Linearity                  | $ax_1[n] + bx_2[n]$                     | $aX_1[k] + bX_2[k]$                                |
| Duality                 | $\tilde{X}[n]$                                     | $N\tilde{x}[-k]$                                             | Duality                    | $X[n]$                                  | $Nx[((-k))_N]$                                     |
| Time Shift              | $\tilde{x}[n-m]$                                   | $W_N^{km}\tilde{X}[k]$                                       | Circular Time Shift        | $x[((n-m))_N]$                          | $W_N^{km}X[k]$                                     |
| Frequency Shift         | $W_N^{-ln}\tilde{x}[n]$                            | $\tilde{X}[k-l]$                                             | Circular Frequency Shift   | $W_N^{-ln}x[n]$                         | $X[((k-l))_N]$                                     |
| Periodic Convolution    | $\sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m]$  | $\tilde{X}_1[k]\tilde{X}_2[k]$                               | Circular Convolution       | $\sum_{m=0}^{N-1} x_1[m]x_2[((n-m))_N]$ | $X_1[k]X_2[k]$                                     |
| Multiplication          | $\tilde{x}_1[n]\tilde{x}_2[n]$                     | $\frac{1}{N}\sum_{l=0}^{N-1} \tilde{X}_1[l]\tilde{X}_2[k-l]$ | Multiplication             | $x_1[n]x_2[n]$                          | $\frac{1}{N}\sum_{l=0}^{N-1} X_1[l]X_2[((k-l))_N]$ |
| Complex Conjugation     | $\tilde{x}^*[n]$                                   | $\tilde{X}^*[-k]$                                            | Complex Conjugation        | $x^*[n]$                                | $X^*[((-k))_N]$                                    |

# Properties (Continued)

|                                       |                                                                                                                                                                                                       |                                                                                                                                                                                                                                                                                                          |                                       |                                                                                                                                                       |                                                                                                                                                                                                                                                        |
|---------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Time-Reversal and Complex Conjugation | $\tilde{x}^*[-n]$                                                                                                                                                                                     | $\tilde{X}^*[k]$                                                                                                                                                                                                                                                                                         | Time-Reversal and Complex Conjugation | $x^*[((-n))_N]$                                                                                                                                       | $X^*[k]$                                                                                                                                                                                                                                               |
| Real Part                             | $\text{Re}\{\tilde{x}[n]\}$                                                                                                                                                                           | $\tilde{X}_{ep}[k] = \frac{1}{2}(\tilde{X}[k] + \tilde{X}^*[-k])$                                                                                                                                                                                                                                        | Real Part                             | $\text{Re}\{x[n]\}$                                                                                                                                   | $X_{ep}[k] = \frac{1}{2}(X[k] + X^*[((-k))_N])$                                                                                                                                                                                                        |
| Imaginary Part                        | $j \text{Im}\{\tilde{x}[n]\}$                                                                                                                                                                         | $\tilde{X}_{op}[k] = \frac{1}{2}(\tilde{X}[k] - \tilde{X}^*[-k])$                                                                                                                                                                                                                                        | Imaginary Part                        | $j \text{Im}\{x[n]\}$                                                                                                                                 | $X_{op}[k] = \frac{1}{2}(X[k] - X^*[((-k))_N])$                                                                                                                                                                                                        |
| Even Part                             | $\tilde{x}_{ep}[n] = \frac{1}{2}(\tilde{x}[n] + \tilde{x}^*[-n])$                                                                                                                                     | $\text{Re}\{\tilde{X}[k]\}$                                                                                                                                                                                                                                                                              | Even Part                             | $x_{ep}[n] = \frac{1}{2}(x[n] + x^*[((-n))_N])$                                                                                                       | $\text{Re}\{X[k]\}$                                                                                                                                                                                                                                    |
| Odd Part                              | $\tilde{x}_{op}[n] = \frac{1}{2}(\tilde{x}[n] - \tilde{x}^*[-n])$                                                                                                                                     | $j \text{Im}\{\tilde{X}[k]\}$                                                                                                                                                                                                                                                                            | Odd Part                              | $x_{op}[n] = \frac{1}{2}(x[n] - x^*[((-n))_N])$                                                                                                       | $j \text{Im}\{X[k]\}$                                                                                                                                                                                                                                  |
| Symmetry for Real Sequence            | $\tilde{x}[n] = \tilde{x}^*[n]$                                                                                                                                                                       | $\tilde{X}[k] = \tilde{X}^*[-k]$<br>$\begin{cases} \text{Re}\{\tilde{X}[k]\} = \text{Re}\{\tilde{X}^*[-k]\} \\ \text{Im}\{\tilde{X}[k]\} = -\text{Im}\{\tilde{X}^*[-k]\} \end{cases}$<br>$\begin{cases}  \tilde{X}[k]  =  \tilde{X}^*[-k]  \\ \angle \tilde{X}[k] = -\angle \tilde{X}^*[-k] \end{cases}$ | Symmetry for Real Sequence            | $x[n] = x^*[n]$                                                                                                                                       | $X[k] = X^*[((-k))_N]$<br>$\begin{cases} \text{Re}\{X[k]\} = \text{Re}\{X^*[((-k))_N]\} \\ \text{Im}\{X[k]\} = -\text{Im}\{X^*[((-k))_N]\} \end{cases}$<br>$\begin{cases}  X[k]  =  X^*[((-k))_N]  \\ \angle X[k] = -\angle X^*[((-k))_N] \end{cases}$ |
| Parseval's Identity                   | $\sum_{n=0}^{N-1} \tilde{x}_1[n] \tilde{x}_2^*[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}_1[k] \tilde{X}_2^*[k]$ $\sum_{n=0}^{N-1}  \tilde{x}[n] ^2 = \frac{1}{N} \sum_{k=0}^{N-1}  \tilde{X}[k] ^2$ |                                                                                                                                                                                                                                                                                                          | Parseval's Identity                   | $\sum_{n=0}^{N-1} x_1[n] x_2^*[n] = \frac{1}{N} \sum_{k=0}^{N-1} X_1[k] X_2^*[k]$ $\sum_{n=0}^{N-1}  x[n] ^2 = \frac{1}{N} \sum_{k=0}^{N-1}  X[k] ^2$ |                                                                                                                                                                                                                                                        |



