ESE 5310: Digital Signal Processing

Lecture 22: April 18, 2024

Spectral Analysis

Penn ESE 5310 Spring 2024 – Khanna
Adapted from M. Lustig, EECS Berkeley

Lecture Outline

- Spectral Analysis with DFT
- Windowing
- Effect of zero-padding
- Time-dependent Fourier transform
  - Aka short-time Fourier transform

Spectral Analysis Using the DFT

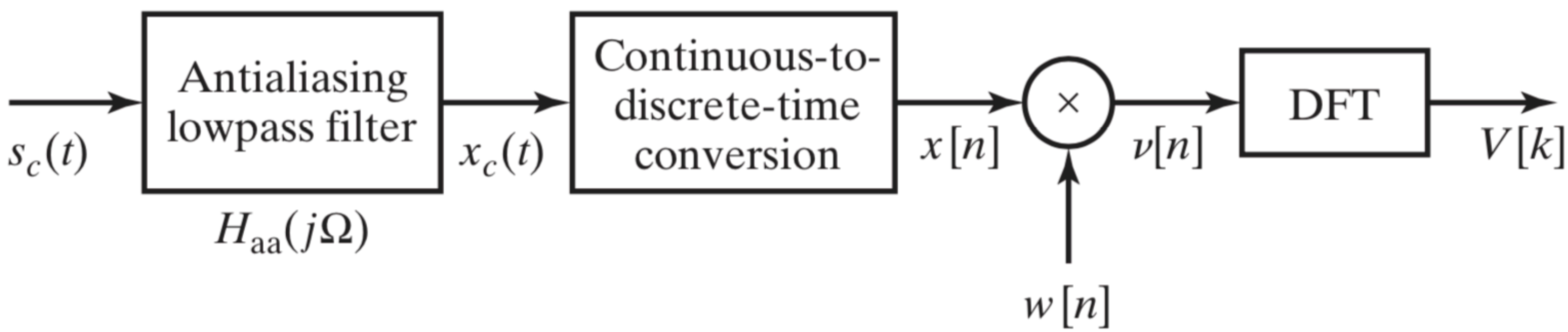
- DFT is a tool for spectrum analysis
  - Find out what frequencies are in your signal
- Should be simple:
  - Take a block, compute spectrum with DFT

Spectral Analysis Using the DFT

- DFT is a tool for spectrum analysis
  - Find out what frequencies are in your signal
- Should be simple:
  - Take a block, compute spectrum with DFT
- But, there are issues and tradeoffs:
  - Signal duration vs spectral resolution
  - Sampling rate vs spectral range
  - Spectral sampling rate
  - Spectral artifacts

Spectral Analysis Using the DFT

- Steps for processing continuous time (CT) signals



Spectral Analysis Using the DFT

- Two important tools:
  - Applying a window → reduced artifacts
  - Zero-padding → increases spectral sampling

Parameter	Symbol	Units
Sampling interval	T	s
Sampling frequency	$\Omega_s = \frac{2\pi}{T}$	rad/s
Window length	L	unitless
Window duration	$L \cdot T$	s
DFT length	$N \geq L$	unitless
DFT duration	$N \cdot T$	s
Spectral resolution	$\frac{\Omega_s}{L} = \frac{2\pi}{L \cdot T}$	rad/s
Spectral sampling interval	$\frac{\Omega_s}{N} = \frac{2\pi}{N \cdot T}$	rad/s

CT Signal Example

$$x_c(t) = A_1 \cos(\omega_1 t) + A_2 \cos(\omega_2 t)$$

$$X_c(j\Omega) = A_1 \pi \left[\delta(\Omega - \omega_1) + \delta(\Omega + \omega_1) \right] + A_2 \pi \left[\delta(\Omega - \omega_2) + \delta(\Omega + \omega_2) \right]$$

CT Signal Example

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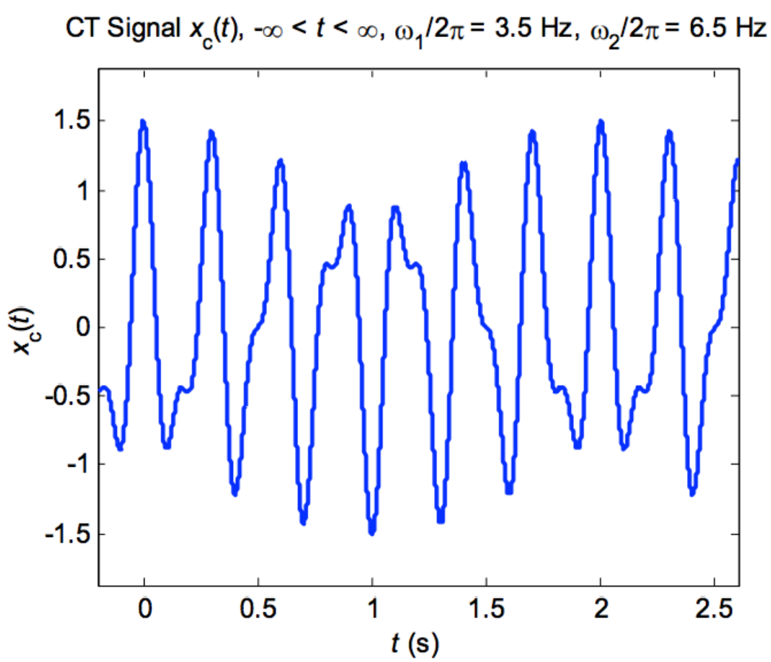

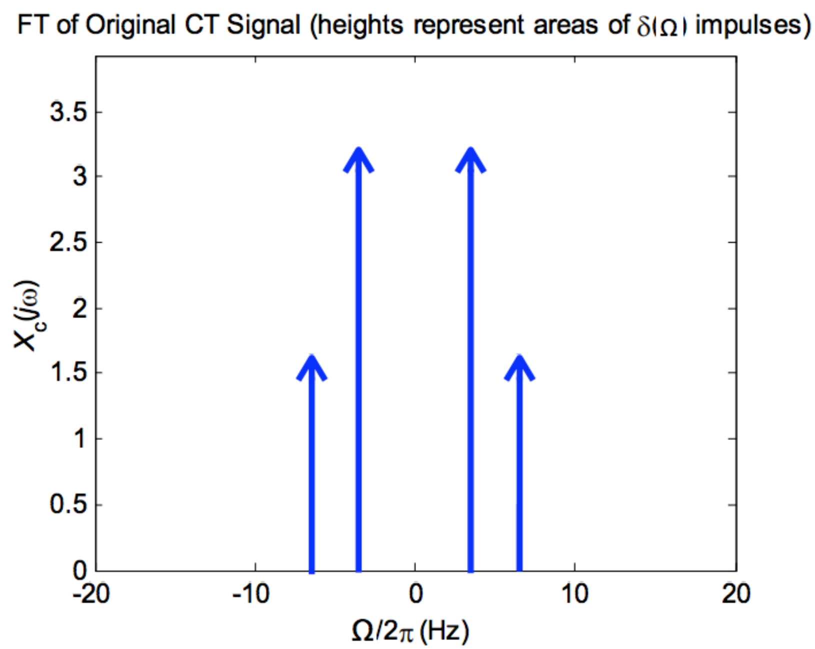

Sampled CT Signal Example

- If we sample the signal over an infinite time duration, we would have:

$$x[n] = x_c(t)\Big|_{t=nT}, \qquad -\infty < n < \infty$$

- With the discrete time Fourier transform (DTFT):

$$X(e^{j\Omega T}) = \frac{1}{T}\sum_{r=-\infty}^{\infty} X_c\left(j\left(\Omega - r\frac{2\pi}{T}\right)\right), \qquad -\infty < \Omega < \infty$$

CT Signal Example

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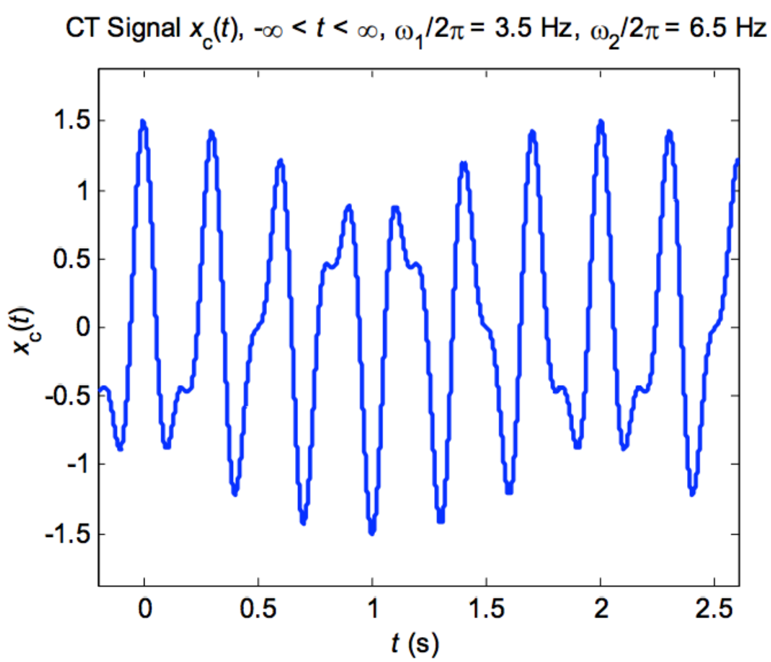
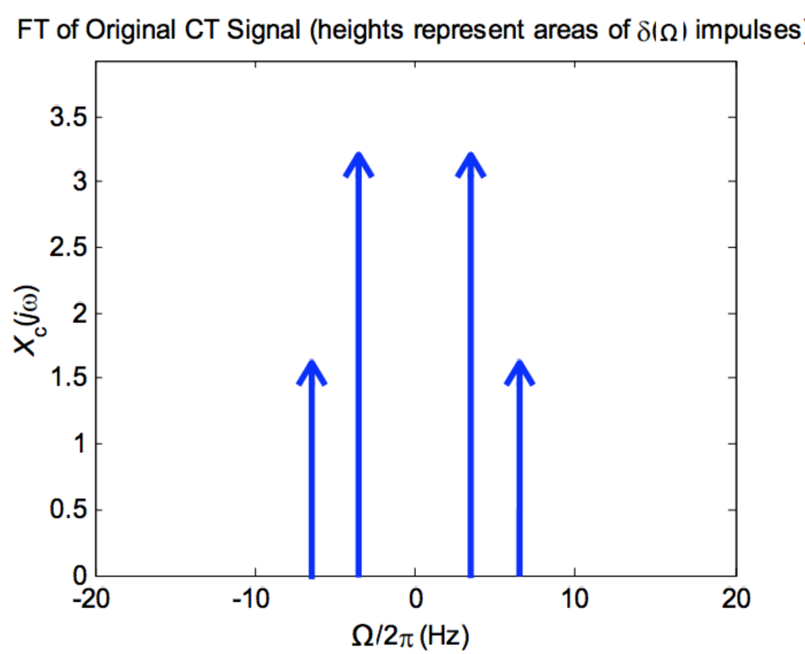
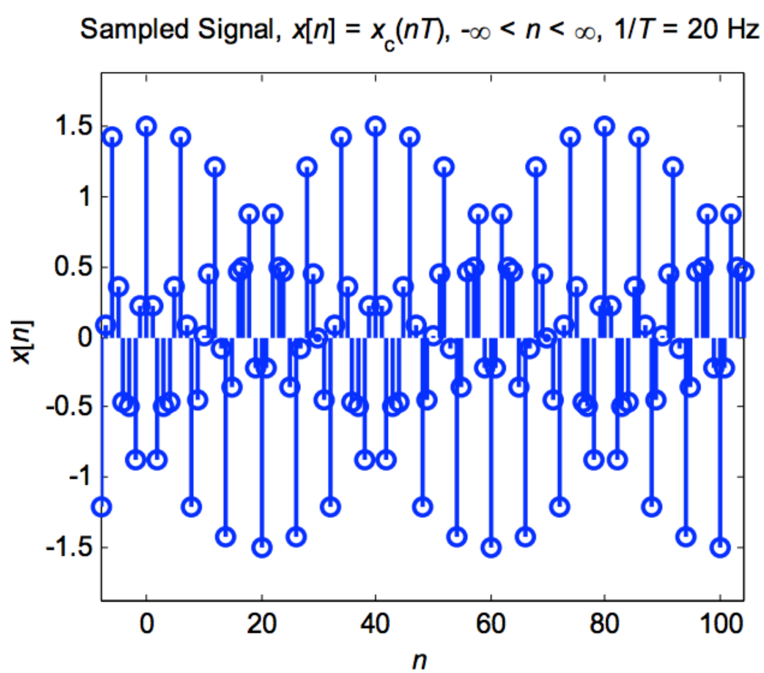
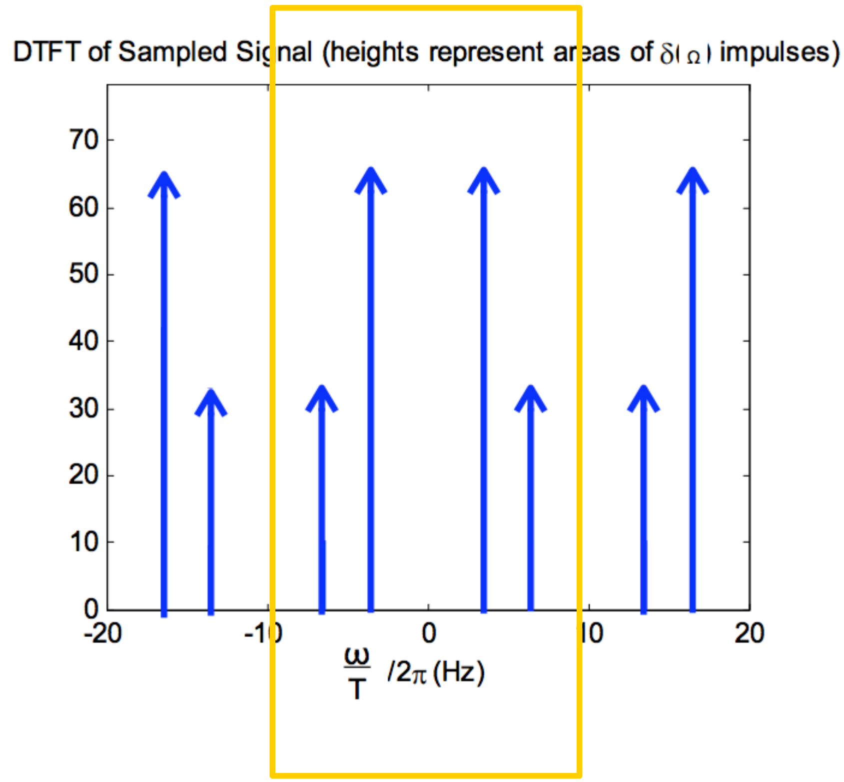




Sampled CT Signal Example

- Sampling with $\Omega s/2\pi=1/T=20Hz$





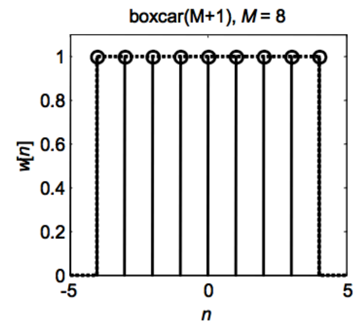
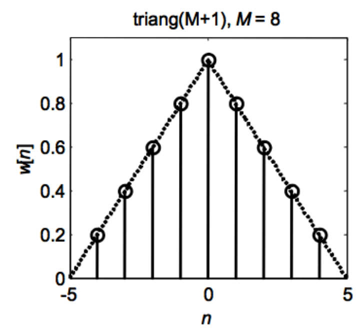
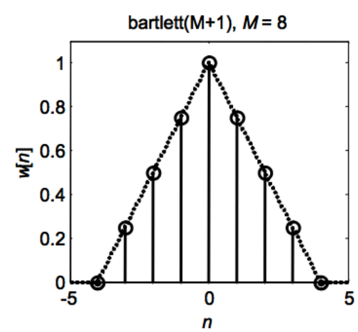
Windowed Sampled CT Signal

- In any real system, we sample only over a finite block of L samples:

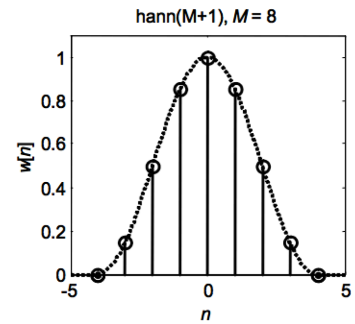
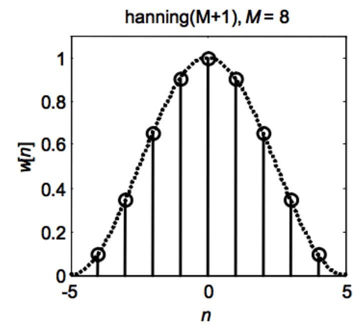
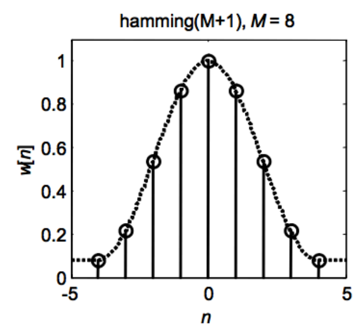
$$x[n] = x_c(t)\Big|_{t=nT}, \qquad 0 < n < L-1$$

- This simply corresponds to a rectangular window of duration L
- Recall there are many other window types
  - Hann, Hamming, Blackman, Kaiser, etc.

