ESE 5310: Digital Signal Processing

Lecture 3: January 30, 2024

Discrete Time Signals and Systems, Pt 2

Lecture Outline

- Discrete Time Systems
- System Properties
- LTI Systems
- Difference Equations
- Eigenfunctions
- Discrete Time Fourier Transform (if time)
  - Definition
  - Window Example

Discrete-Time Systems

Discrete Time Systems

DEFINITION

A discrete-time **system** $\mathcal{H}$ is a transformation (a rule or formula) that maps a discrete-time input signal x into a discrete-time output signal y

$$y = \mathcal{H}\{x\}$$

$$x \longrightarrow \boxed{\mathcal{H}} \longrightarrow y$$

- Systems manipulate the information in signals
- Examples
  - Speech recognition system that converts acoustic waves into text
  - Radar system transforms radar pulse into position and velocity
  - fMRI system transform frequency into images of brain activity
  - Moving average system smooths out the day-to-day variability in a stock price

System Properties

- Causality
  - y[n] only depends on x[m] for m<=n
- Linearity
  - Scaled sum of arbitrary inputs results in output that is a scaled sum of corresponding outputs
    - $Ax_1[n]+Bx_2[n] \rightarrow Ay_1[n]+By_2[n]$
- Memoryless
  - y[n] depends only on x[n]
- Time Invariance
  - Shifted input results in shifted output
    - $x[n-q] \rightarrow y[n-q]$
- BIBO Stability
  - A bounded input results in a bounded output (ie. max signal value exists for output if max)

Examples

- Causal? Linear? Time-invariant? Memoryless? BIBO Stable?
- Time Shift:

$$y[n] = x[n-m]$$

- Accumulator:

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

- Compressor (M>1):

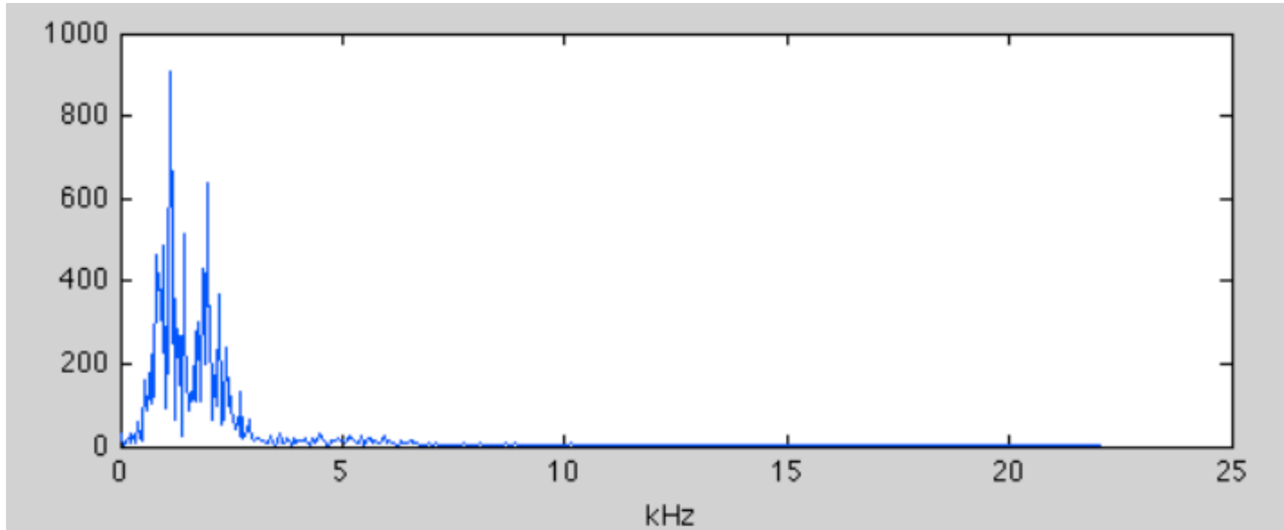
$$y[n] = x[Mn]$$

Non-Linear System Example

- Median Filter
  - $y[n]=MED\{x[n-k], \ldots x[n+k]\}$
  - Let $k=1$
  - $y[n]=MED\{x[n-1], x[n], x[n+1]\}$