# Circular Convolution

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□ Circular Convolution:

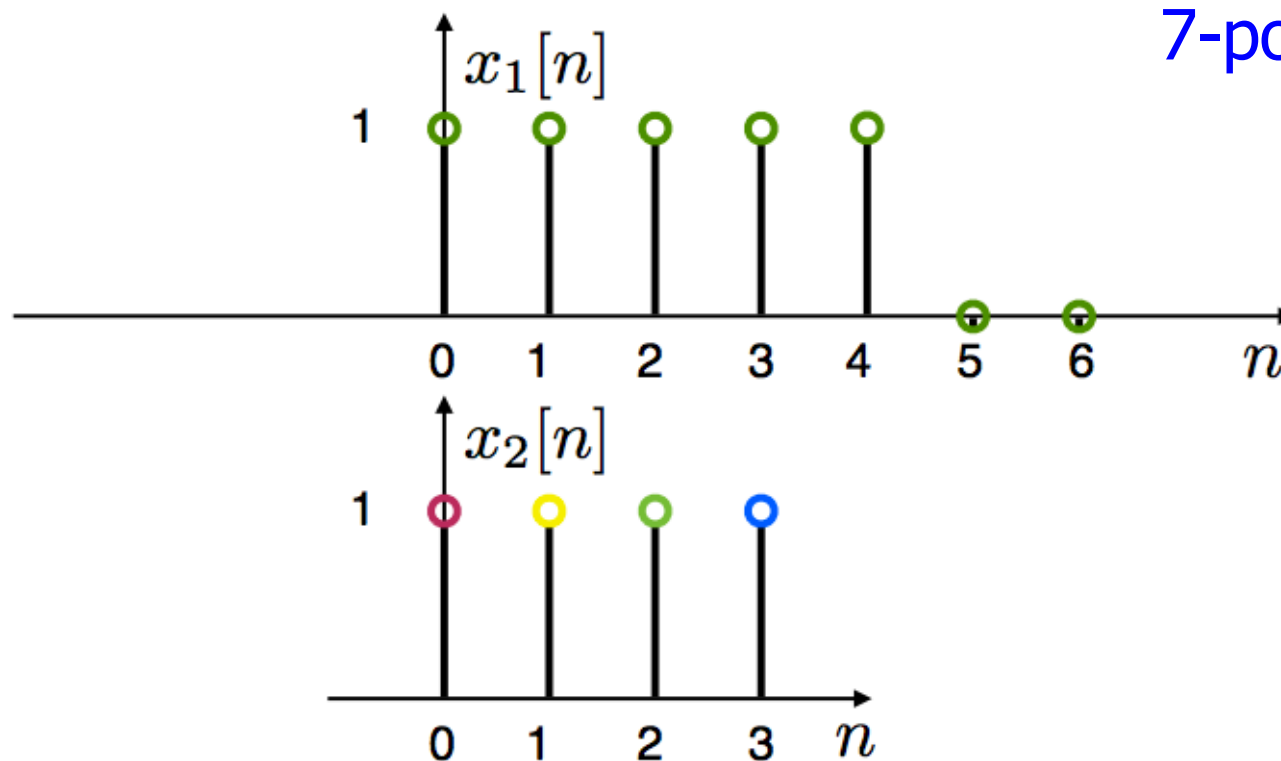
$$x_1[n] \circledN x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m] x_2[((n - m))_N]$$