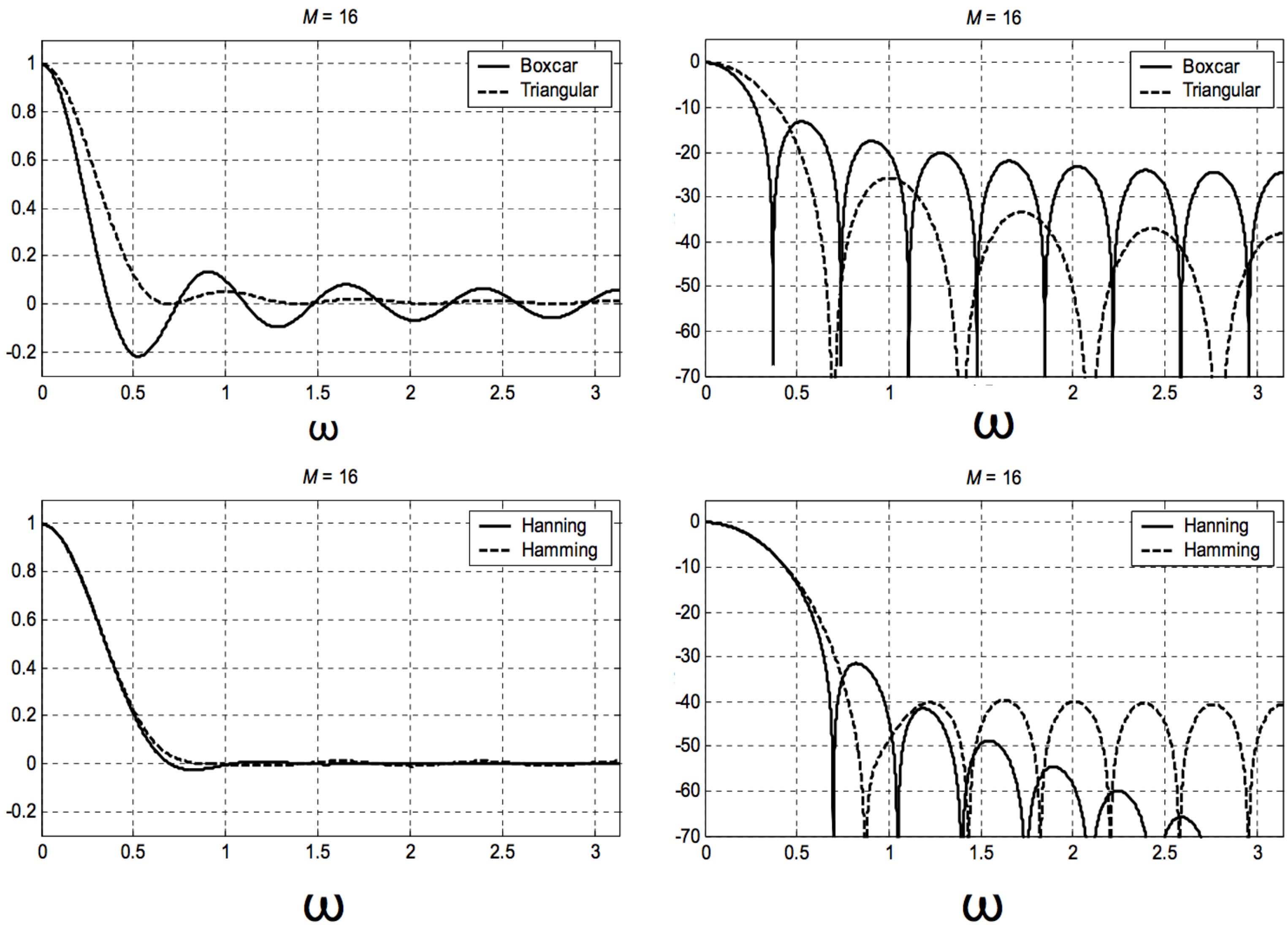
Windows

Name(s)	Definition	MATLAB Command	Graph ($M = 8$)
Rectangular Boxcar Fourier	$w[n] = \begin{cases} 1 & \lvert n \rvert \le M/2 \\ 0 & \lvert n \rvert > M/2 \end{cases}$	`boxcar(M+1)`	
Triangular	$w[n] = \begin{cases} 1 - \dfrac{\lvert n \rvert}{M/2+1} & \lvert n \rvert \le M/2 \\ 0 & \lvert n \rvert > M/2 \end{cases}$	`triang(M+1)`	
Bartlett	$w[n] = \begin{cases} 1 - \dfrac{\lvert n \rvert}{M/2} & \lvert n \rvert \le M/2 \\ 0 & \lvert n \rvert > M/2 \end{cases}$	`bartlett(M+1)`	

Windows

Name(s)	Definition	MATLAB Command	Graph ($M = 8$)
Hann	$w[n] = \begin{cases} \frac{1}{2}\left[1+\cos\left(\frac{\pi n}{M/2}\right)\right] & \lvert n\rvert \le M/2 \\ 0 & \lvert n\rvert > M/2 \end{cases}$	`hann(M+1)`	
Hanning	$w[n] = \begin{cases} \frac{1}{2}\left[1+\cos\left(\frac{\pi n}{M/2+1}\right)\right] & \lvert n\rvert \le M/2 \\ 0 & \lvert n\rvert > M/2 \end{cases}$	`hanning(M+1)`	
Hamming	$w[n] = \begin{cases} 0.54+0.46\cos\left(\frac{\pi n}{M/2}\right) & \lvert n\rvert \le M/2 \\ 0 & \lvert n\rvert > M/2 \end{cases}$	`hamming(M+1)`	

Windows



M = 16
Boxcar
Triangular
M = 16
Boxcar
Triangular
M = 16
Hanning
Hamming
M = 16
Hanning
Hamming
ω
ω
ω
ω

Windowed Sampled CT Signal

- We take the block of signal samples and multiply by a window of duration L, obtaining:

$$v[n] = x[n] \cdot w[n], \qquad 0 < n < L-1$$

- If the window w[n] has DTFT, $W(e^{j\omega})$, then the windowed block of signal samples has a DTFT given by the periodic convolution between $X(e^{j\omega})$ and $W(e^{j\omega})$:

$$V(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})W(e^{j(\omega-\theta)})d\theta$$

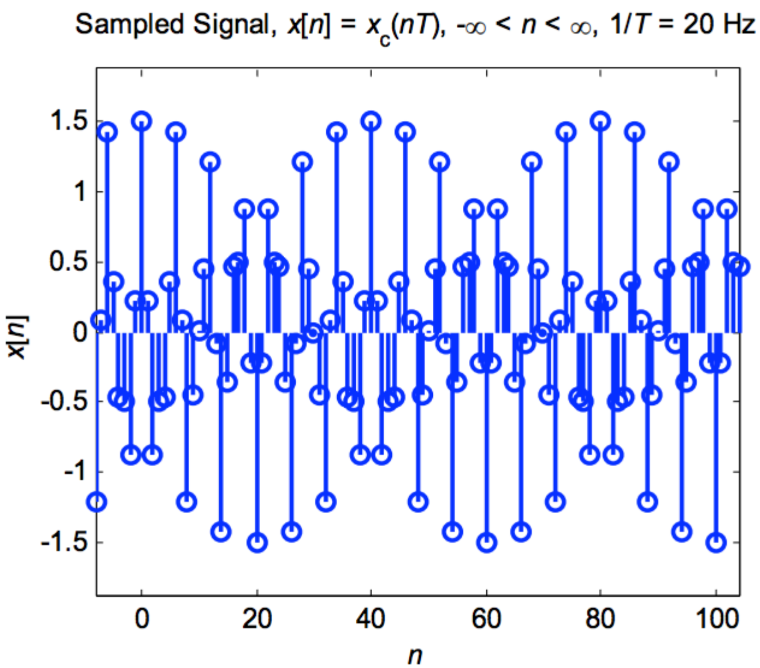
Windowed Sampled CT Signal

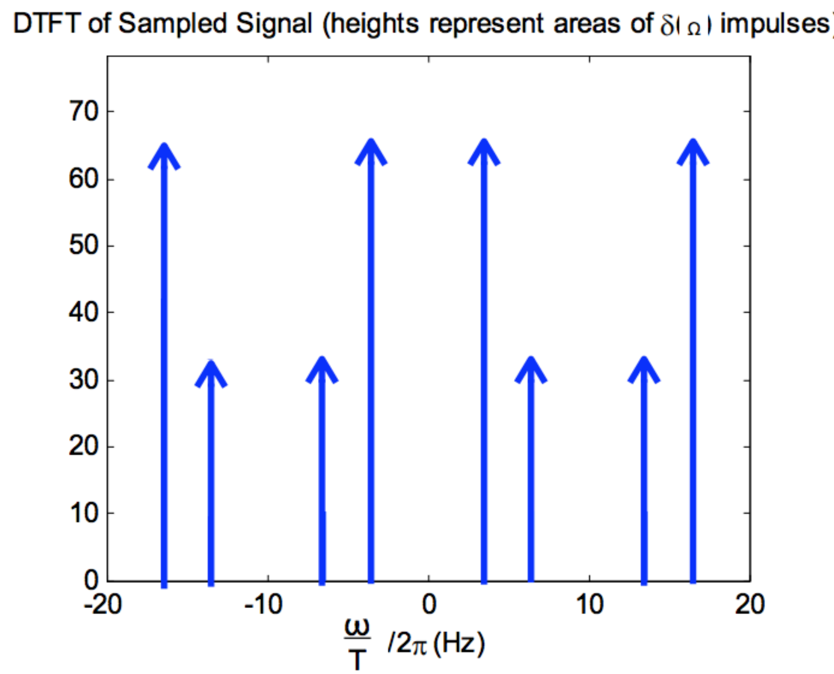
- Convolution with $W(e^{j\omega})$ has two effects in the spectrum:
  - It limits the spectral resolution (spectral spreading)
    - Main lobes of the DTFT of the window
  - The window can produce spectral leakage
    - Side lobes of the DTFT of the window

- These two are always a tradeoff
  - time-frequency uncertainty principle
    - More later…

Sampled CT Signal Example

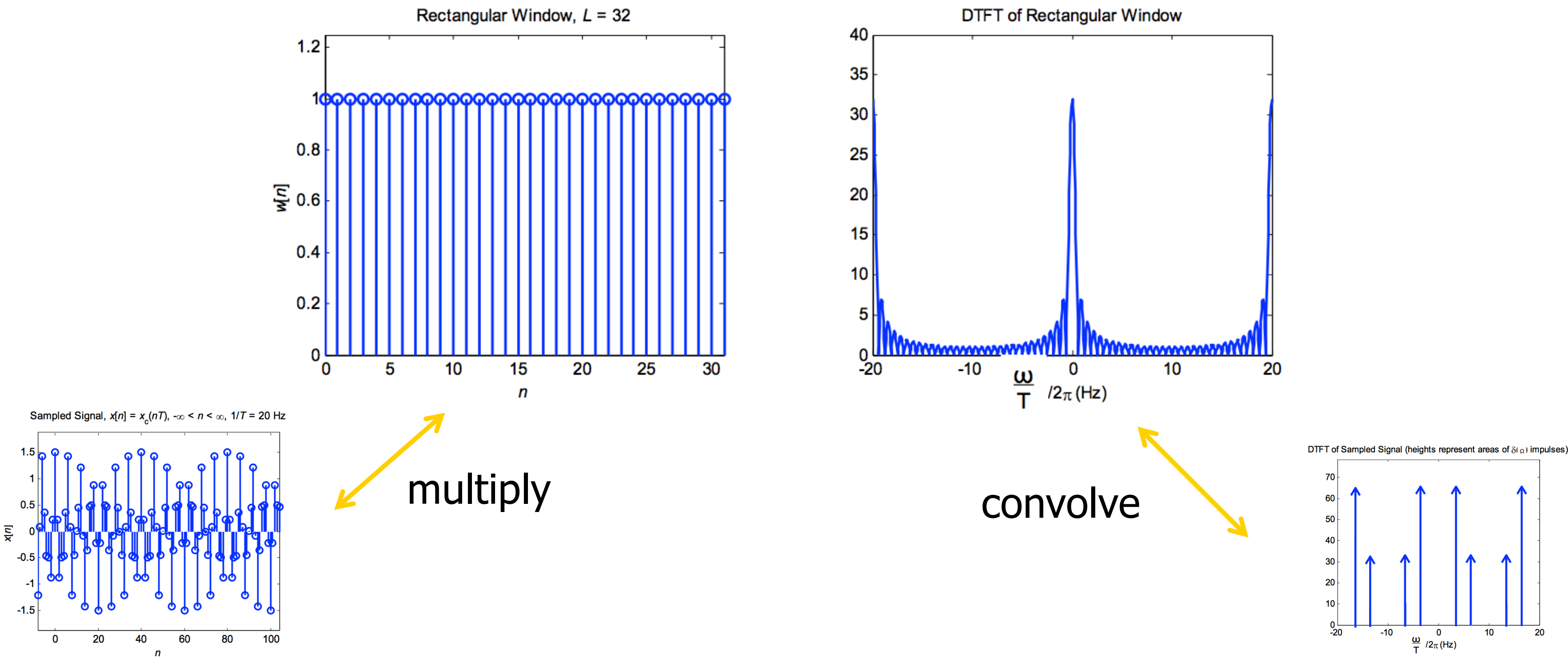
- Sampling with $\Omega s/2\pi = 1/T = 20$Hz





Windowed Sampled CT Signal Example

- As before, the sampling rate is $\Omega s/2\pi=1/T=20$Hz
- Rectangular Window, L = 32



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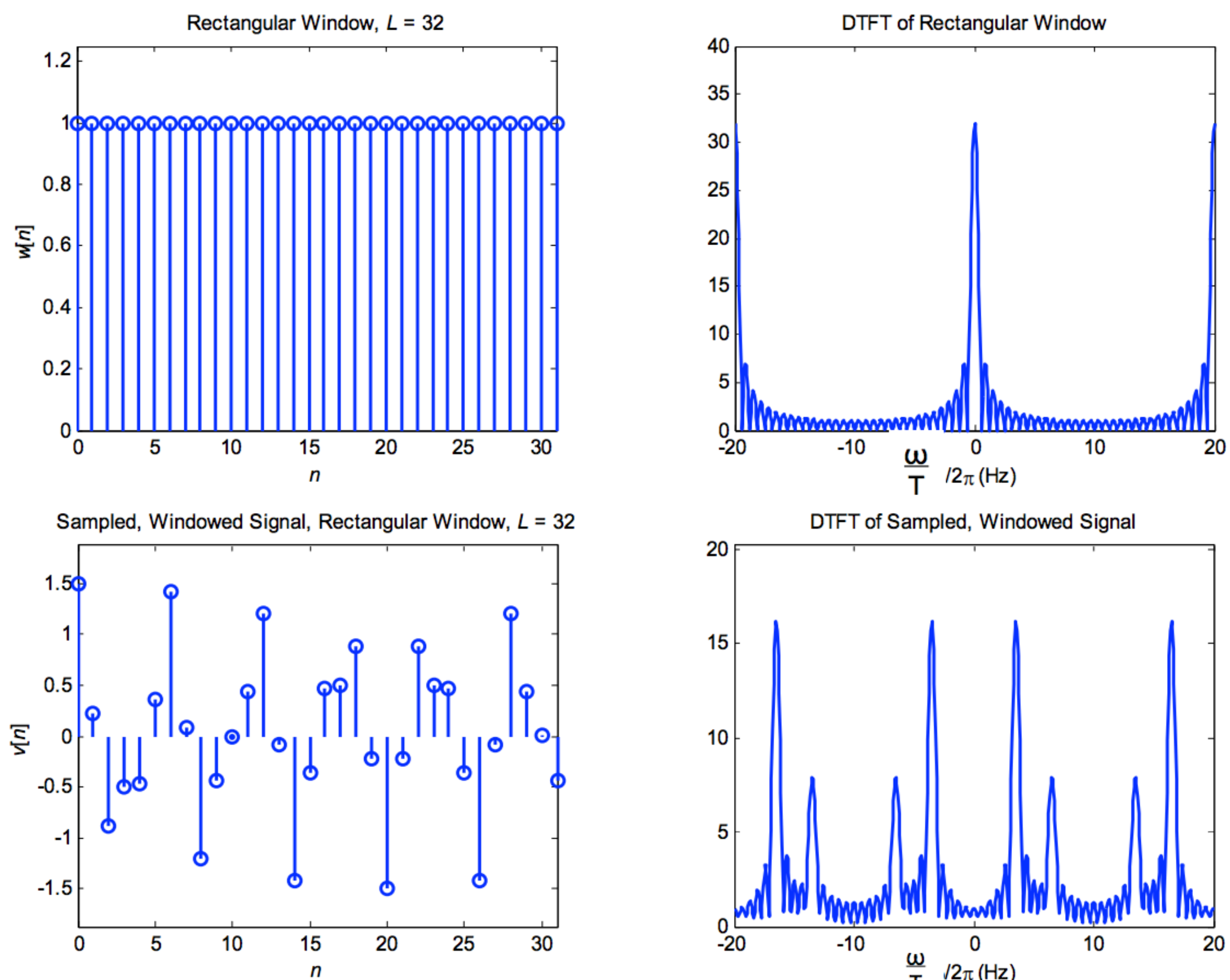