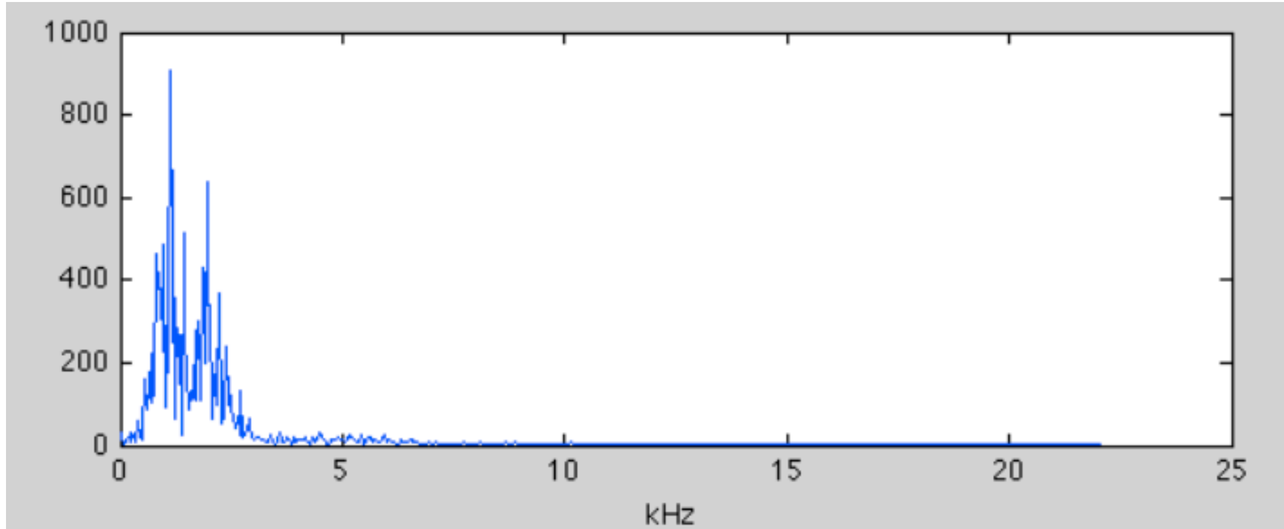
Spectrum of Speech

Speech

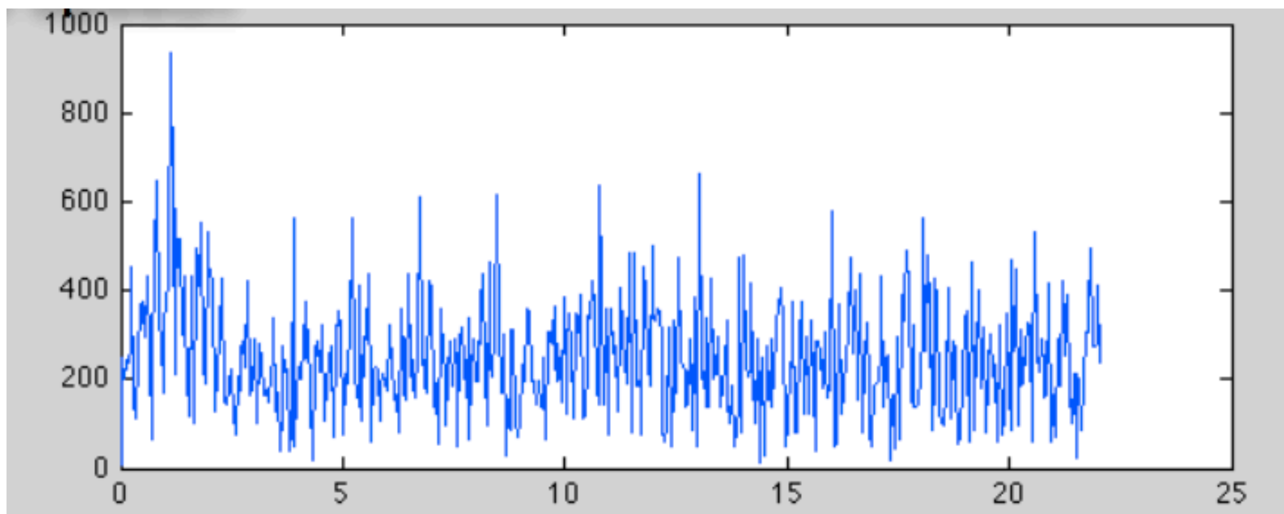


Spectrum of Speech

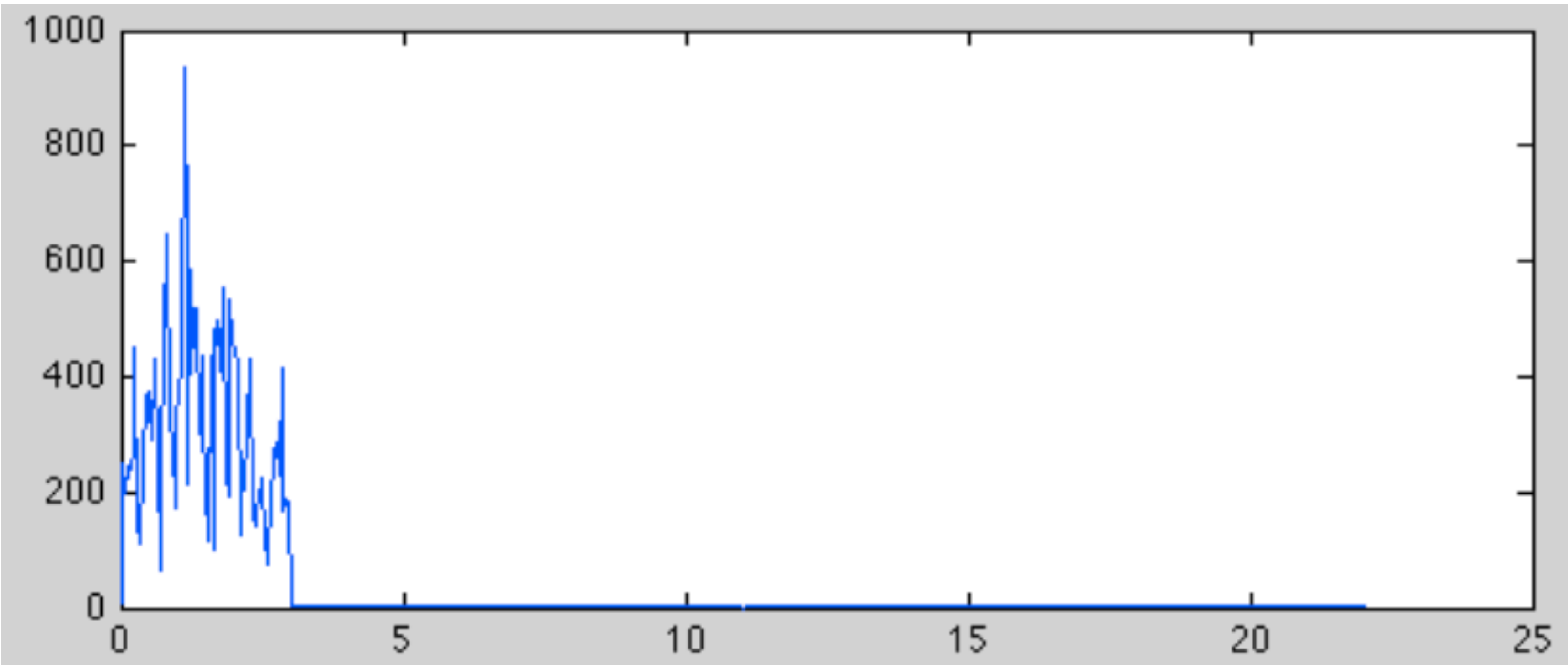
Speech



Corrupted Speech

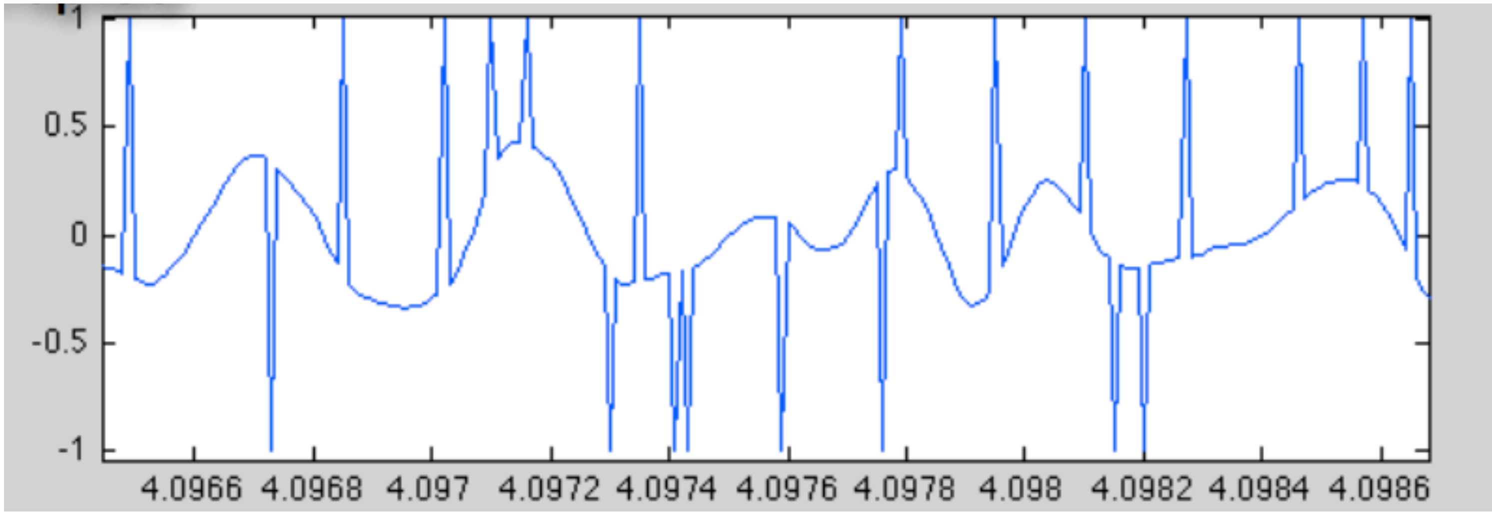