For two signals of length  $N$

**Note: Circular convolution is commutative**

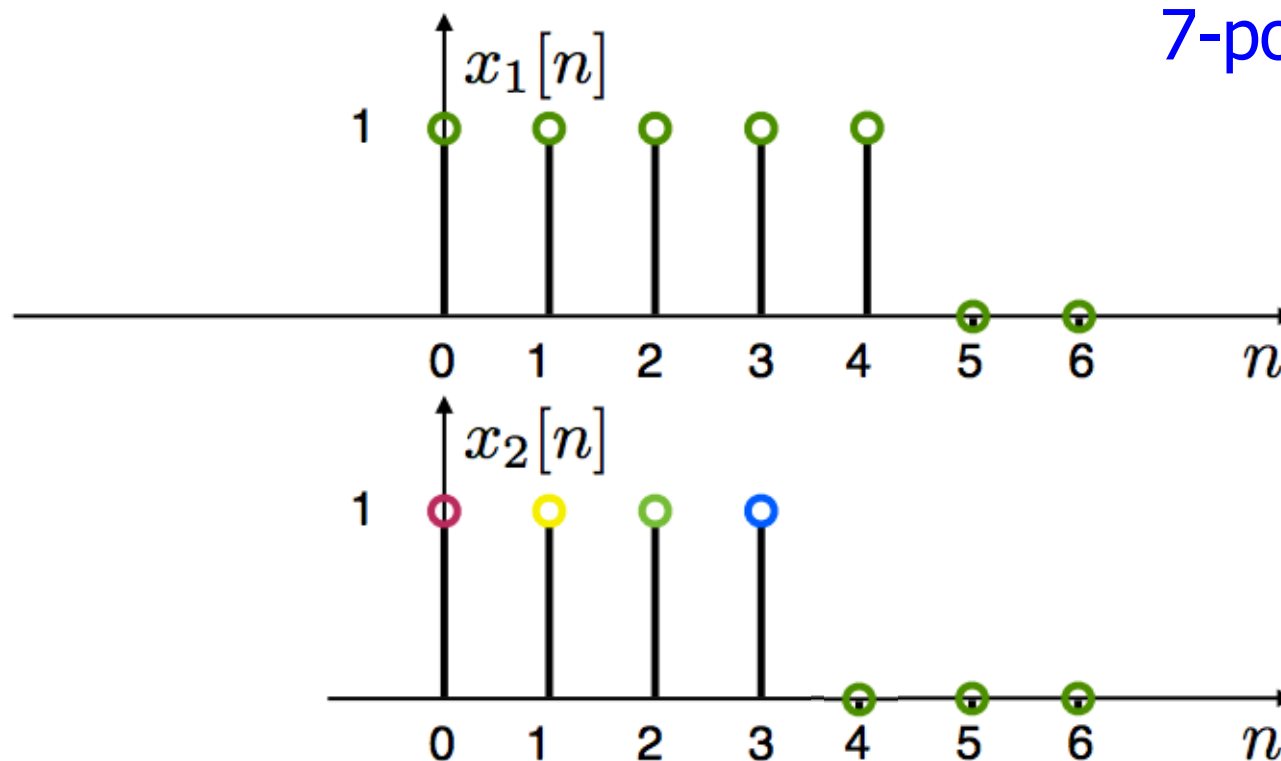
$$x_2[n] \circledN x_1[n] = x_1[n] \circledN x_2[n]$$

# Compute Circular Convolution Sum



$$x_1[n] \circledcirc x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m] x_2[((n - m))_N]$$

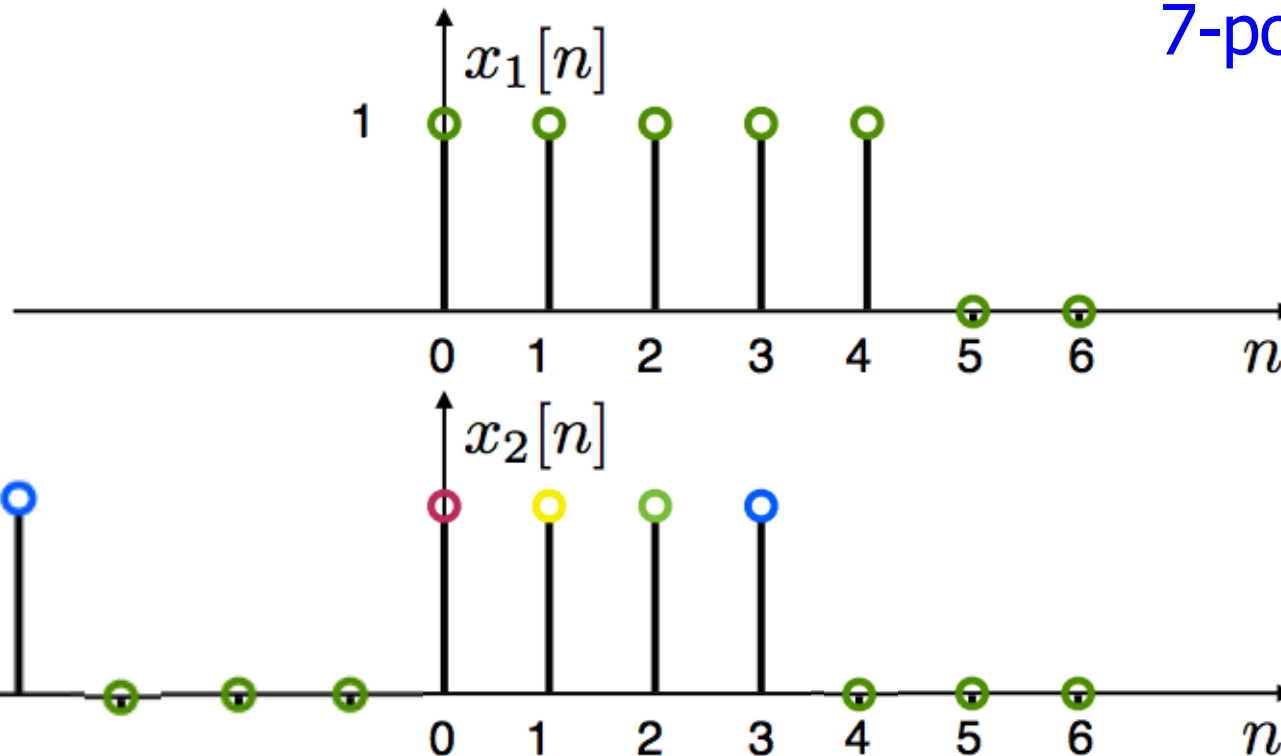
# Compute Circular Convolution Sum



$$x_1[n] \circledN x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m] x_2[((n - m))_N]$$