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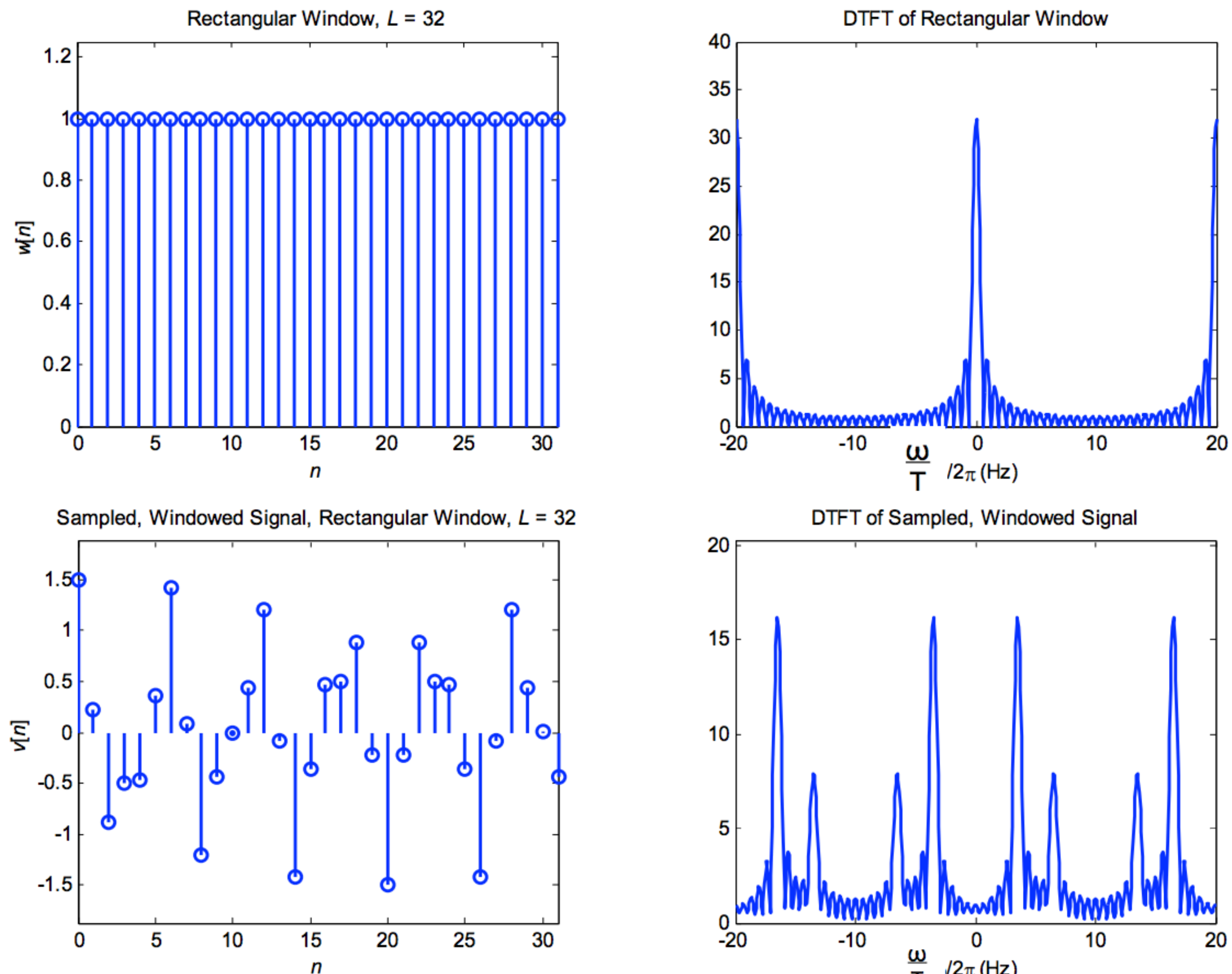

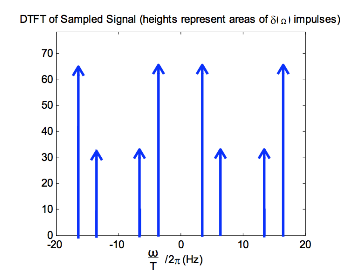

Windowed Sampled CT Signal Example

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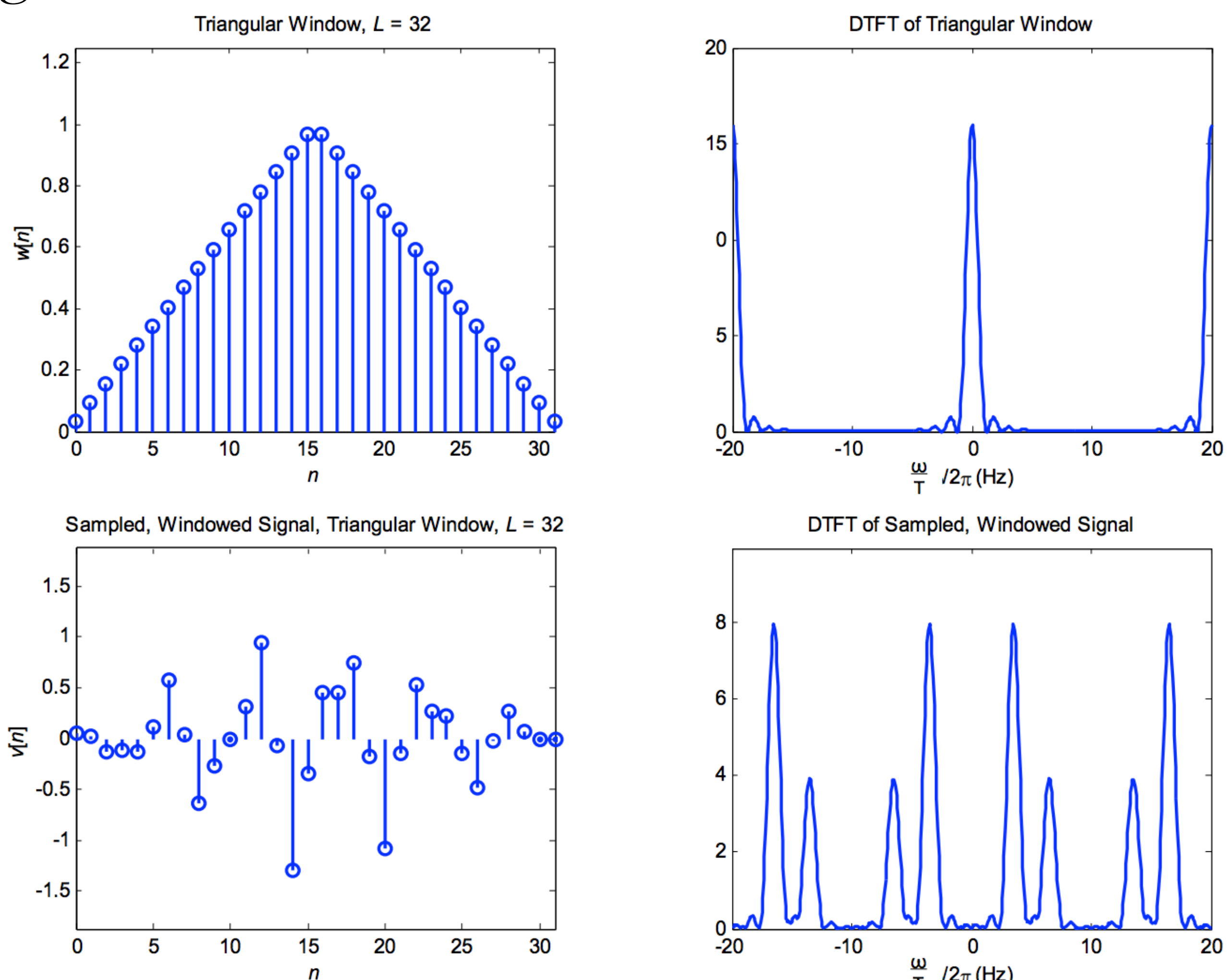
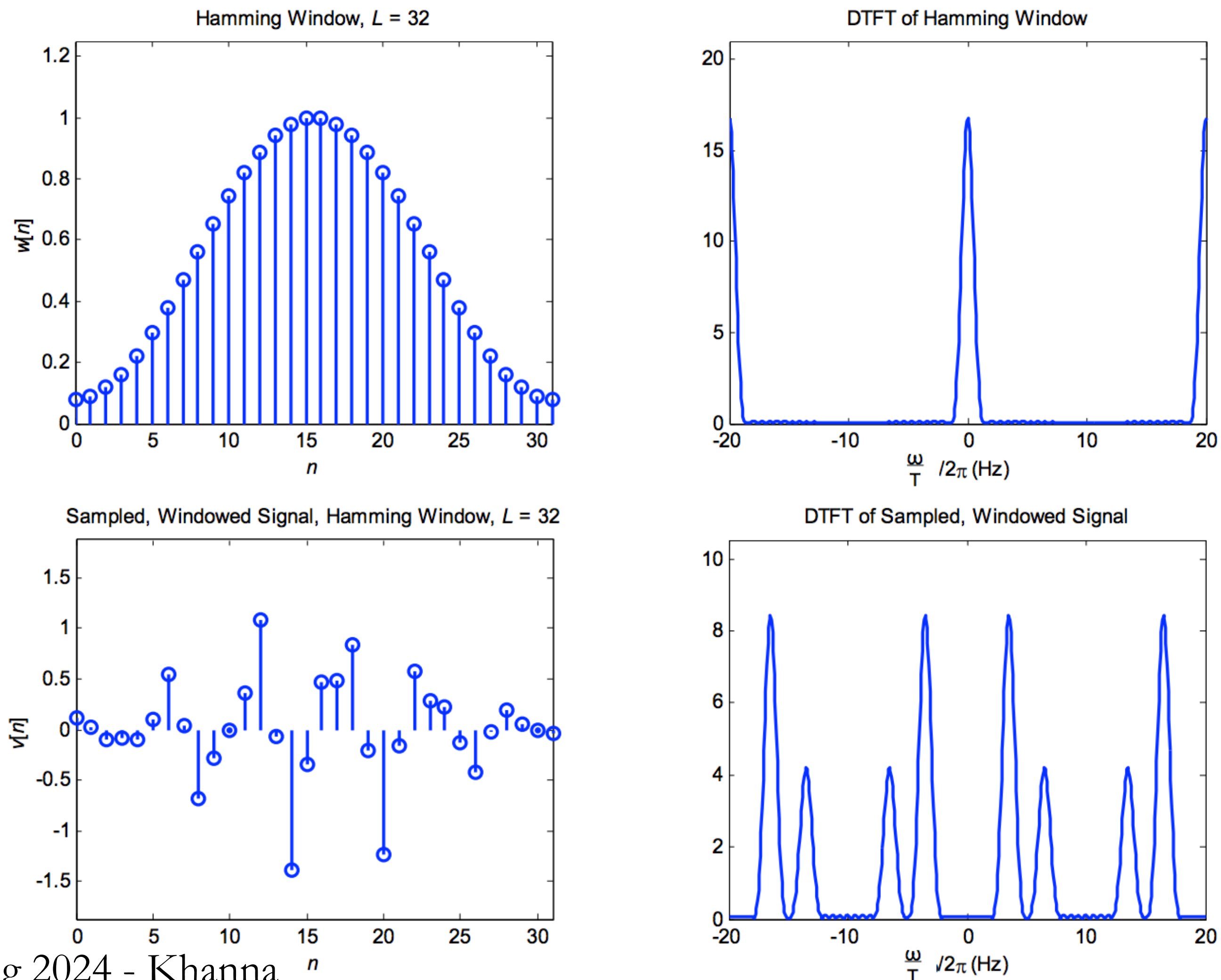
- Triangular Window, L = 32



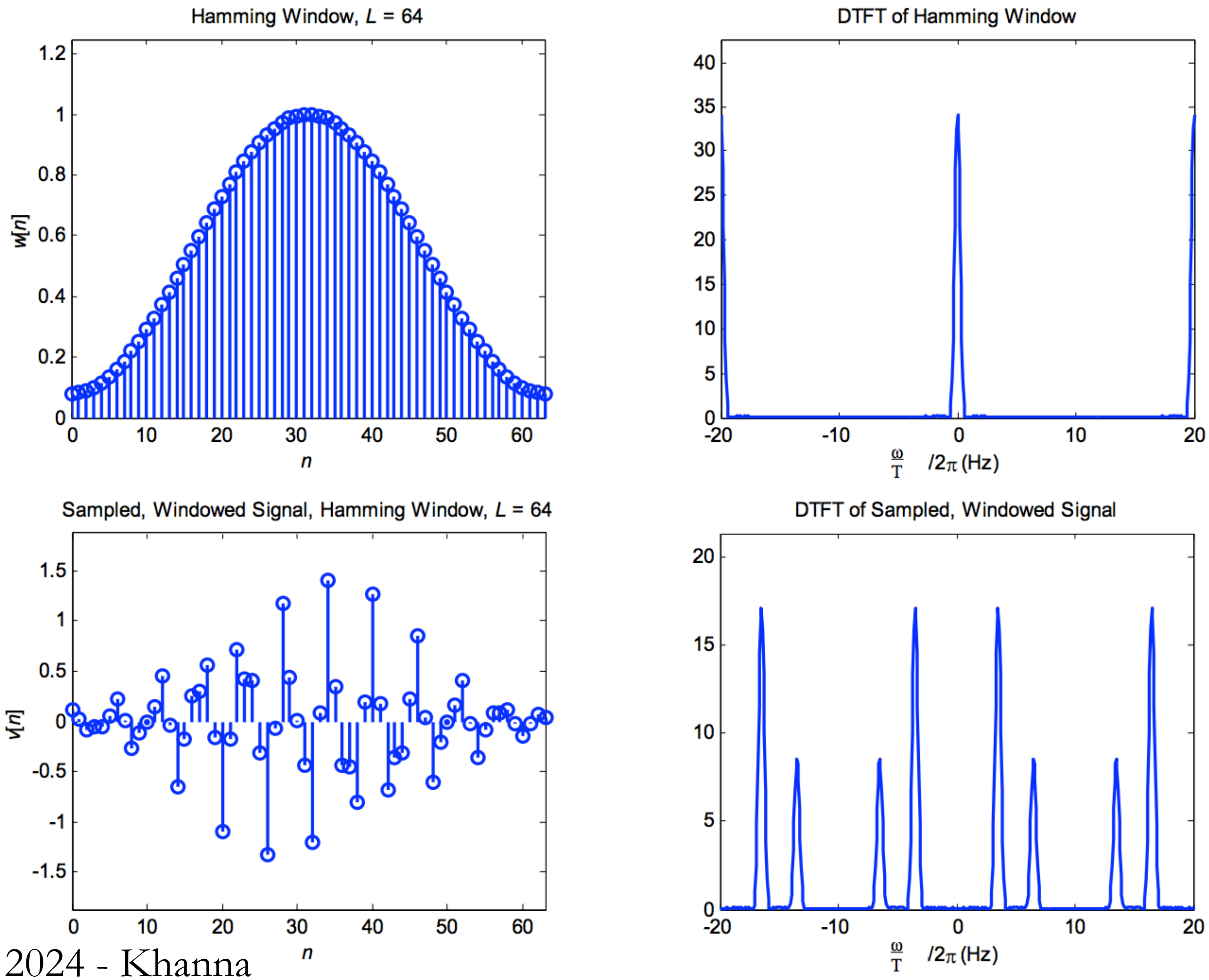
Windowed Sampled CT Signal Example

- As before, the sampling rate is $\Omega s/2\pi=1/T=20\text{Hz}$
- Hamming Window, L = 32



Windowed Sampled CT Signal Example

- As before, the sampling rate is $\Omega s/2\pi=1/T=20Hz$
- Hamming Window, L = 64



Optimal Window: Kaiser

- Minimum main-lobe width for a given sidelobe energy percentage

$$\frac{\int_{\text{sidelobes}} |H(e^{j\omega})|^2 d\omega}{\int_{-\pi}^{\pi} |H(e^{j\omega})|^2 d\omega}$$

- Window is parameterized with M and β
  - β determines side-lobe level
  - M determines main-lobe width

Window Comparison Example

$$y[n] = \underbrace{\sin(2\pi 0.1992 n)}_{1.25} + 0.005 \underbrace{\sin(2\pi 0.25 n)}_{1.57} \mid 0 \le n \le 128$$