Low Pass Filtering



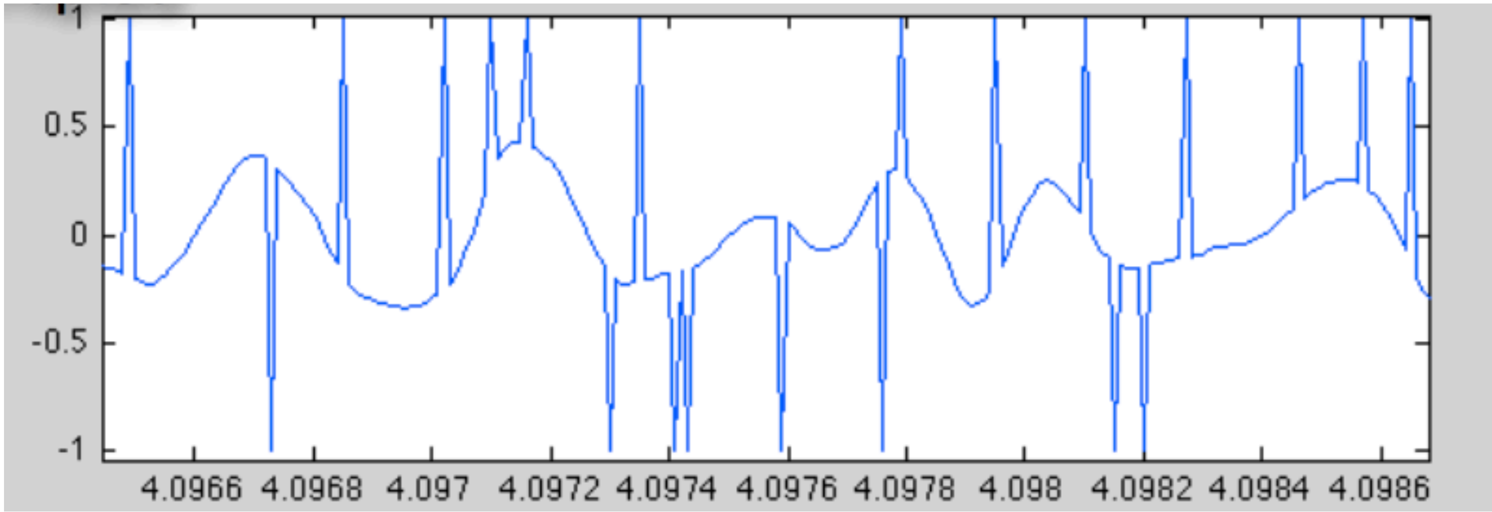
Speech in Time

Corrupted Speech

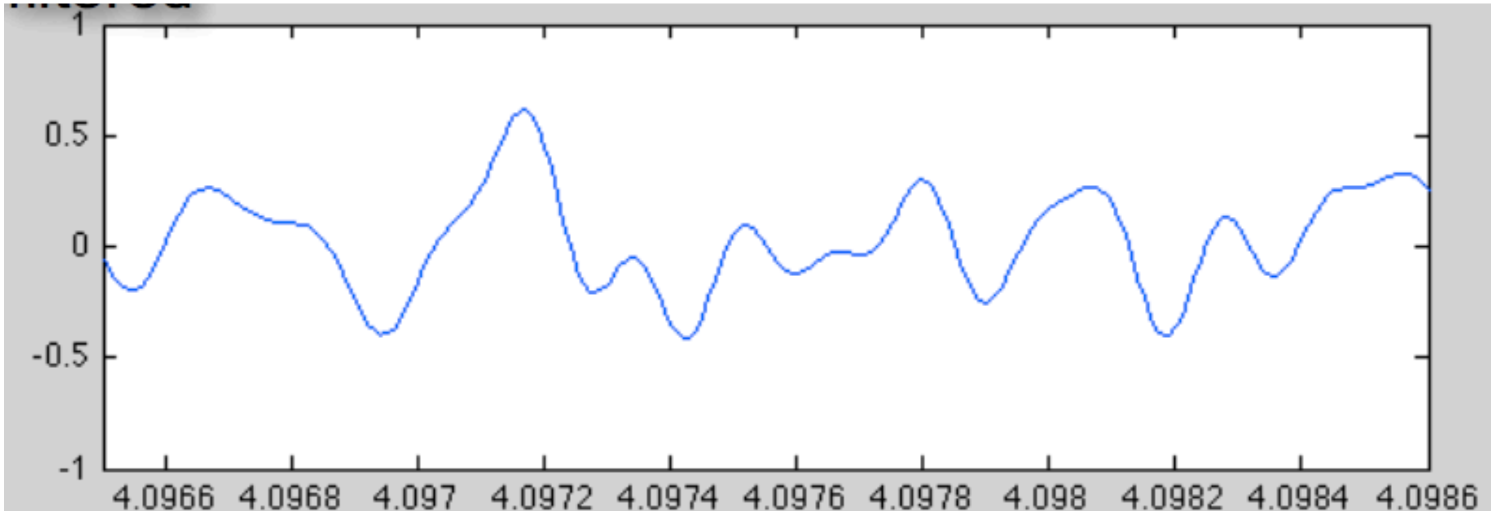


Low Pass Filtering

Corrupted Speech

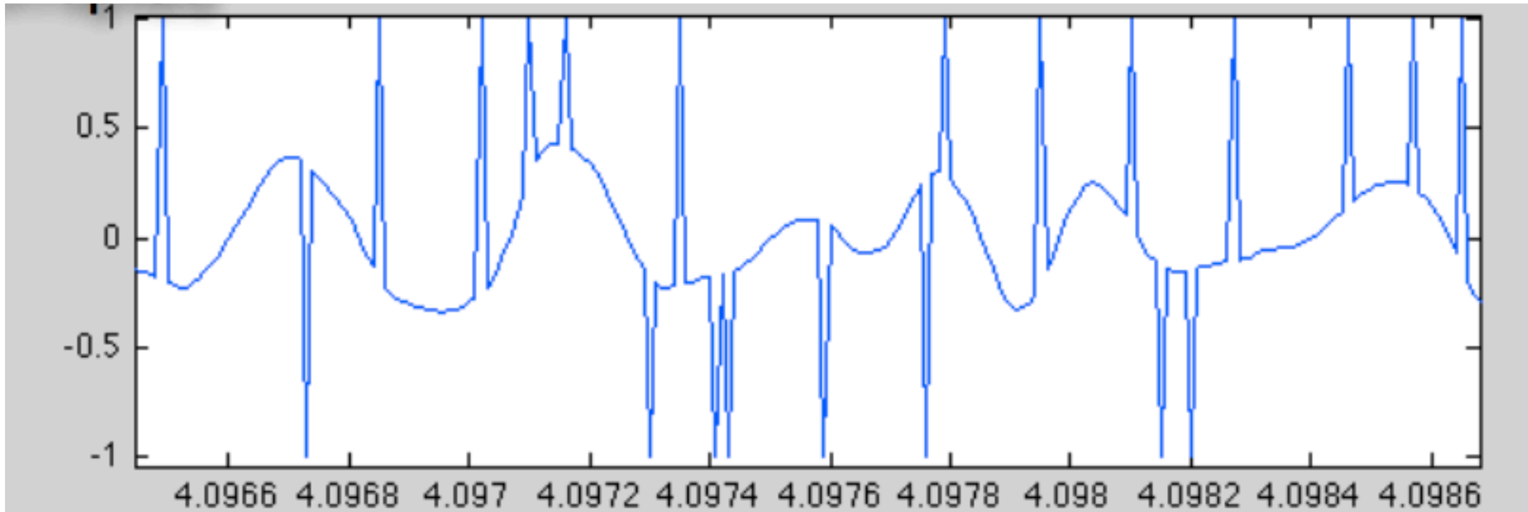


LP-Filtered Speech