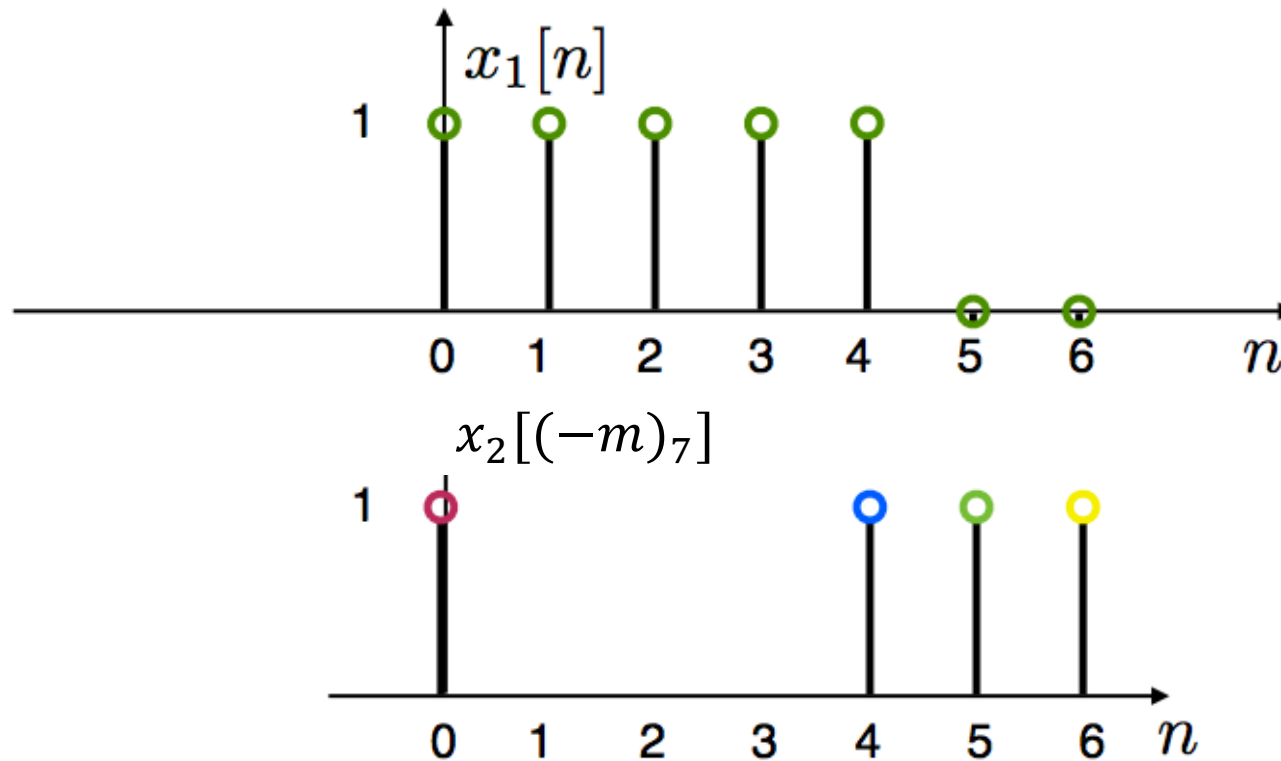
# Compute Circular Convolution Sum

7-point convolution



$$x_1[n] \circledcirc x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m] x_2[((n - m))_N]$$

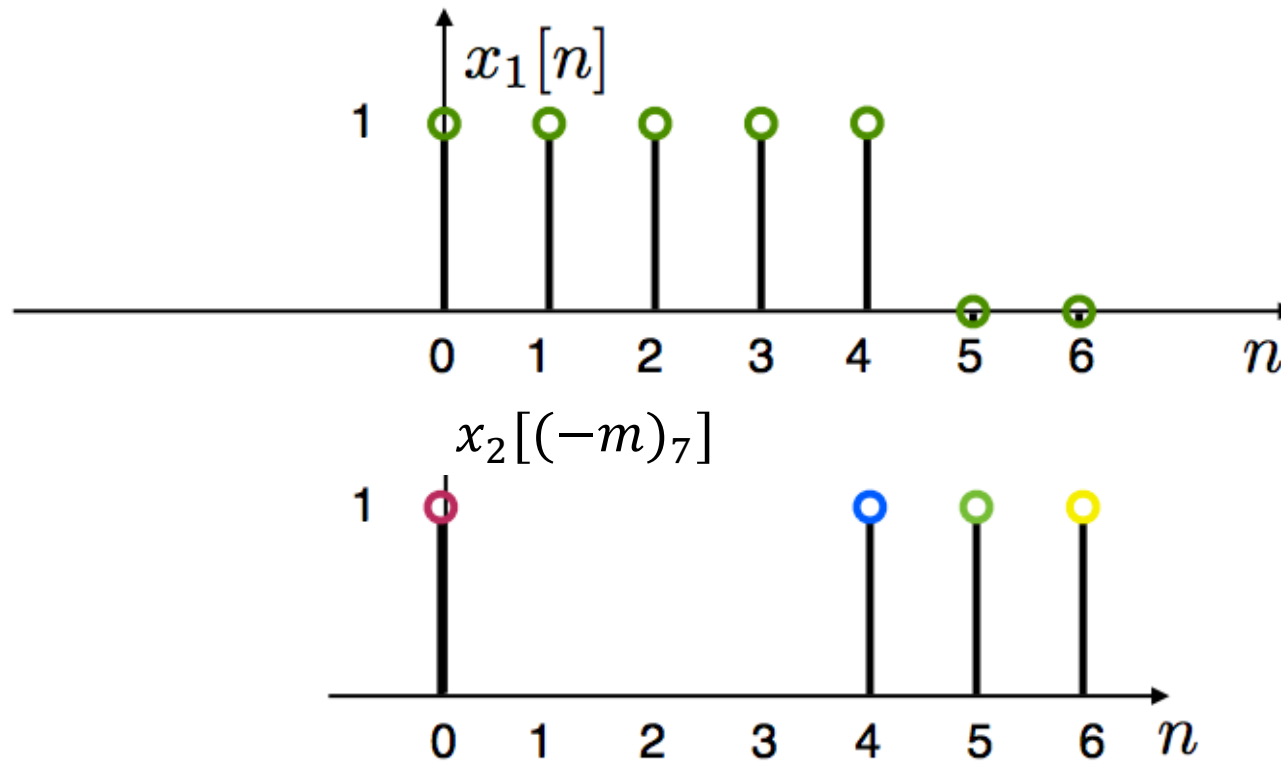
# Compute Circular Convolution Sum



$$x_1[n] \circledN x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m] x_2[((n - m))_N]$$