200x smaller → -46dB

Window Comparison Example

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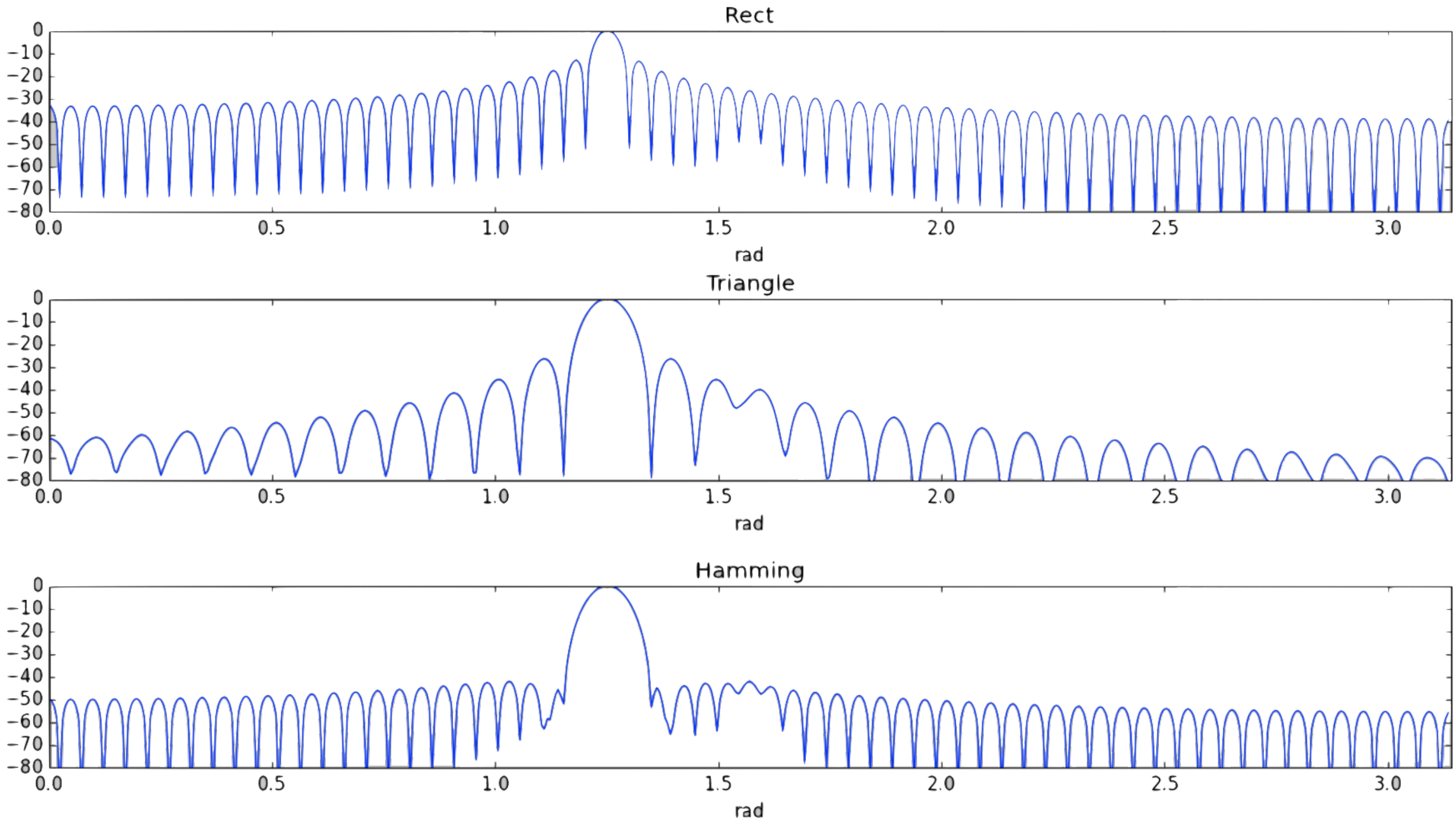
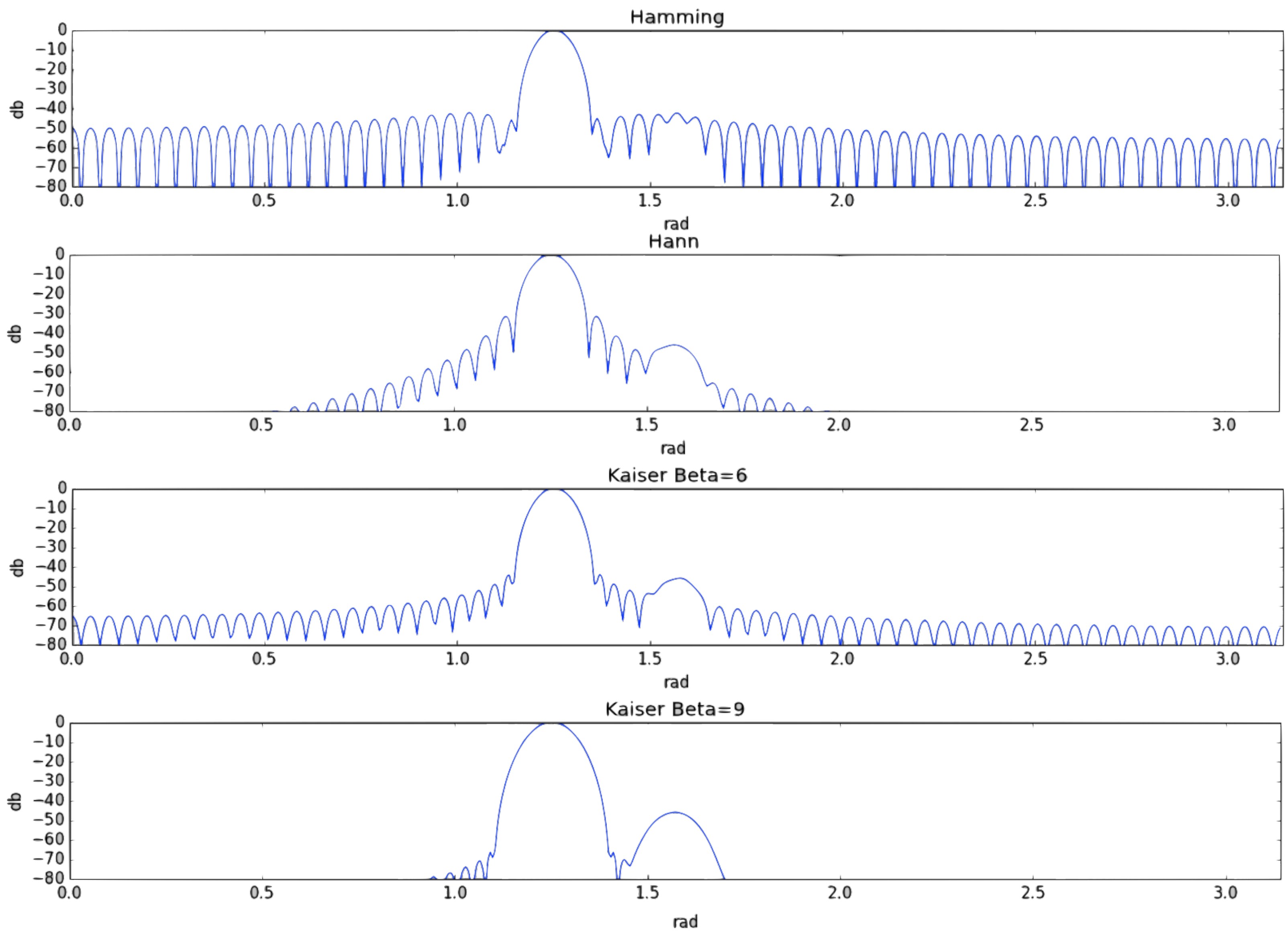


Window Comparison Example



Zero-Padding

- In preparation for taking an N-point DFT, we may zero-pad the windowed block of signal samples

$$\begin{cases} v[n] & 0 \le n \le L-1 \\ 0 & L \le n \le N-1 \end{cases}$$

- This zero-padding has no effect on the DTFT of v[n], since the DTFT is computed by summing over infinity
- Effect of Zero Padding
  - We take the N-point DFT of the zero-padded v[n], to obtain the block of N spectral samples:

Zero-Padding

- Consider the DTFT of the zero-padded v[n]. Since the zero-padded v[n] is of length N, its DTFT can be written:

$$V(e^{j\omega}) = \sum_{n=0}^{N-1} v[n]e^{-nj\omega}, \qquad -\infty < \omega < \infty$$

- The N-point DFT of v[n] is given by:

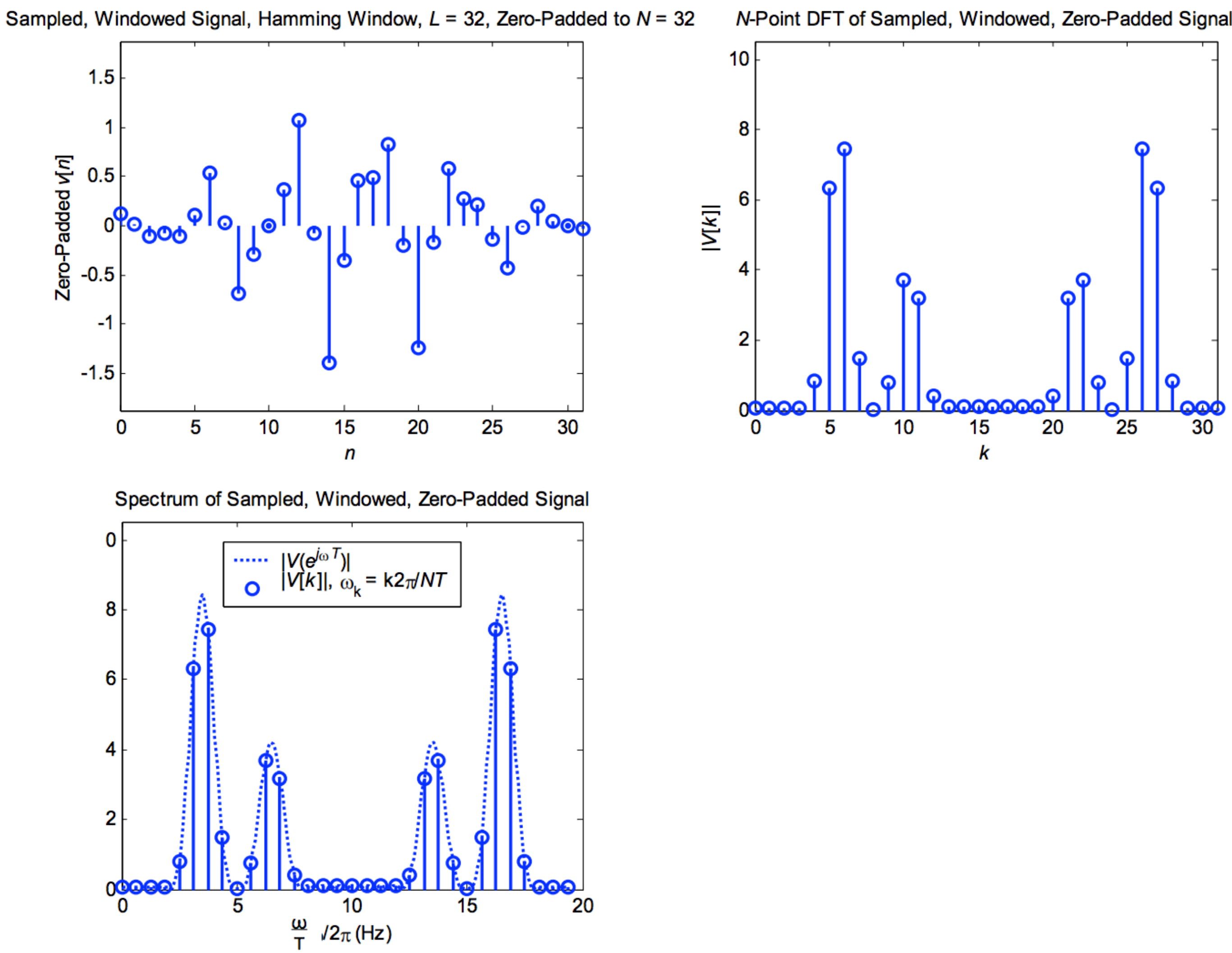
$$V[k] = \sum_{n=0}^{N-1} v[n]W_N^{kn} = \sum_{n=0}^{N-1} v[n]e^{-j(2\pi/N)nk}, \qquad 0 \le k \le N-1$$

- We know that the DFT is a sampled $V(e^{j\omega})$:

$$V[k] = V(e^{j\omega})\big|_{\omega=k\frac{2\pi}{N}}, \qquad 0 \le k \le N-1$$

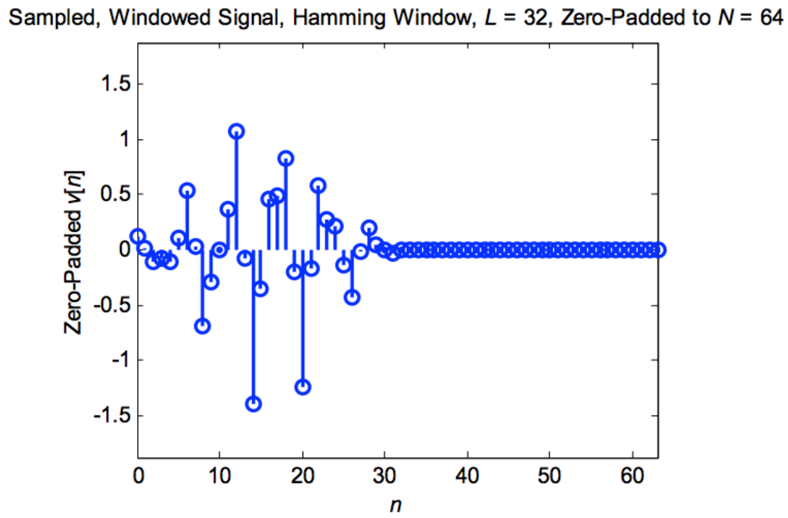
Frequency Analysis with DFT

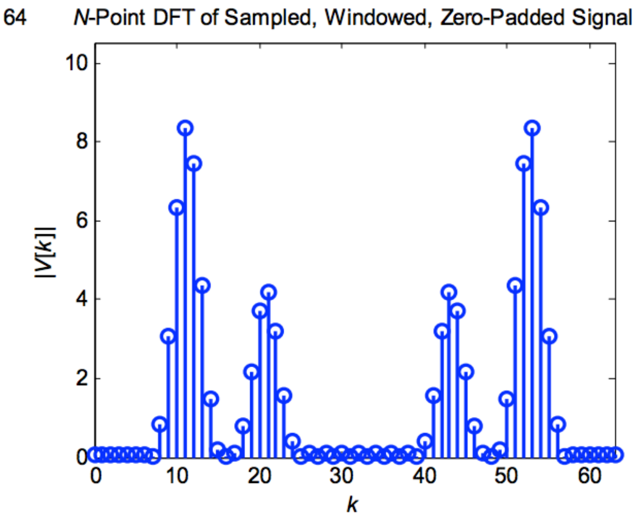
- Hamming window, L = N = 32

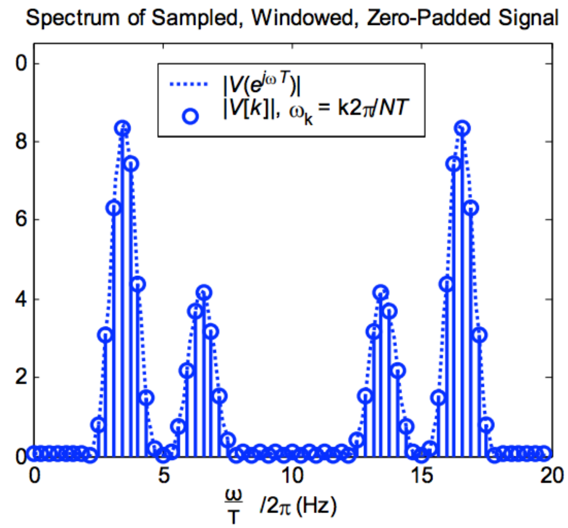


Frequency Analysis with DFT

- Hamming window, L = 32, Zero-padded to N = 64







Frequency Analysis with DFT

- Length of window determines spectral resolution
- Type of window determines side-lobe amplitude/main-lobe width (spectral leakage/spreading)
  - Some windows have better tradeoff between resolution and side-lobe height
- Zero-padding approximates the DTFT better (spectral sampling). Does not introduce new information!

Potential Problems and Solutions

- 1. Spectral error from aliasing?
  - a. Filter signal to reduce frequency content above $\Omega_s/2 = \pi/T$.
  - b. Increase sampling frequency $\Omega_s = 2\pi/T$.
- 2. Insufficient frequency resolution?
  - a. Increase L
  - b. Use window having narrow main lobe.
- 3. Spectral error from leakage?
  - a. Use window having low side lobes.
  - b. Increase L
- 4. Missing features due to spectral sampling?
  - a. Increase N by zero-padding v[n] to length N > L
  - b. Increase L

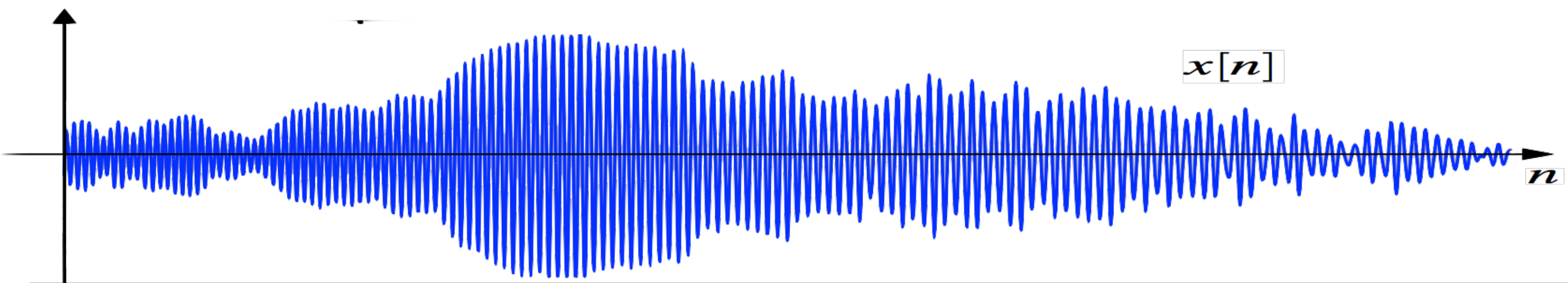
Time Dependent DFT

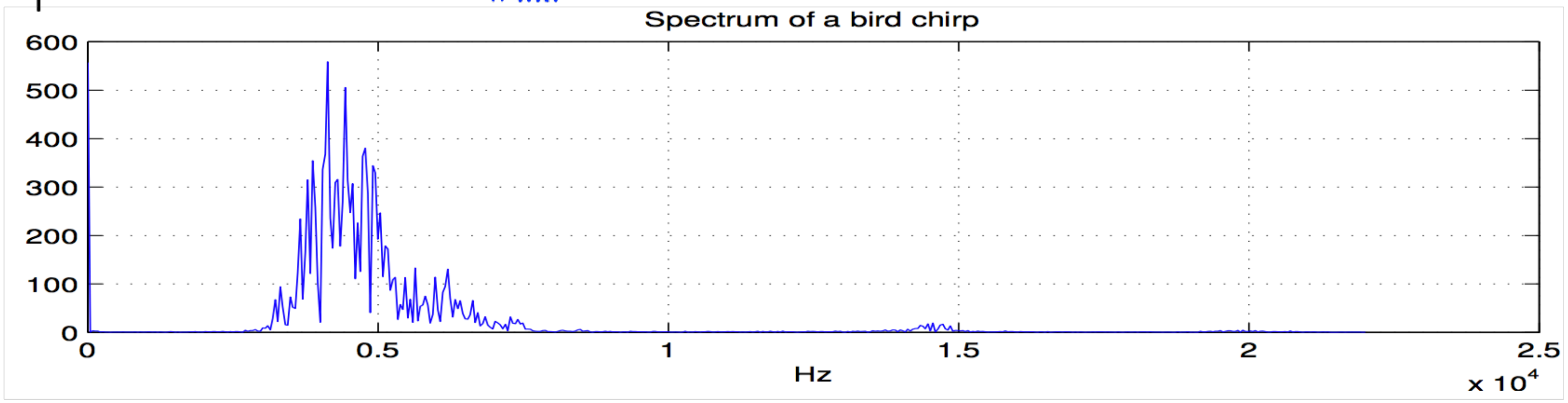
DFT

- DFT is only one out of a LARGE class of transforms
- Used for:
  - Analysis
  - Compression
  - Denoising
  - Detection
  - Recognition
  - Approximation (Sparse)