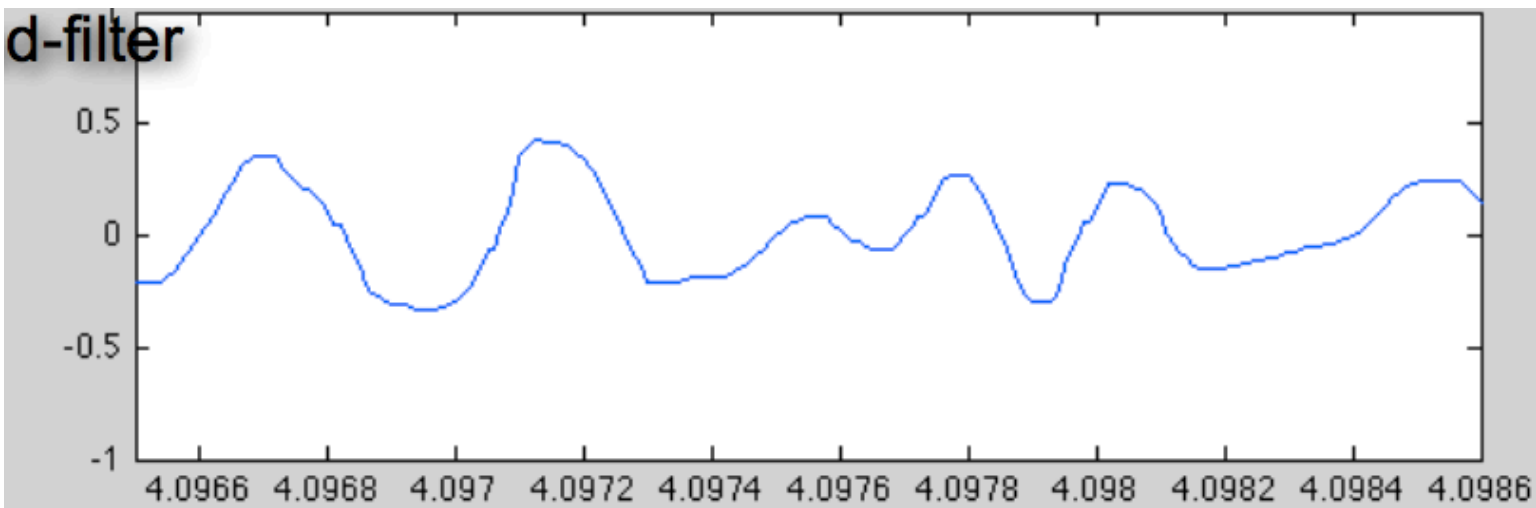


Median Filtering

Corrupted Speech



Med-Filter Speech



LTI Systems

LTI Systems

DEFINITION

A system $\mathcal{H}$ is **linear time-invariant** (LTI) if it is both linear and time-invariant