# Compute Circular Convolution Sum

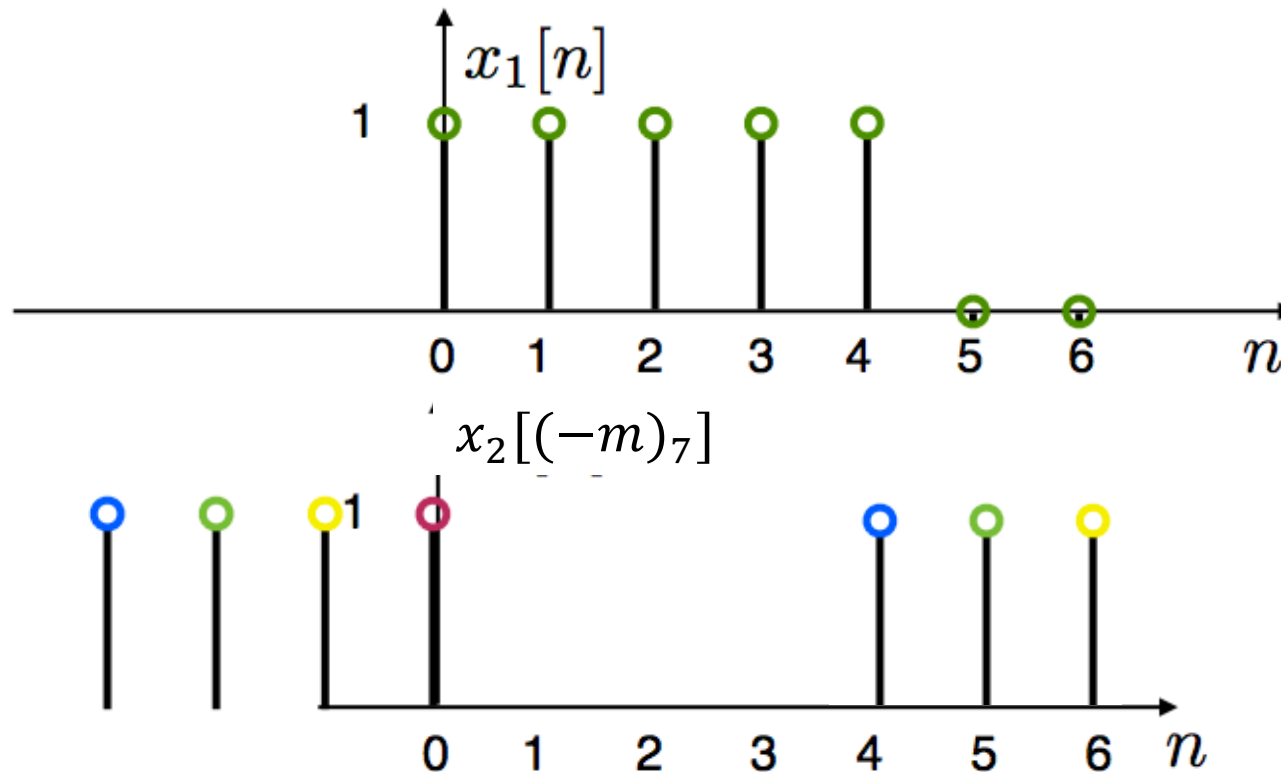
$$y[0]=2$$



$$x_1[n] \circledast x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m] x_2[((n-m))_N]$$