Example of Spectral Analysis

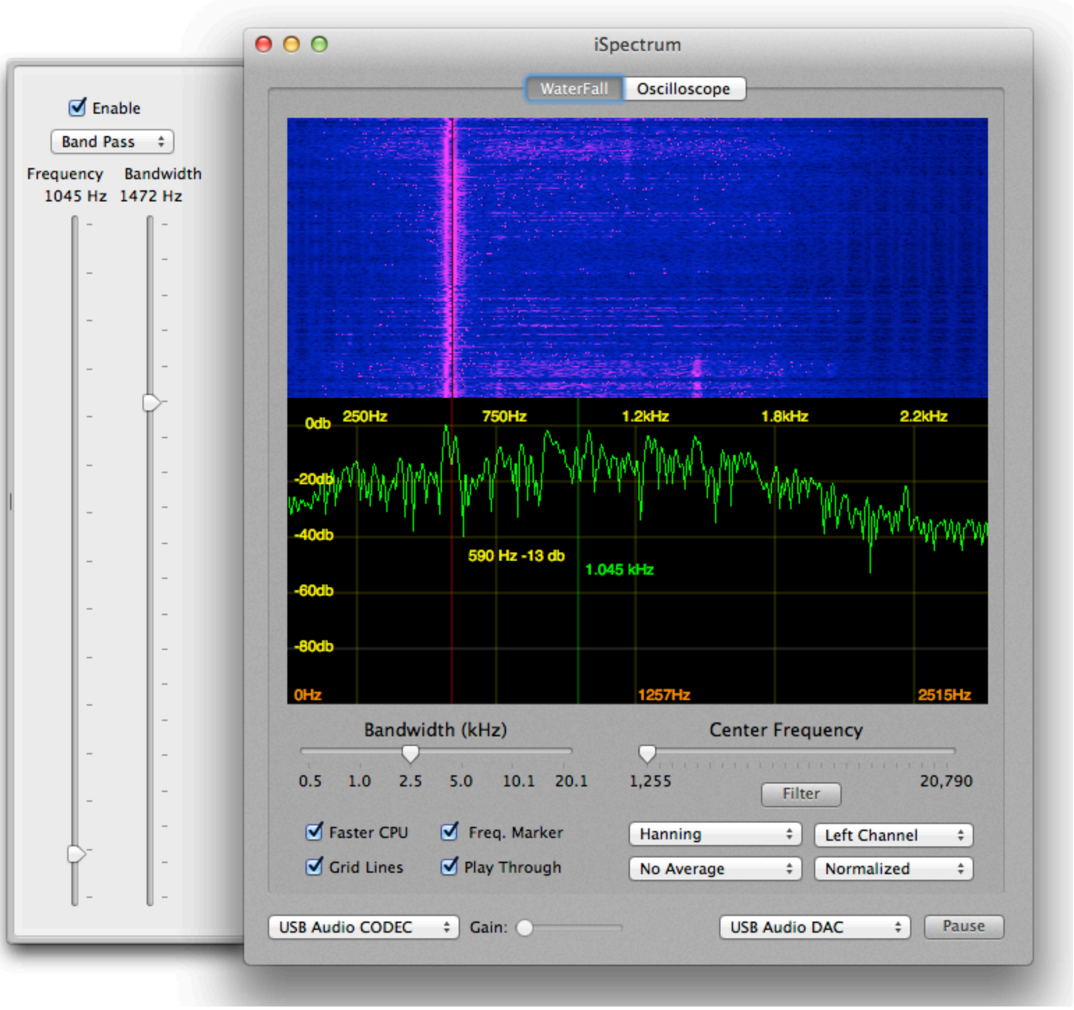
- Spectrum of a bird chirping
  - Interesting,.... but...
  - Does not tell the whole story
  - No temporal information!





iSpectrum Demo

- https://dogparksoftware.com/iSpectrum.html



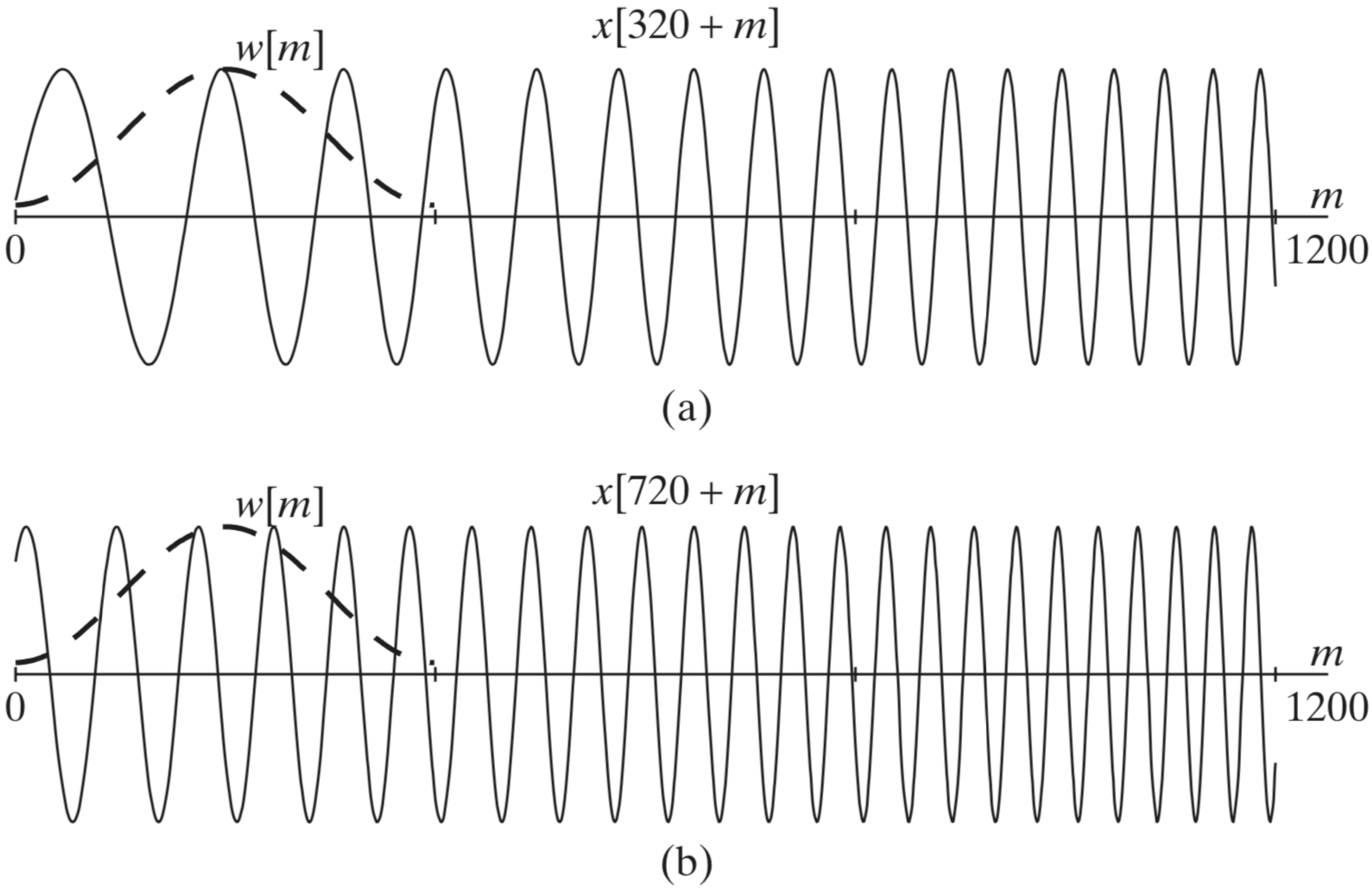
Time Dependent Fourier Transform

- Also called short-time Fourier transform
- To get temporal information, use part of the signal around every time point

$$X[n,\lambda) = \sum_{m=-\infty}^{\infty} x[n+m]w[m]e^{-j\lambda m}$$

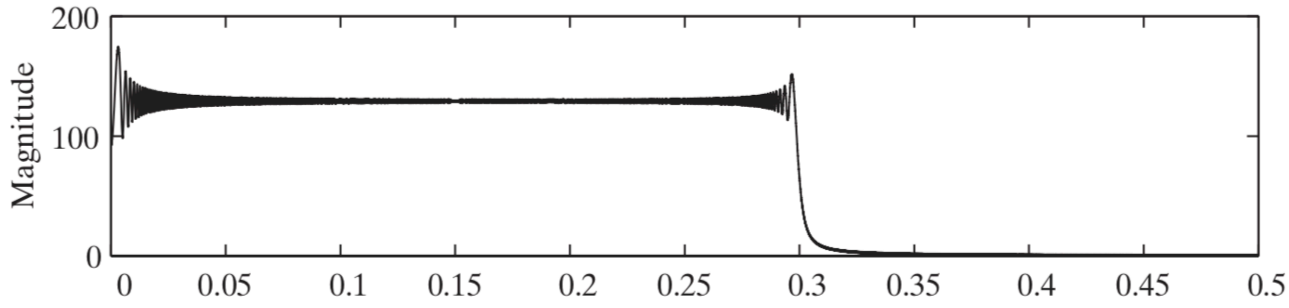
- Mapping from 1D → 2D, n discrete, λ cont.
- Simply slide a window and compute DTFT

Time Dependent Fourier Transform



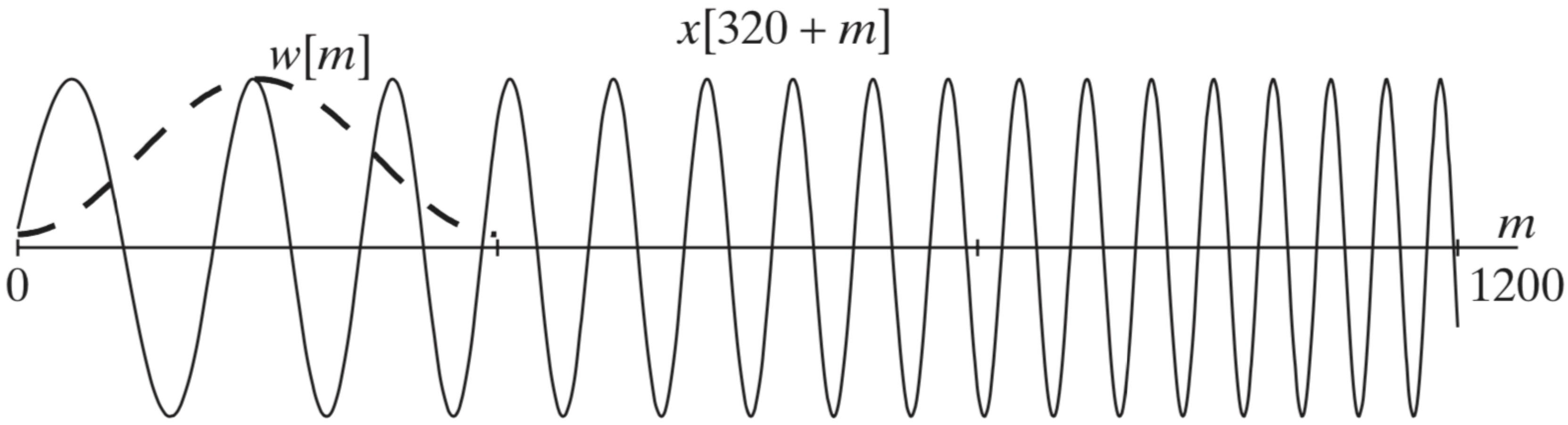
Time Dependent Fourier Transform

DTFT of 20,000 samples of $x[n] = \cos(\alpha_0 n^2)$

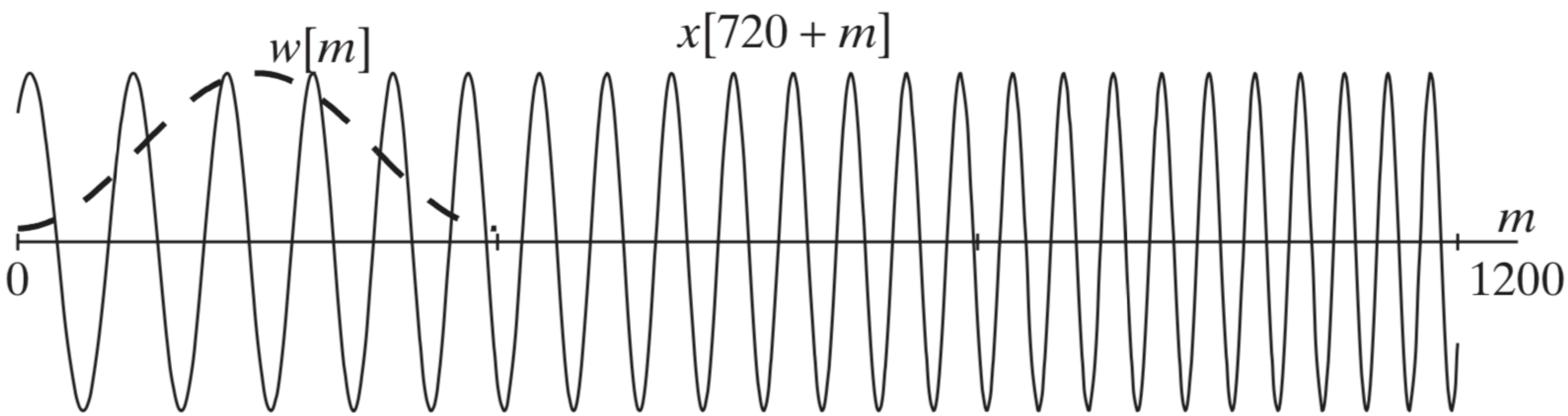


(a)

Time Dependent Fourier Transform



(a)



(b)

Time Dependent Fourier Transform

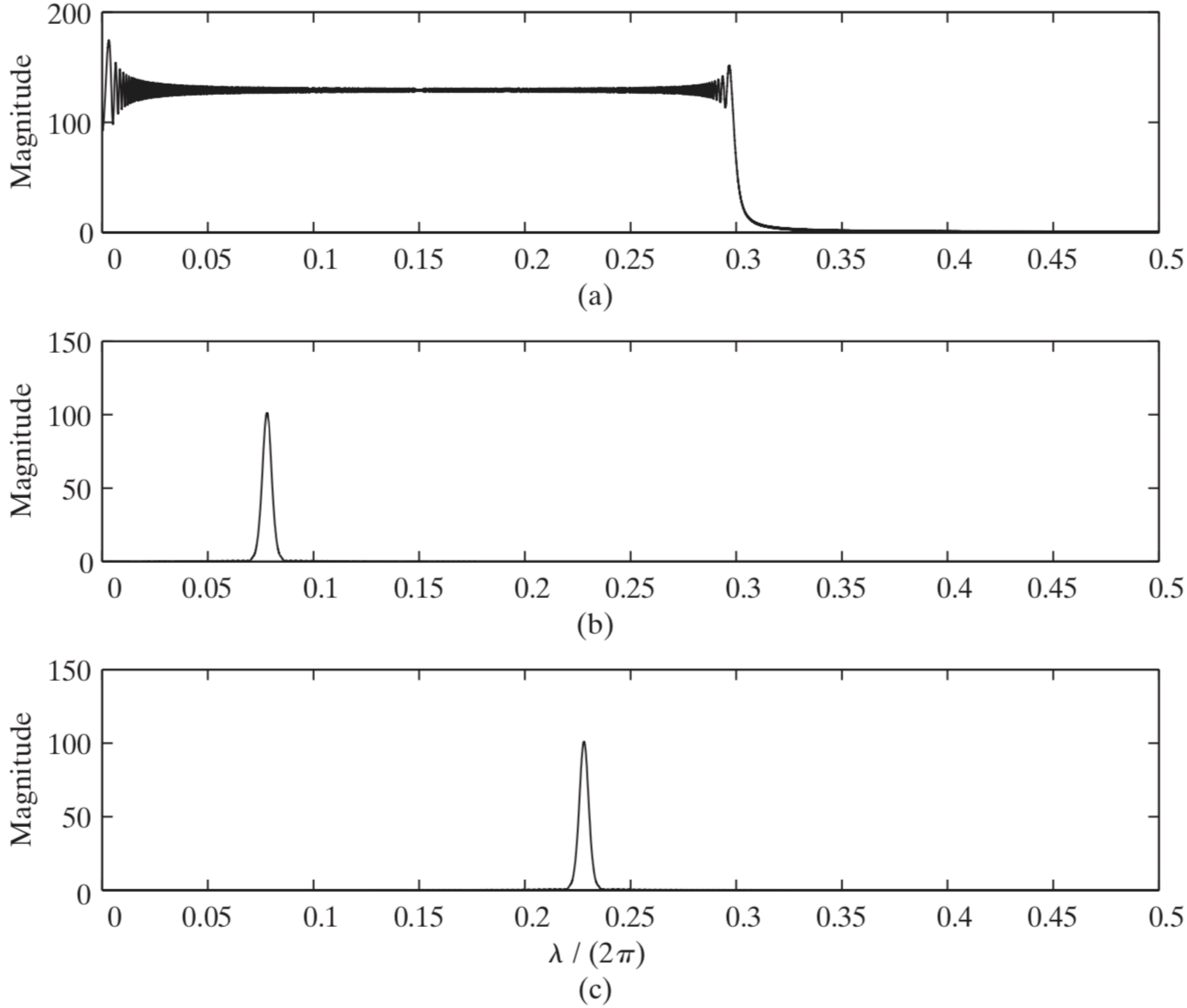
DTFT of 20,000 samples of $x[n] = \cos(\alpha_0 n^2)$

X[5000,λ)

DTFT of x[5000+m]w[m], where w[m] is Hamming window of length L=401

X[15000,λ)

DTFT of x[15000+m]w[m], where w[m] is Hamming window of length L=401



Spectrogram

- Plotting Y[n,λ)

$$y[n] = \begin{cases} 0 & n < 0 \\ \cos(\alpha_0 n^2) & 0 \le n \le 20{,}000 \\ \cos(0.2\pi n) & 20{,}000 < n \le 25{,}000 \\ \cos(0.2\pi n) + \cos(0.23\pi n) & 25{,}000 < n. \end{cases}$$