- LTI system can be completely characterized by its impulse response

$$\delta \longrightarrow \boxed{\mathcal{H}} \longrightarrow h$$

- Then the output for an arbitrary input is a sum of weighted, delay impulse responses

$$x \longrightarrow \boxed{h} \longrightarrow y \qquad y[n] = \sum_{m=-\infty}^{\infty} h[n-m]\, x[m]$$

$$y[n] = x[n] * h[n]$$

Convolution

$$x \longrightarrow \boxed{h} \longrightarrow y$$

- Convolution formula:

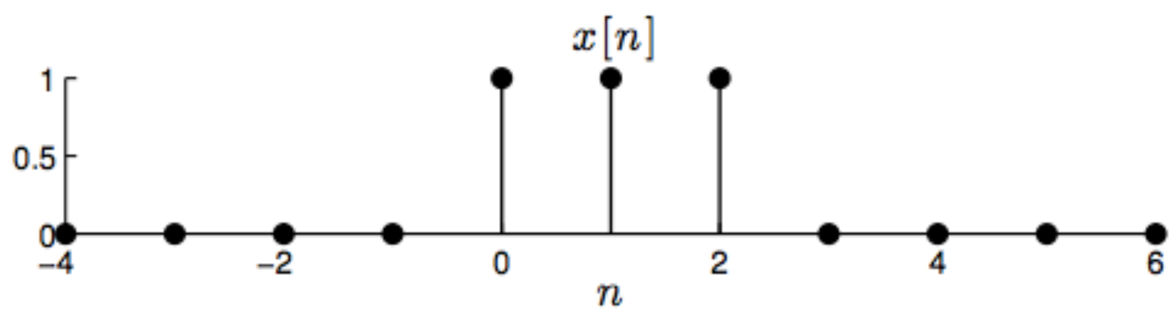
$$y[n] = x[n] * h[n] = \sum_{m=-\infty}^{\infty} h[n-m]\, x[m]$$

- Convolution method:
  - 1) Time reverse the impulse response and shift it *n* time steps to the right
  - 2) Compute the inner product between the shifted impulse response and the input vector
  - Repeat for every *n*