# Compute Circular Convolution Sum

$$y[0]=2$$

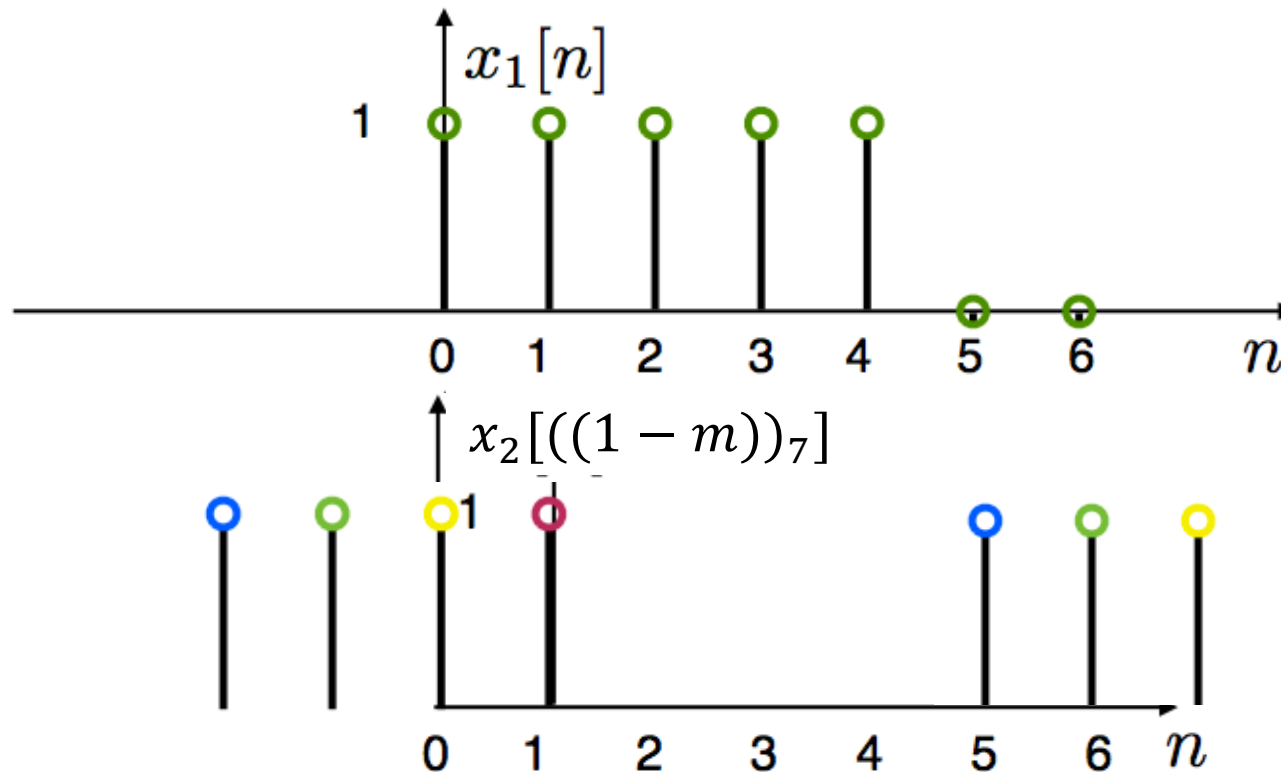


$$x_1[n] \circledN x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m] x_2[((n - m))_N]$$

# Compute Circular Convolution Sum

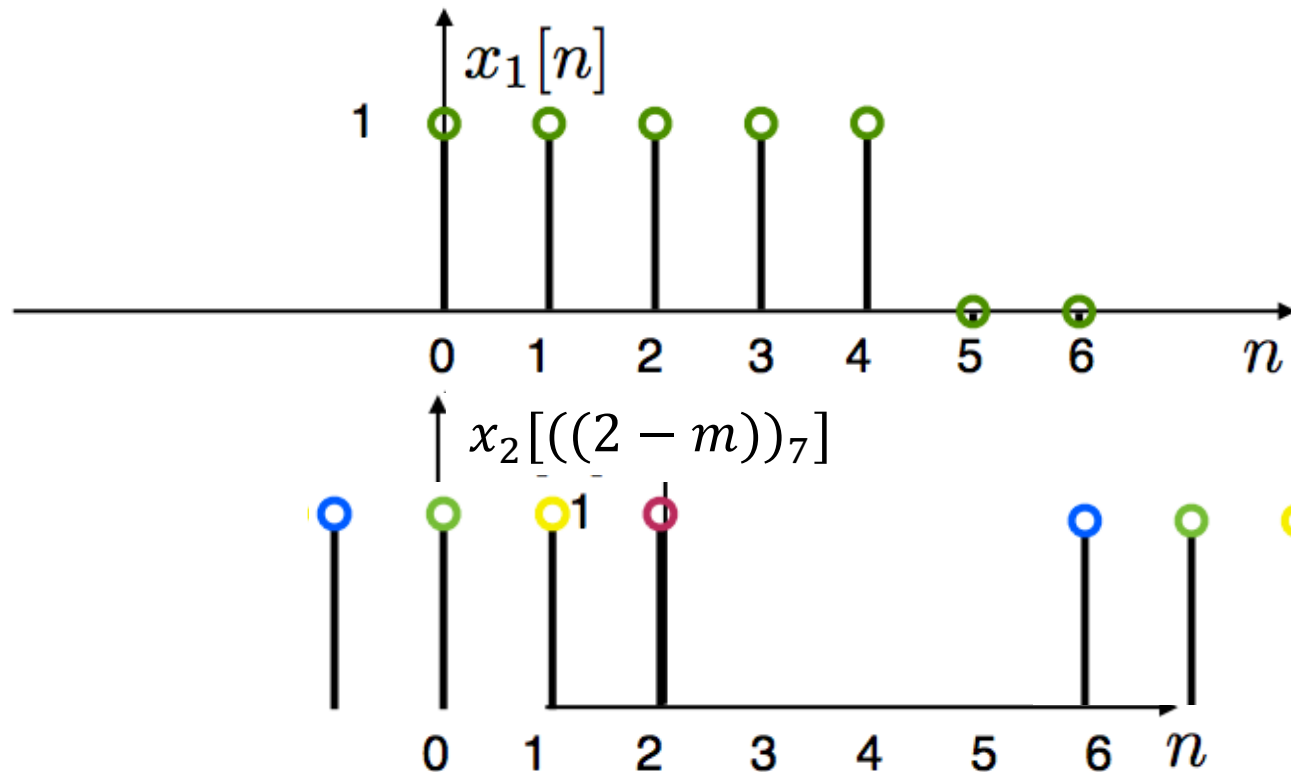
$$y[0]=2$$

$$y[1]=2$$



$$x_1[n] \circledN x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m] x_2[((n-m))_N]$$