Spectrogram

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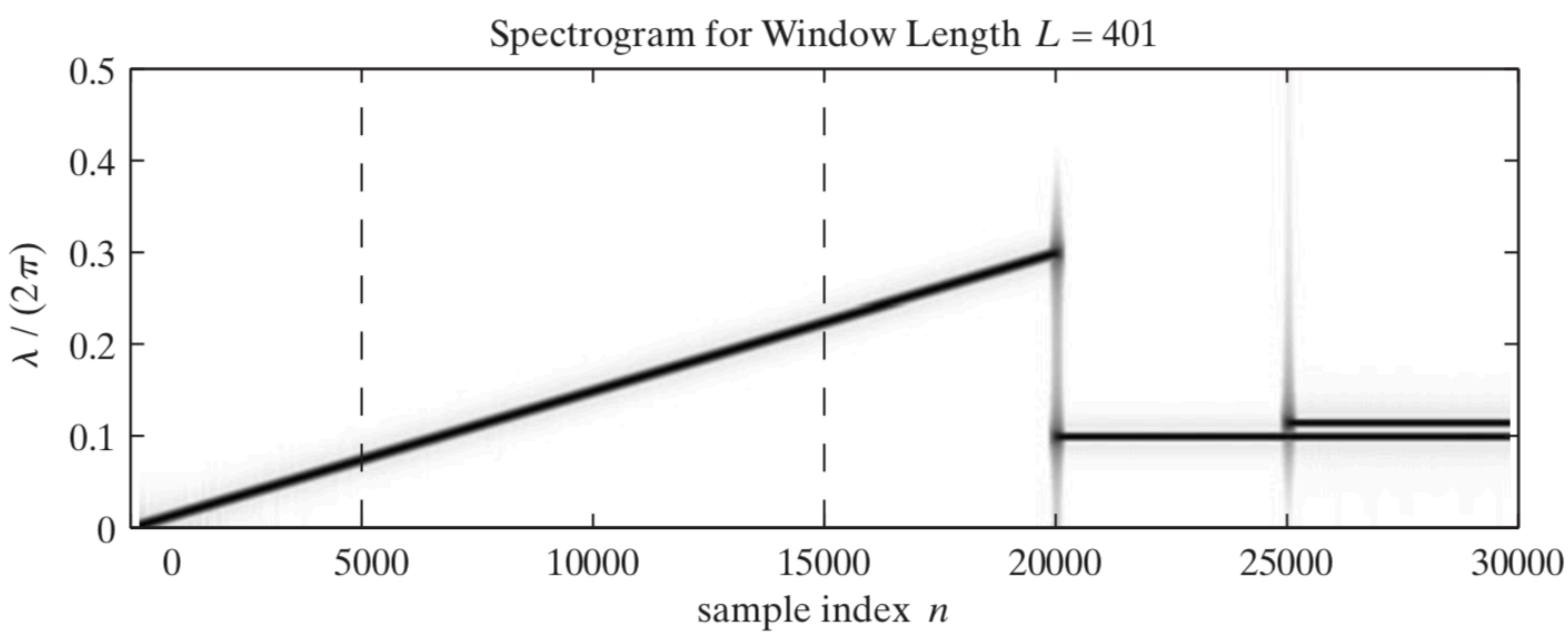

Spectrogram

- Plotting $Y[n,\lambda)$

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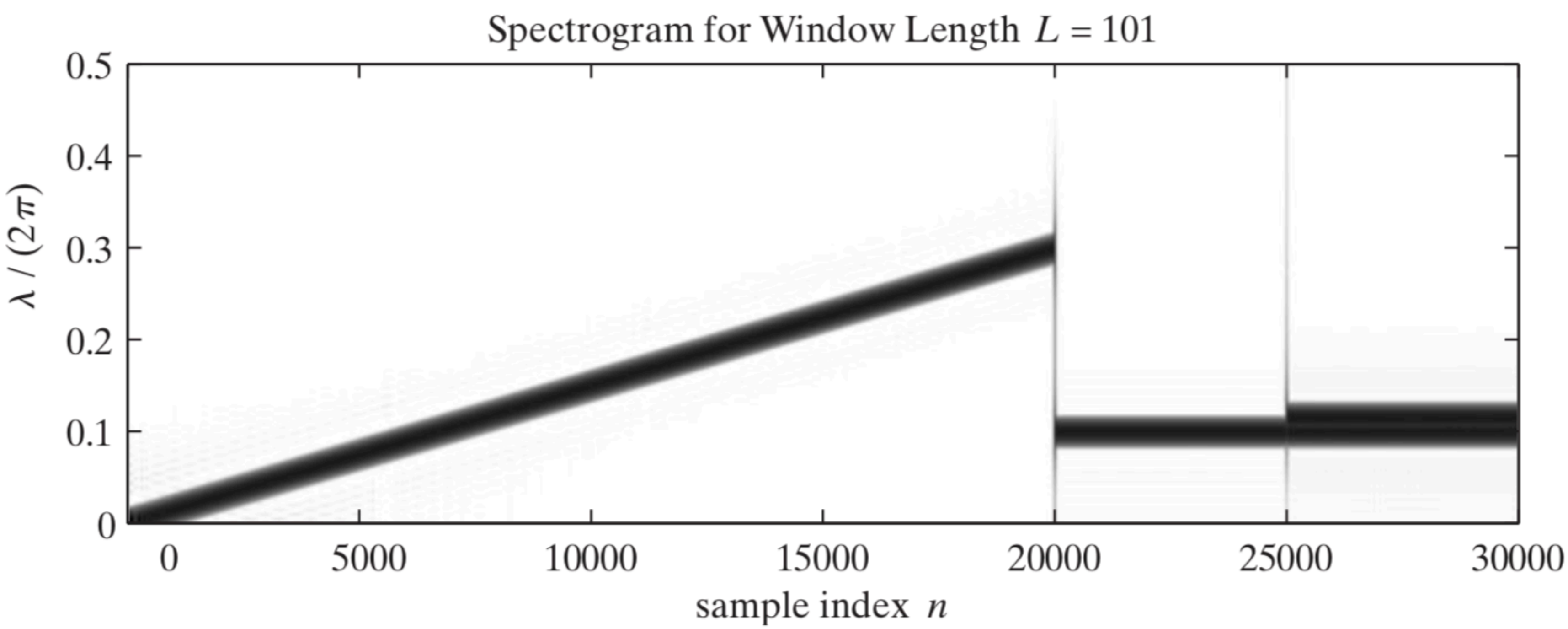
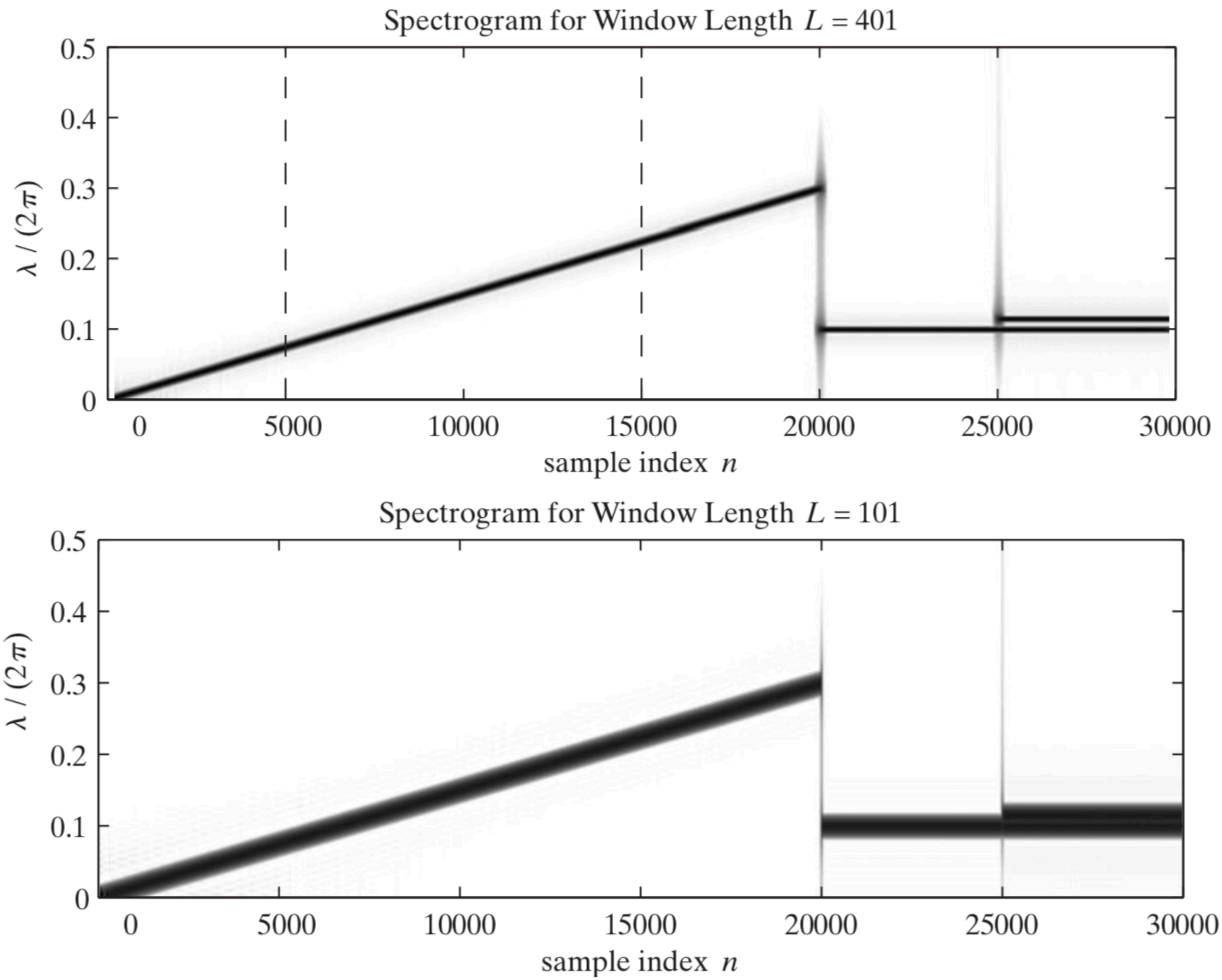
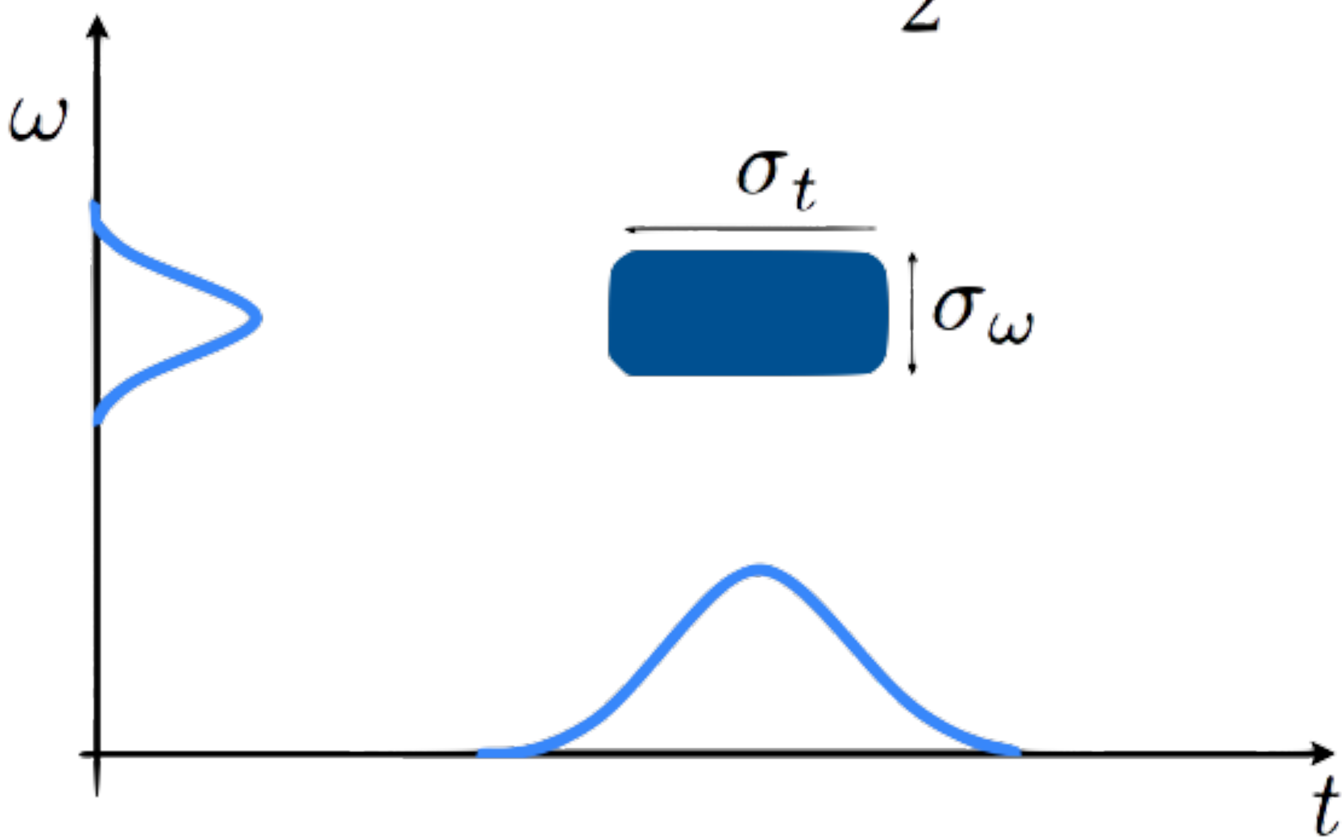


Spectrogram

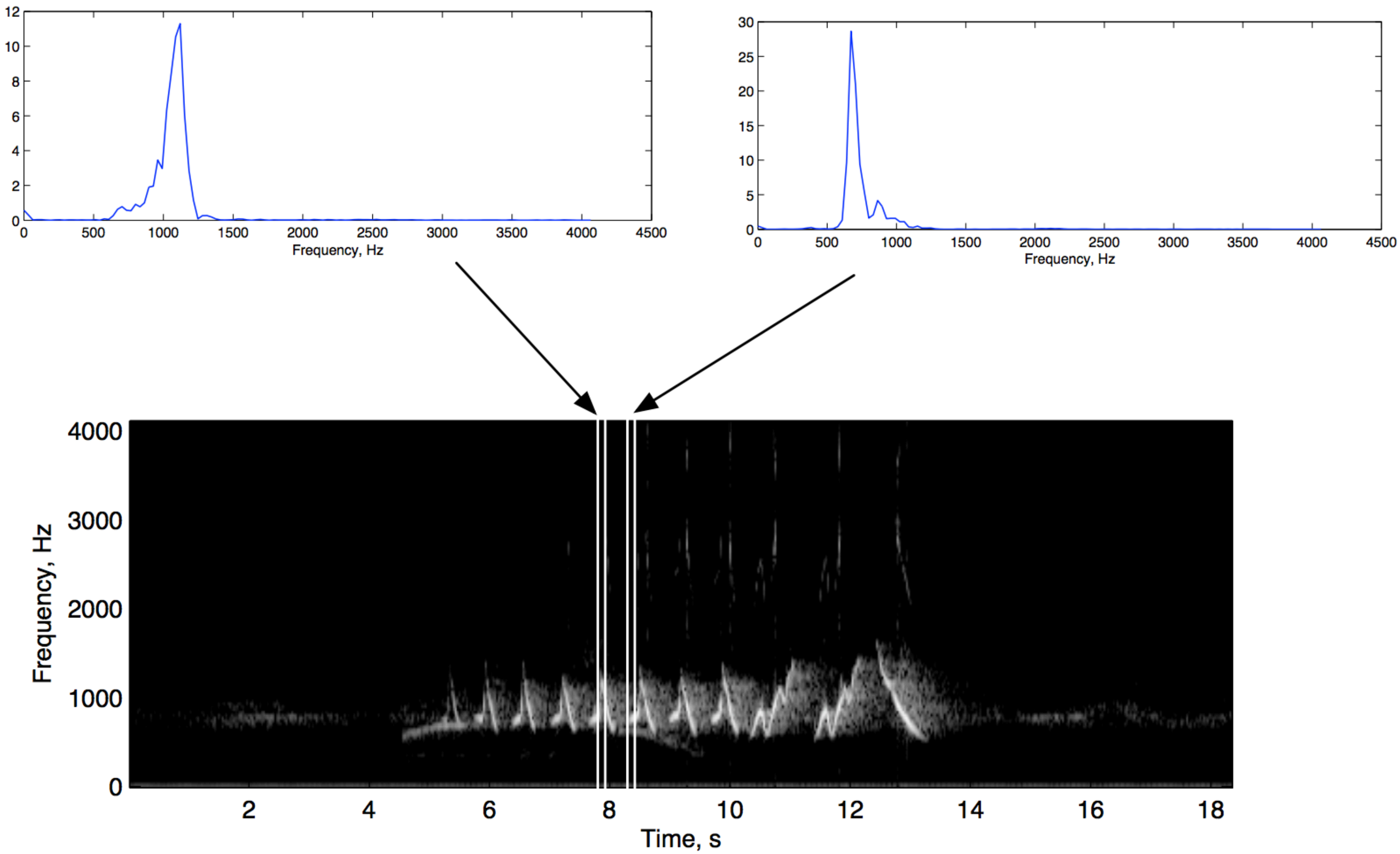


Time-Frequency Uncertainty Principle

$$\sigma_t \cdot \sigma_\omega \geq \frac{1}{2}$$



Spectrogram Example



Discrete Time-Dependent Fourier Transform

$$X[n,\lambda) = \sum_{m=-\infty}^{\infty} x[n+m]w[m]e^{-j\lambda m}$$

$$X[rR,k] = X[rR, 2\pi k / N) = \sum_{m=0}^{L-1} x[rR+m]w[m]e^{-j(2\pi/N)km}$$

- L - Window length
- R - Jump of samples
- N - DFT length

Discrete Time-Dependent Fourier Transform

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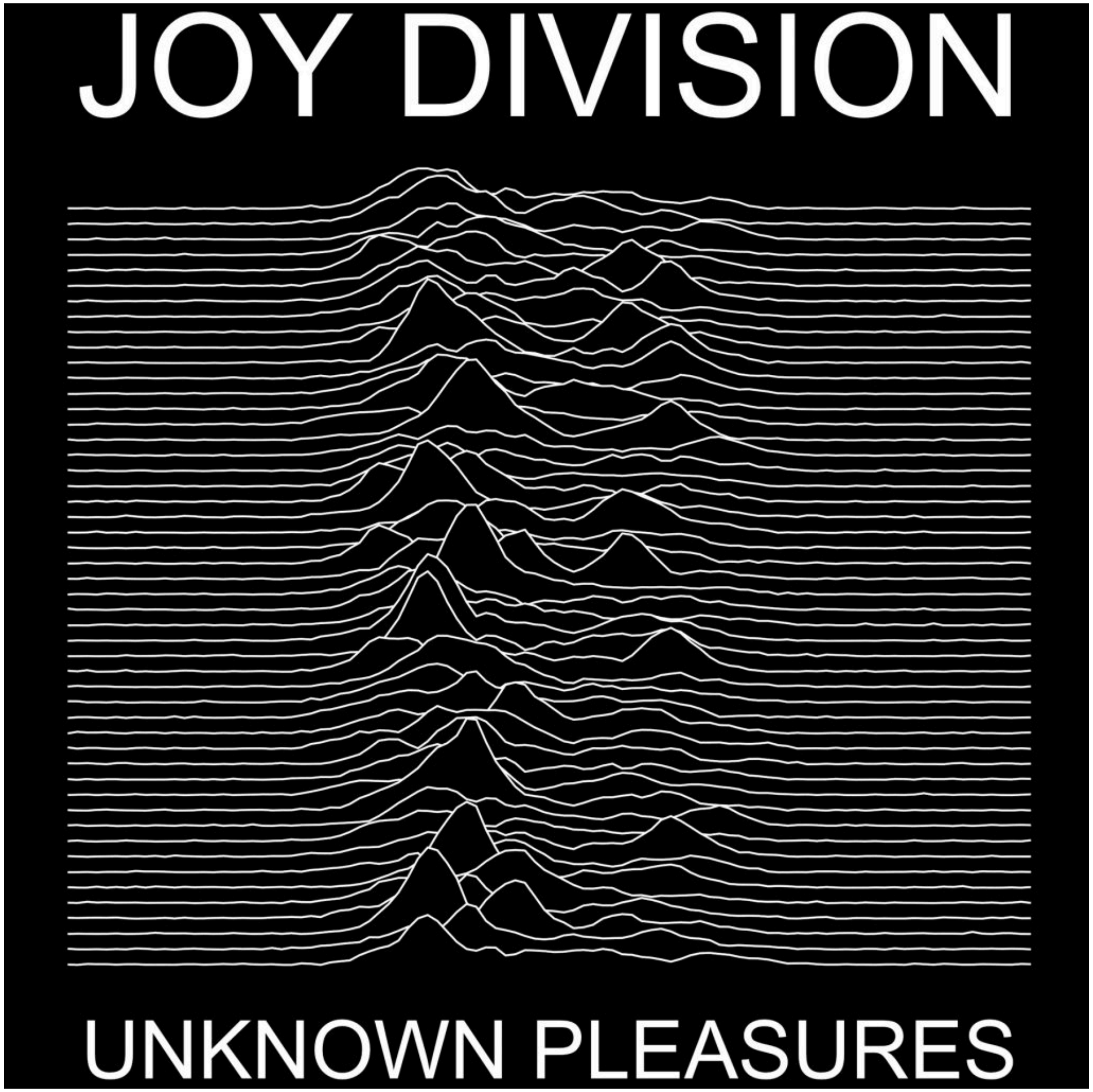
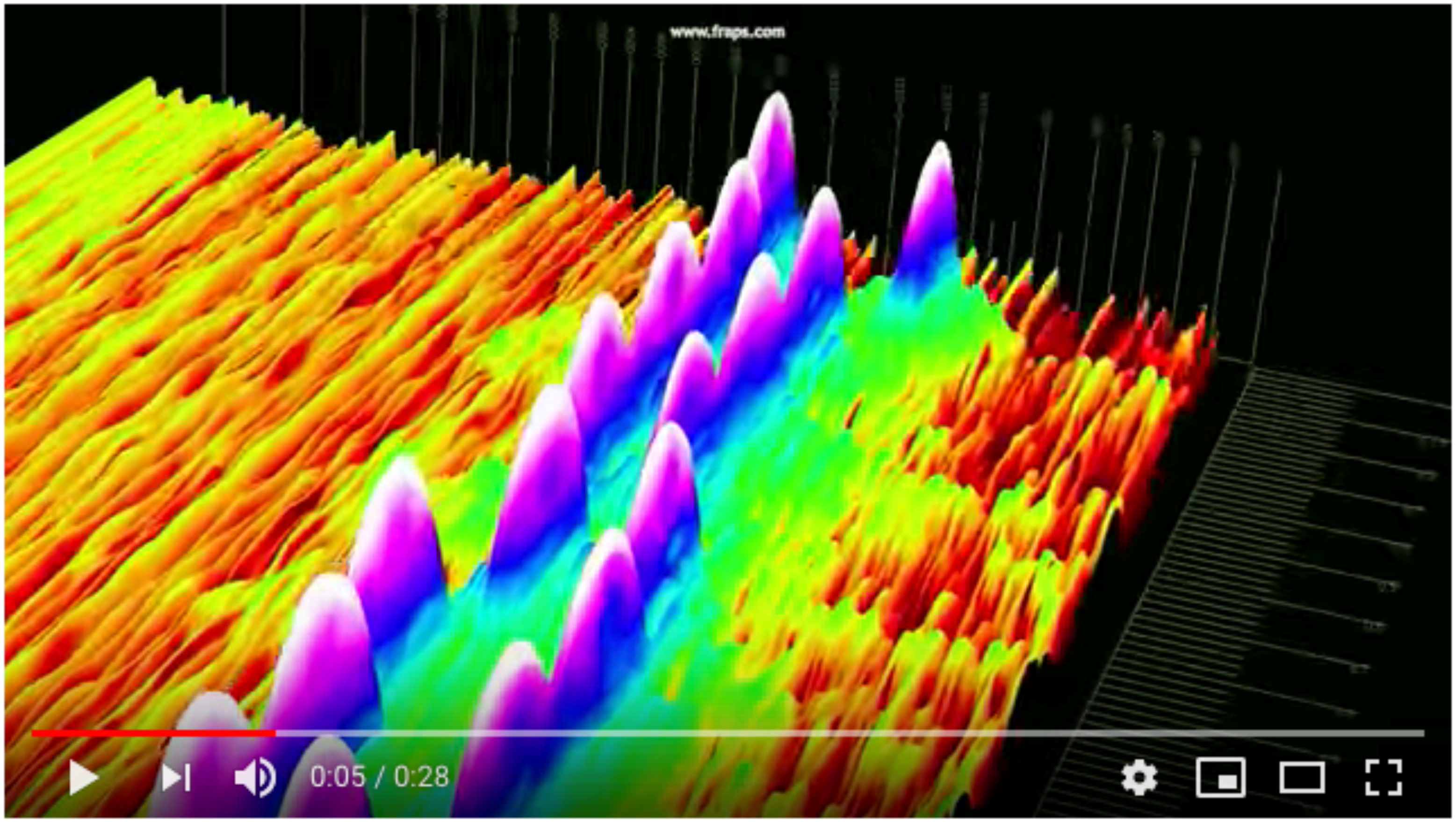
$$X_r[k] = \sum_{m=0}^{L-1} x[rR+m]w[m]e^{-j(2\pi/N)km}$$

Joy Division



Audio Visualization



- https://www.youtube.com/watch?v=vvr9AMWEU-c

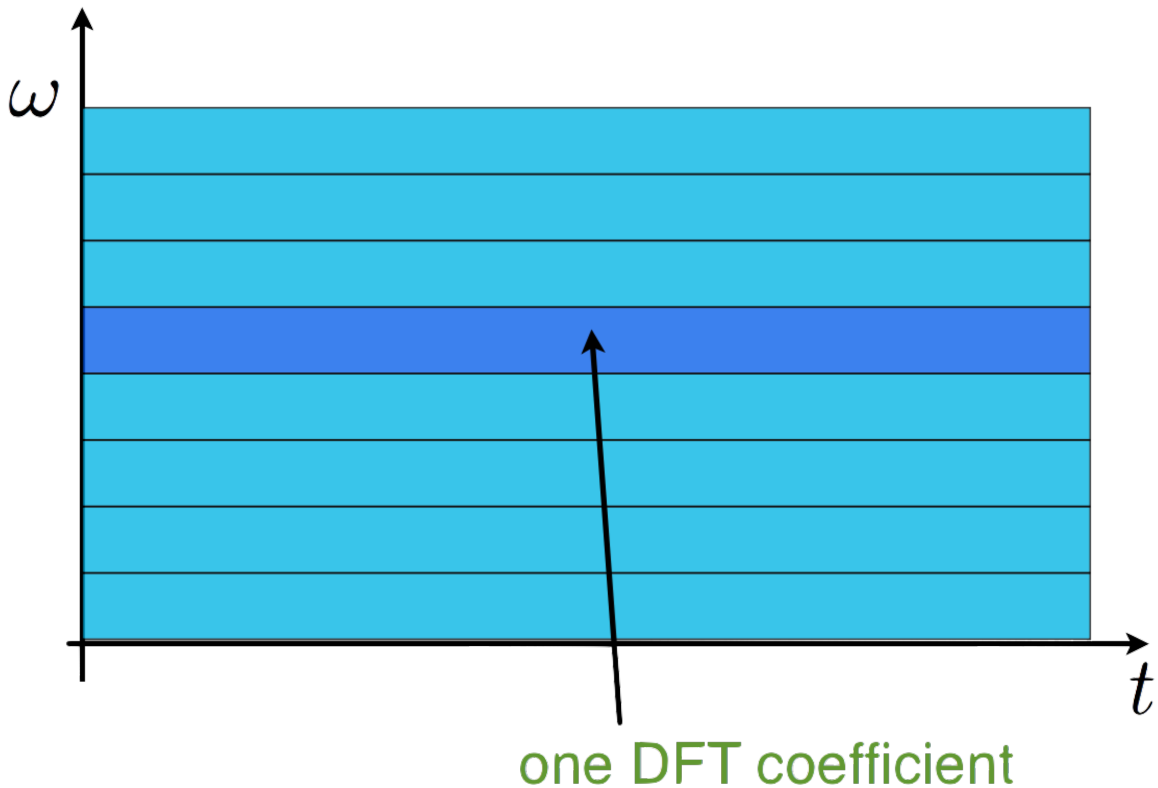
DFT

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}$$

$$\Delta\omega = \frac{2\pi}{N}$$

$$\Delta t = N$$

$$\Delta\omega \cdot \Delta t = 2\pi$$



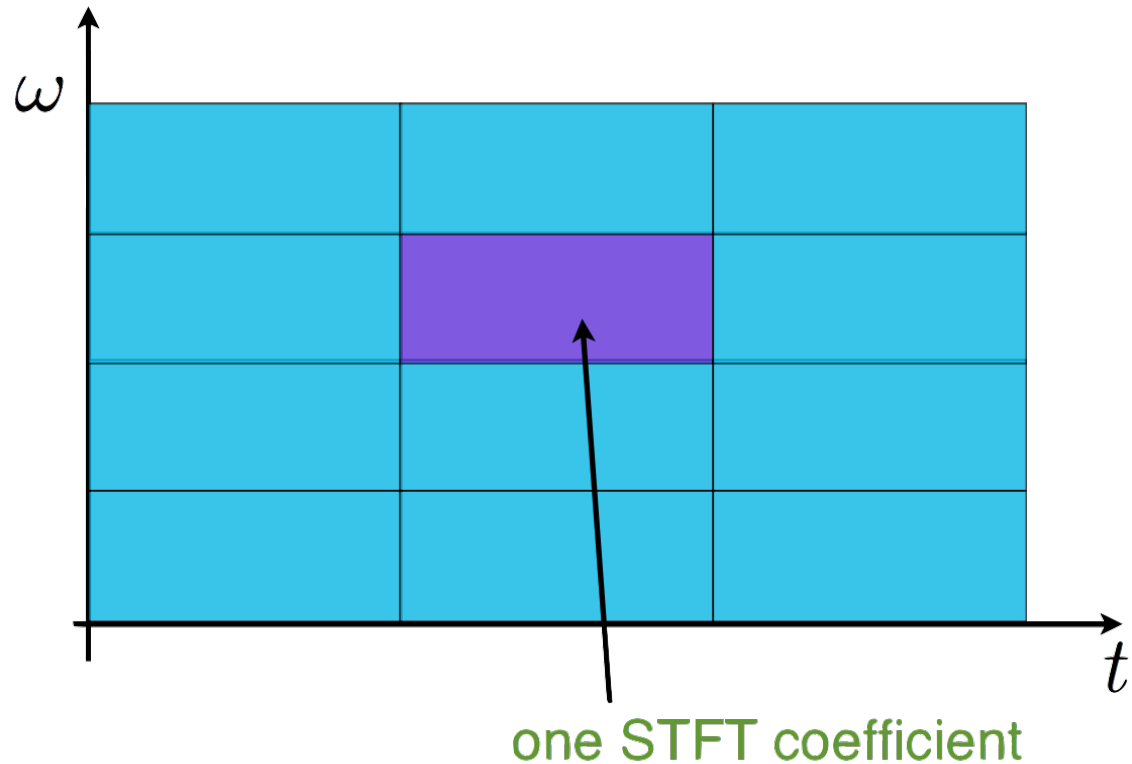
Discrete STFT

optional

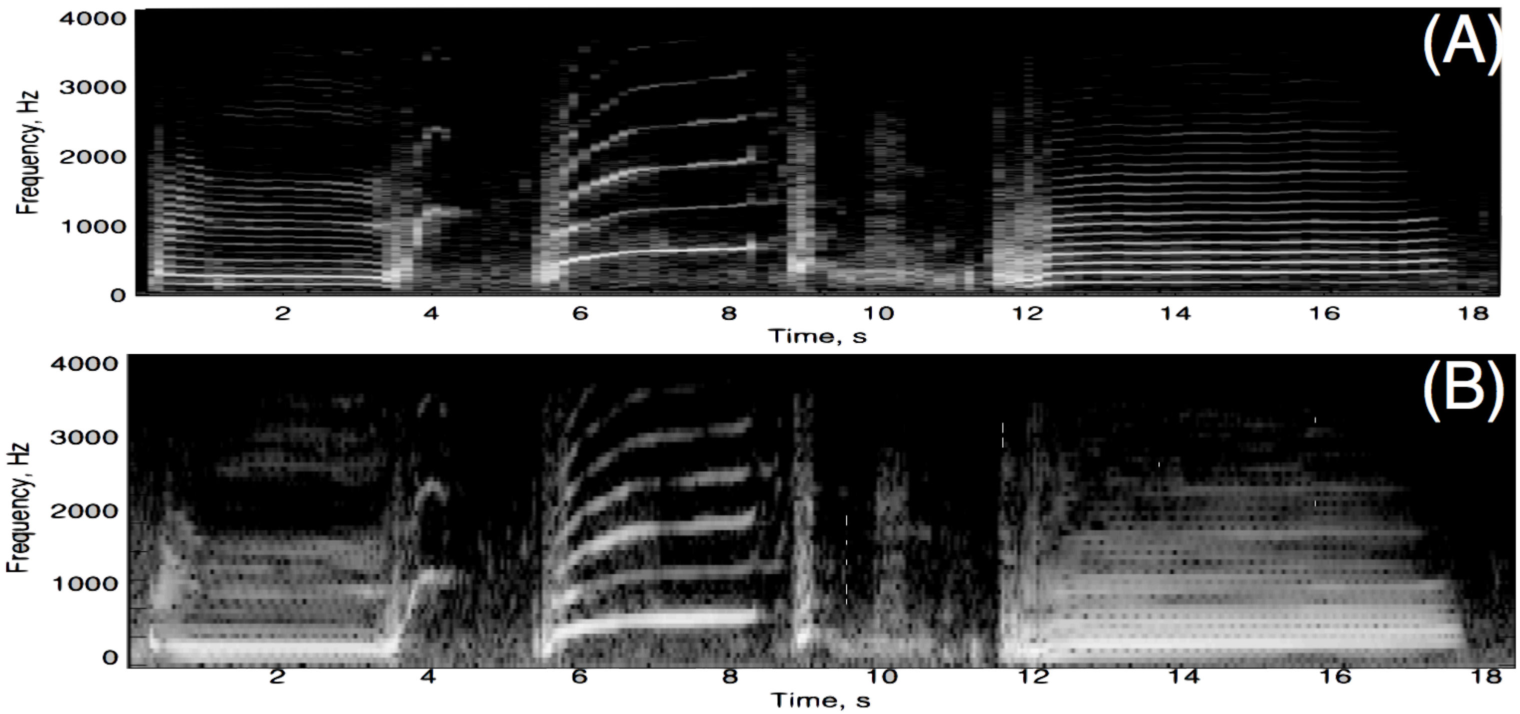
$$X[r,k] = \sum_{m=0}^{L-1} x[r\textcolor{green}{R} + m]w[m]e^{-j2\pi km/N}$$

$$\Delta\omega = \frac{2\pi}{L}$$

$$\Delta t = L$$



Spectrogram



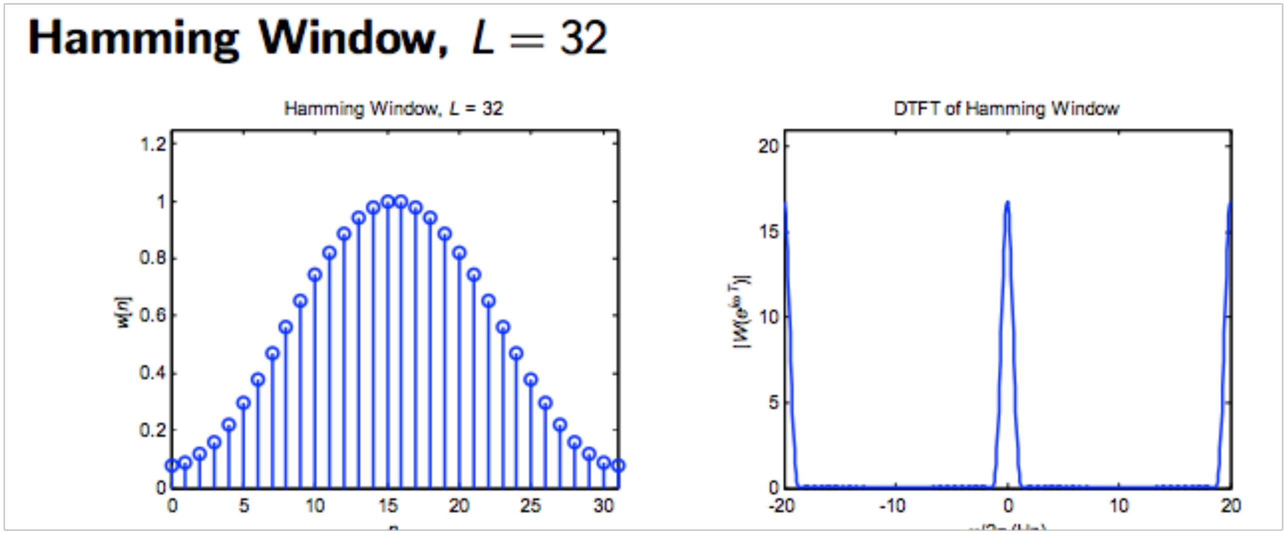
- What is the difference between the spectrograms?