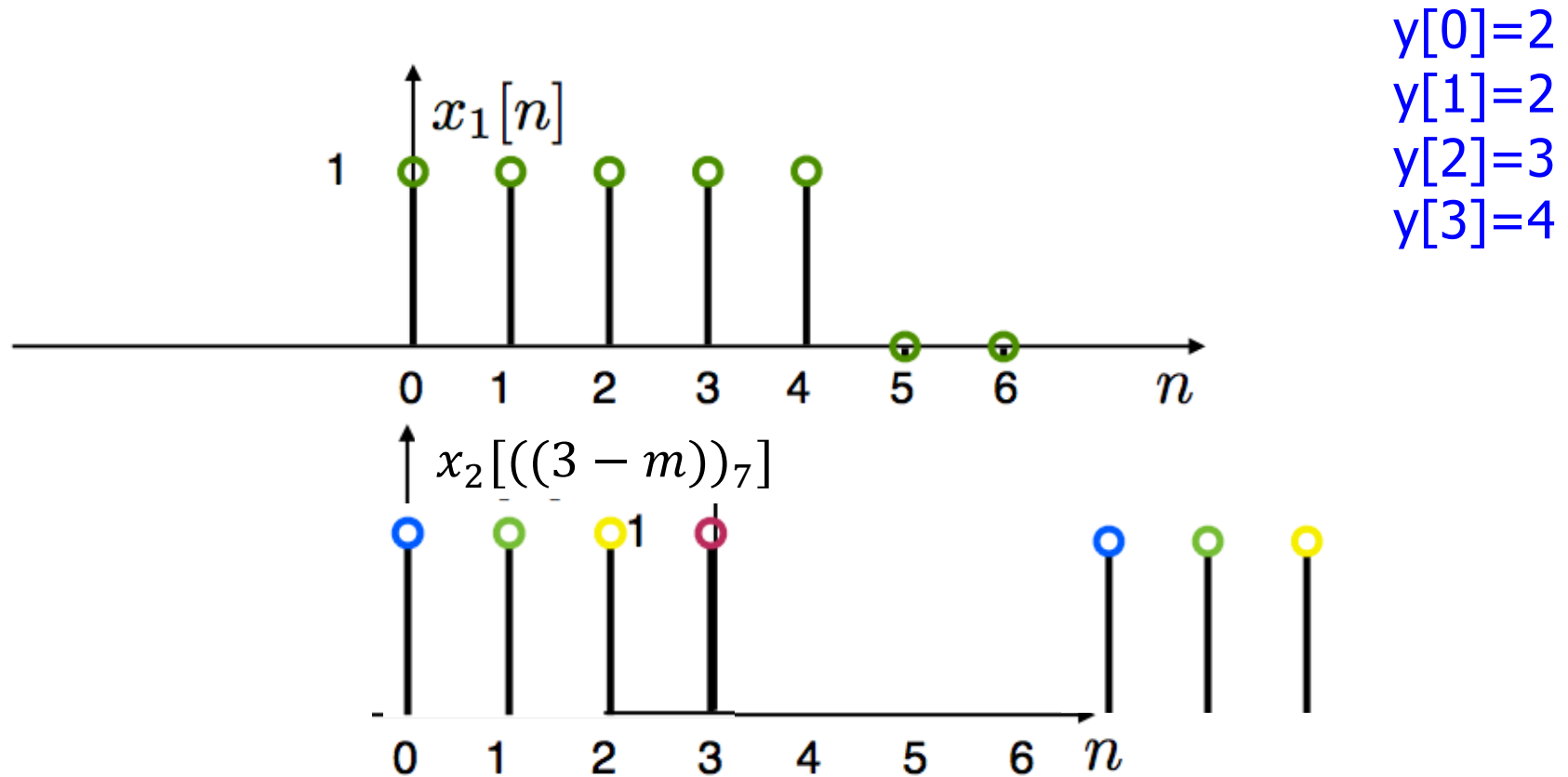
# Compute Circular Convolution Sum



$$\begin{aligned} y[0] &= 2 \\ y[1] &= 2 \\ y[2] &= 3 \end{aligned}$$

$$x_1[n] \circledast x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m] x_2[((n - m))_N]$$