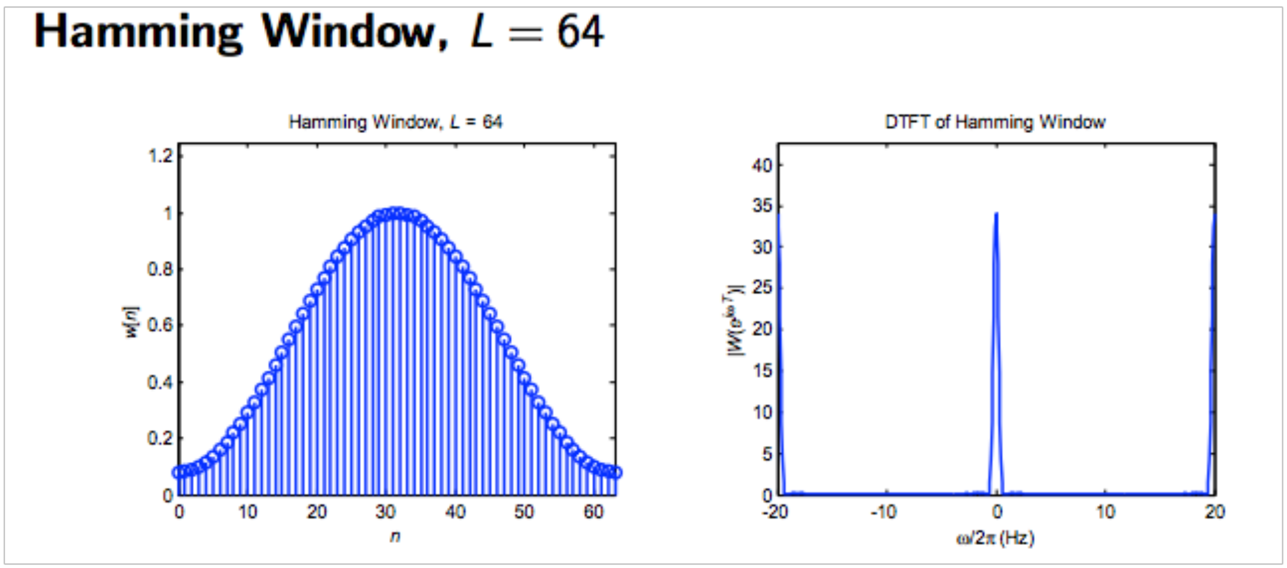
a) Window size B<A

b) Window size B>A

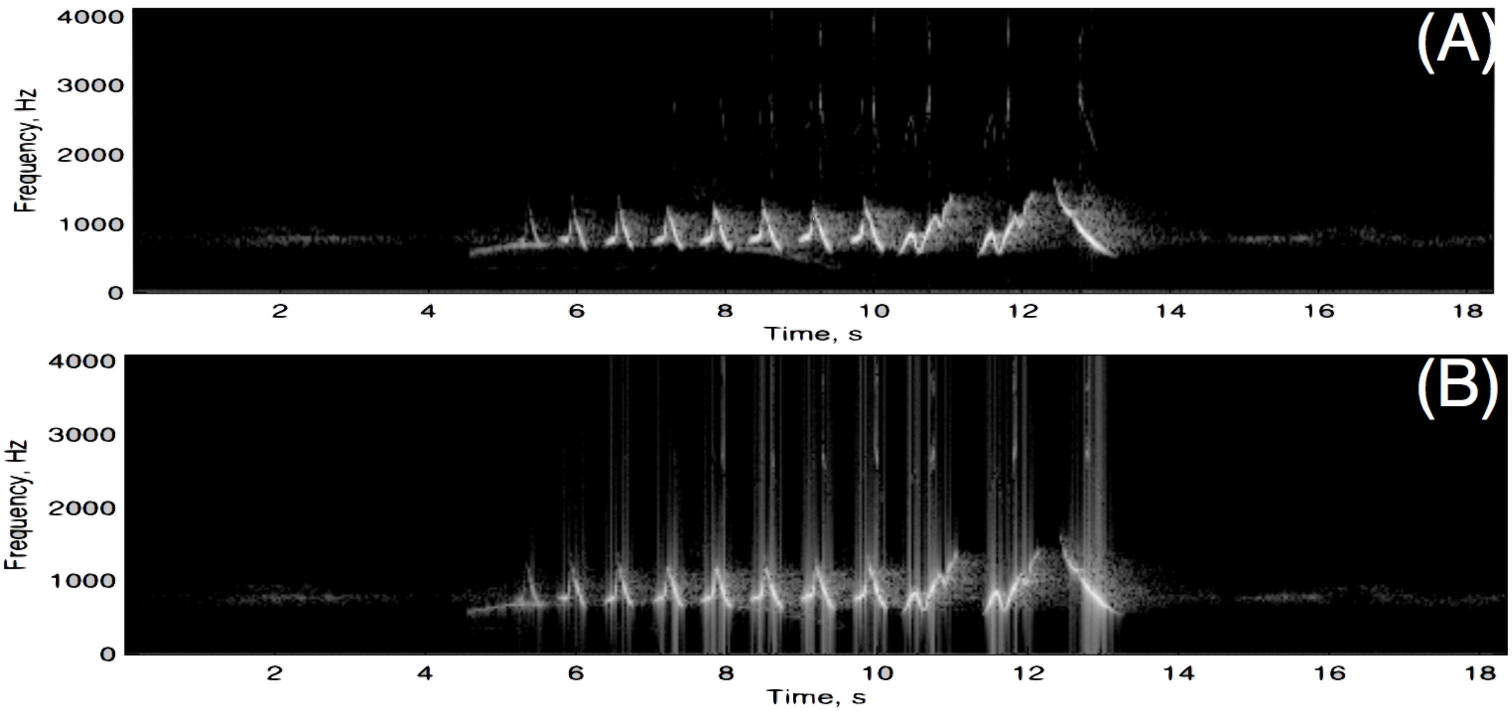
c) Window type is different

Window Size


Hamming Window, $L = 32$
Hamming Window, L = 32
DTFT of Hamming Window
w[n]
$|W(e^{j\omega T})|$

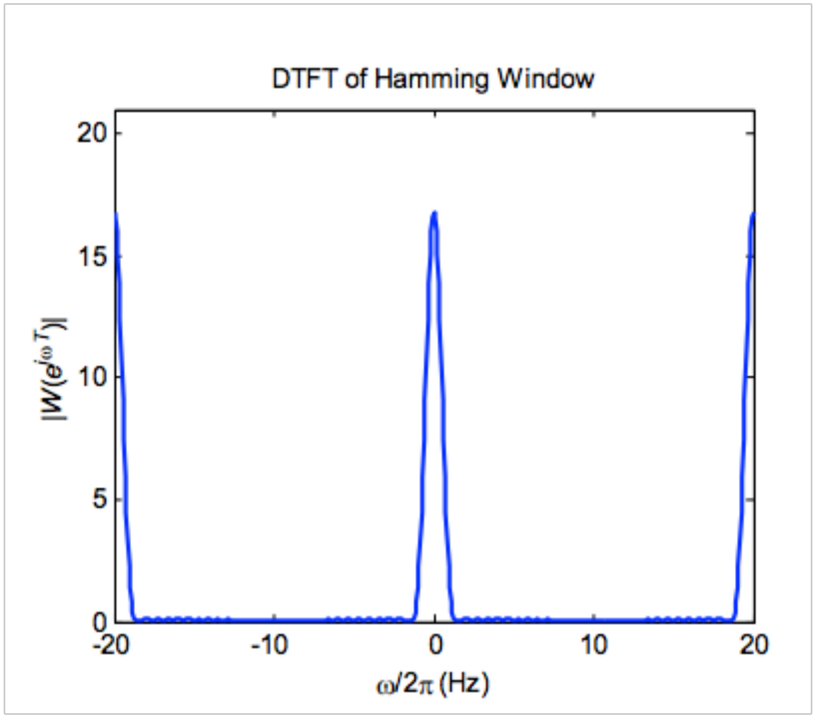

Hamming Window, $L = 64$
Hamming Window, L = 64
DTFT of Hamming Window
w[n]
n
$|W(e^{j\omega T})|$
ω/2π (Hz)

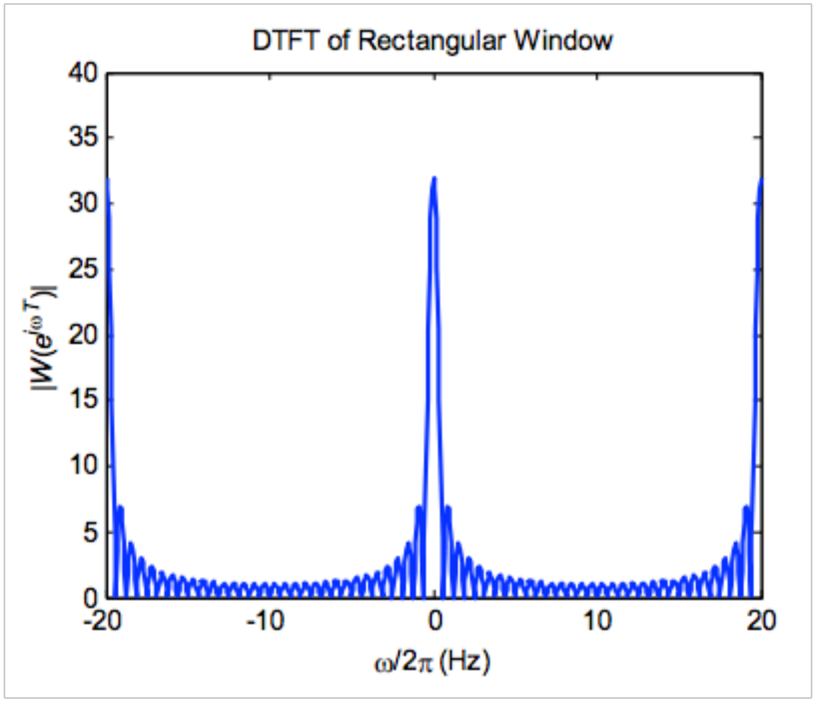
Spectrogram



- What is the difference between the spectrograms?
  - a) Window size B<A
  - b) Window size B>A
  - c) Window type is different

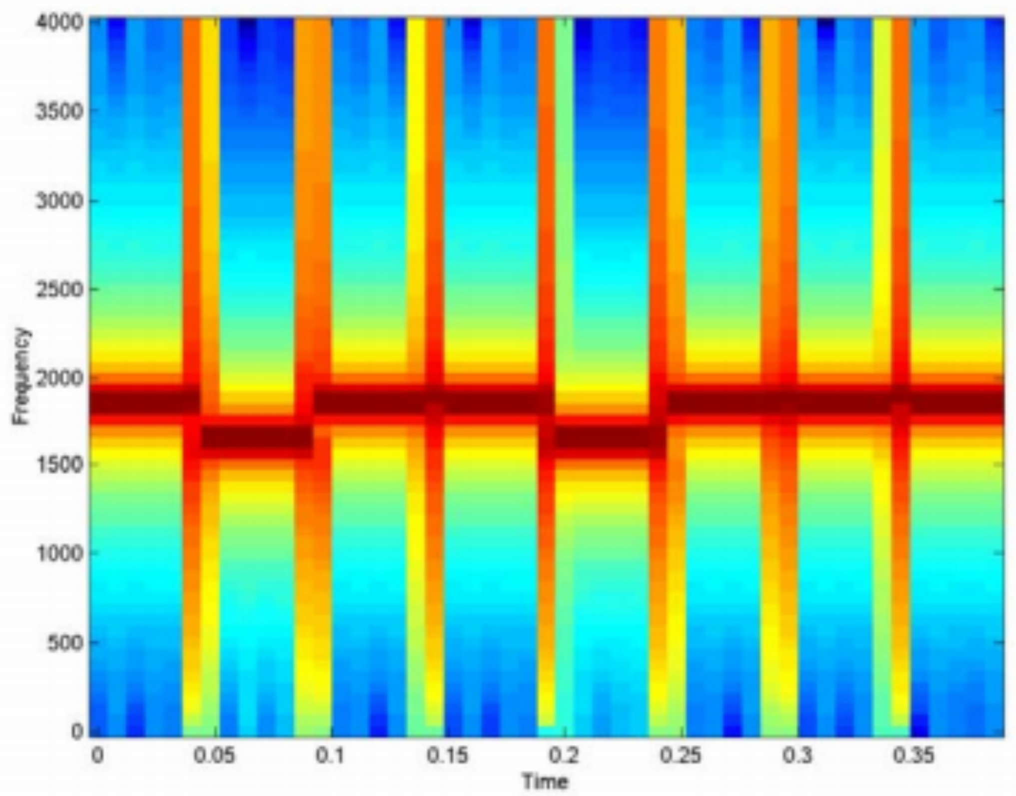
Sidelobes of Windows





Application – Frequency Shift Keying

- FSK Communications
  - Spectrogram transmitting 'H' (ASCII H = 01001000)



STFT Reconstruction

- If R ≤ L ≤ N, then we can recover x[n] block-by-block from $X_r[k]$
- For non-overlapping windows, R=L

$$x_r[m] = \frac{1}{N}\sum_{k=0}^{N-1} X_r[k] e^{j2\pi km/N}$$

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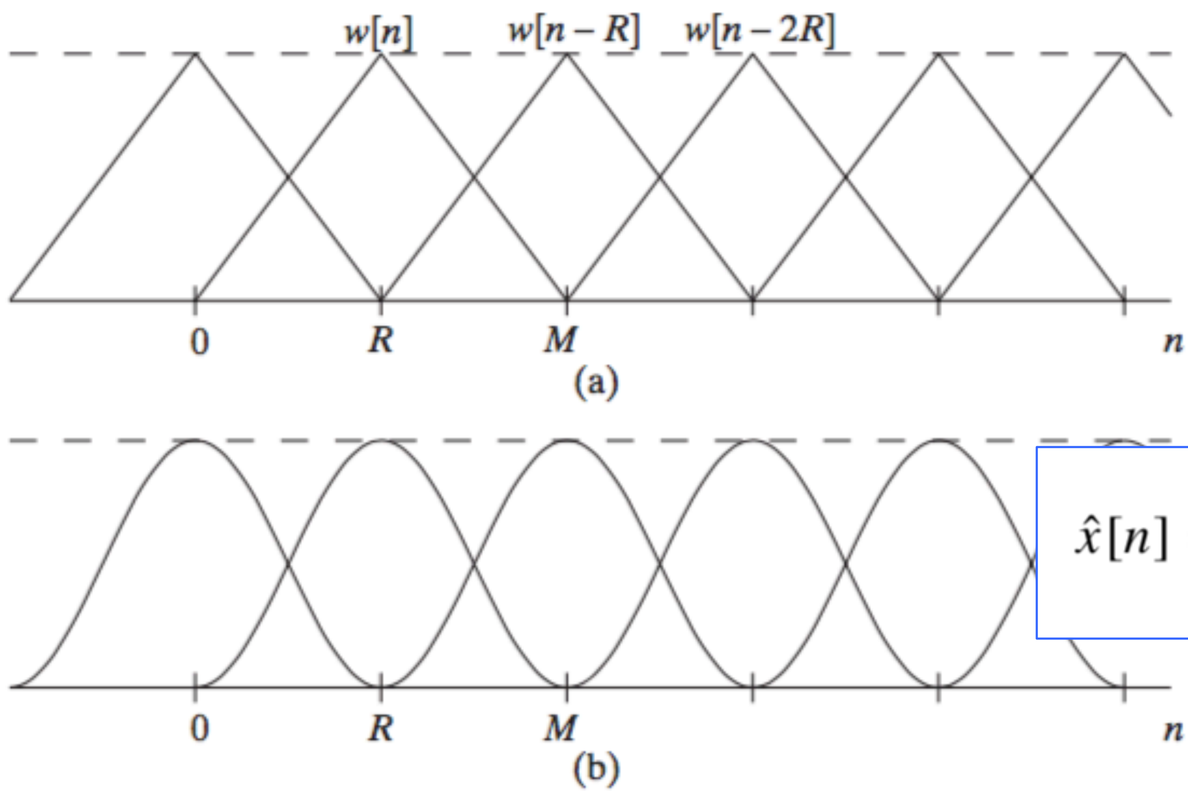
$$x_r[m] = \frac{1}{N}\sum_{k=0}^{N-1} X_r[k]e^{j2\pi km/N}$$

$$x[n] = \frac{x_r[n-rR]}{w[n-rR]} \qquad \forall \quad rR \le n \le (r+1)R-1$$

SFTF Reconstruction with overlap

- Practically make R<L<N
- If we choose R, L, and N appropriately with window, the overlap-add will negate the window effects



$$\hat{x}[n] = \sum_{r=-\infty}^{\infty} x_r[n - rR].$$

Big Ideas

- Frequency analysis with DFT
  - Nontrivial to choose sampling frequency, signal length, window type, DFT length (zero-padding)
  - Get accurate representation of DFT
- Time-dependent Fourier transform
  - Aka short-time Fourier transform
  - Includes temporal information about signal
  - Useful for many applications
    - Analysis, Compression, Denoising, Detection, Recognition, Approximation (Sparse)
  - Overlap for reconstruction

Admin

- Project 2
  - Out now
  - Due 5/1
- Final Exam – 5/1 0
  - 3-5pm
  - DRLB A8