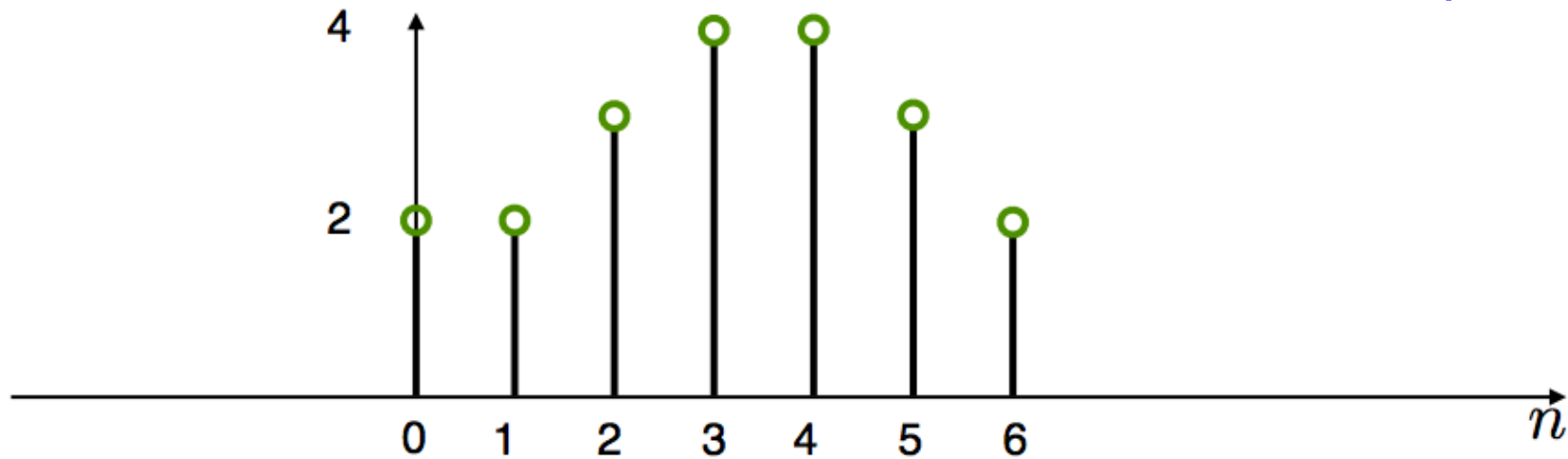
# Compute Circular Convolution Sum



$$x_1[n] \circledN x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m] x_2[((n-m))_N]$$

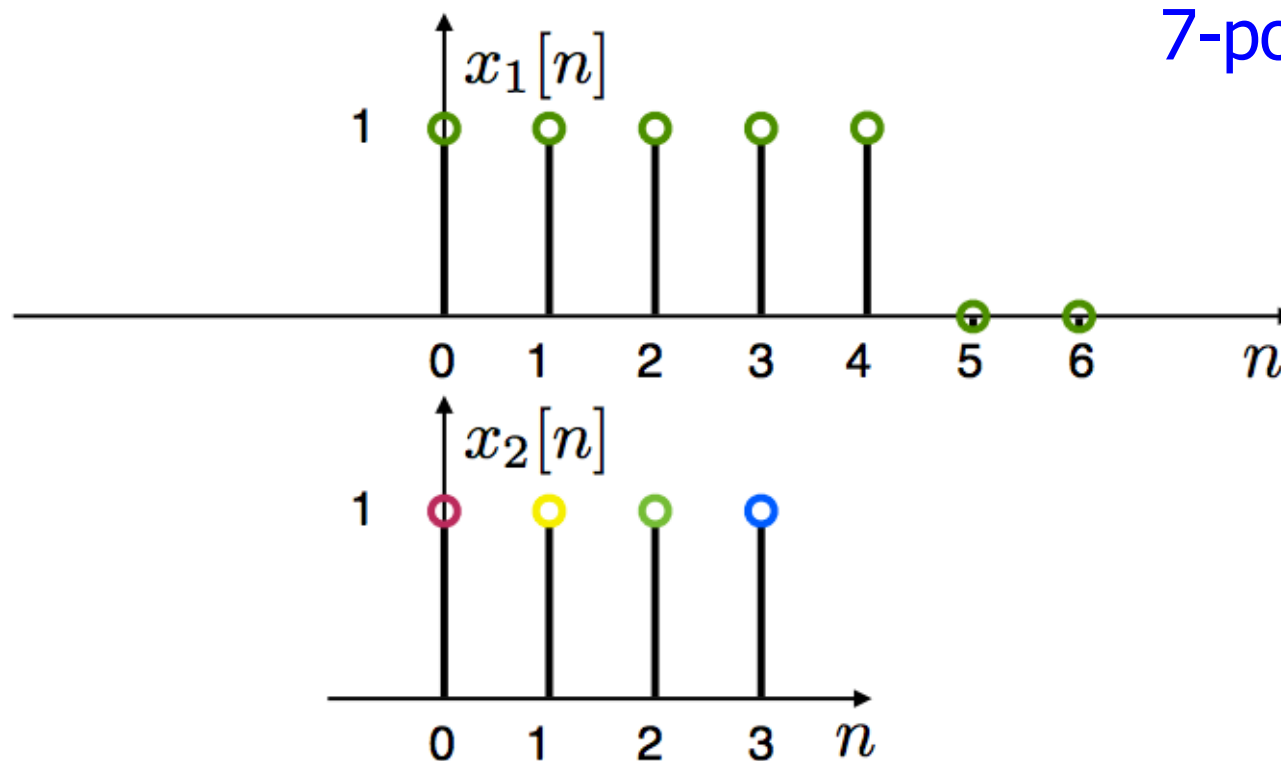
# Result

$y[0]=2$   
 $y[1]=2$   
 $y[2]=3$   
 $y[3]=4$



$$x_1[n] \circledN x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m] x_2[((n-m))_N]$$

# Compute Circular Convolution Sum



$$x_1[n] \circledN x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m] x_2[((n - m))_N]$$



# Circular Convolution

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- For  $x_1[n]$  and  $x_2[n]$  with length  $N$

$$x_1[n] \circledR x_2[n] \leftrightarrow X_1[k] \cdot X_2[k]$$

- Very useful!! (for linear convolutions with DFT)



# Multiplication

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- For  $x_1[n]$  and  $x_2[n]$  with length  $N$

$$x_1[n] \cdot x_2[n] \leftrightarrow \frac{1}{N} X_1[k] \circledast X_2[k]$$



# Linear Convolution

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- Next....
  - Using DFT, circular convolution is easy
  - But, linear convolution is useful, not circular
  - So, show how to perform linear convolution with circular convolution
  - Use DFT to do linear convolution



# Big Ideas

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- ❑ Discrete Fourier Transform (DFT)
  - For finite signals assumed to be zero outside of defined length
  - N-point DFT is sampled DTFT at N points
  - Useful properties allow easier linear convolution
- ❑ DFT Properties
  - Inherited from DFS, but circular operations!



# Admin

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- HW 7 due 4/9
  - Should be able to do it all now