Convolution Example

$$y[n] = x[n] * h[n] = \sum_{m=-\infty}^{\infty} h[n-m]\, x[m]$$

- Convolve a unit pulse with itself



Convolution is Commutative

- Convolution is commutative:

$$x * h = h * x$$

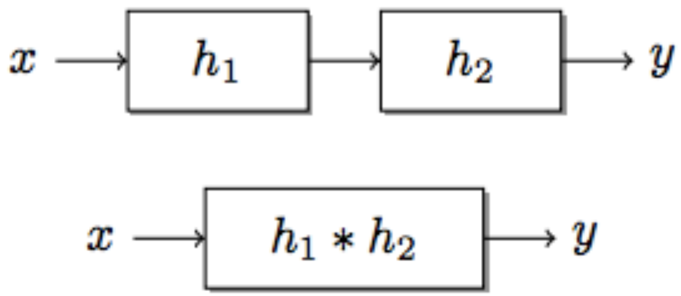
- These block diagrams are equivalent

$$x \longrightarrow \boxed{h} \longrightarrow y \qquad\qquad h \longrightarrow \boxed{x} \longrightarrow y$$

- Implication: pick either h or x to flip and shift when convolving

LTI Systems in Series

- Impulse response of the cascade of two LTI systems:



LTI Systems in Series

- Impulse response of the cascade of two LTI systems:

$$x \longrightarrow \boxed{h_1} \longrightarrow \boxed{h_2} \longrightarrow y$$

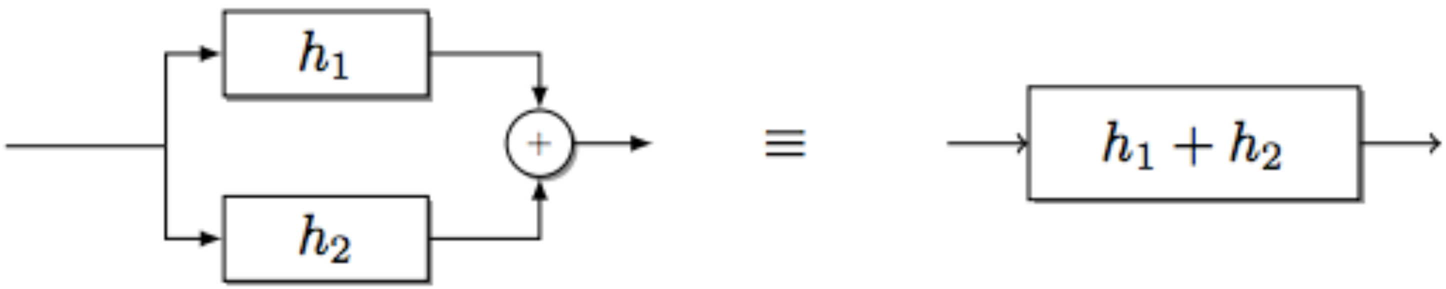
$$x \longrightarrow \boxed{h_1 * h_2} \longrightarrow y$$

- Proof by picture

$$\delta \longrightarrow \boxed{h_1} \longrightarrow h_1 \qquad \longrightarrow \boxed{h_2} \longrightarrow h_1 * h_2$$

LTI Systems in Parallel

- Impulse response of the parallel connection of two LTI systems:



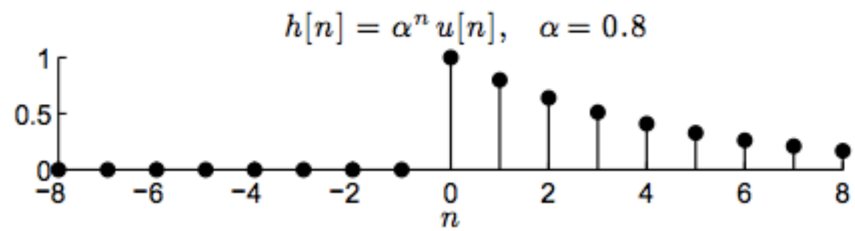
Causal System Revisited

DEFINITION

A system $\mathcal{H}$ is **causal** if the output $y[n]$ at time n depends only the input $x[m]$ for times $m \leq n$. In words, causal systems do not look into the future

- An LTI system is causal if its impulse response is causal:

$$h[n] = 0 \text{ for } n < 0$$



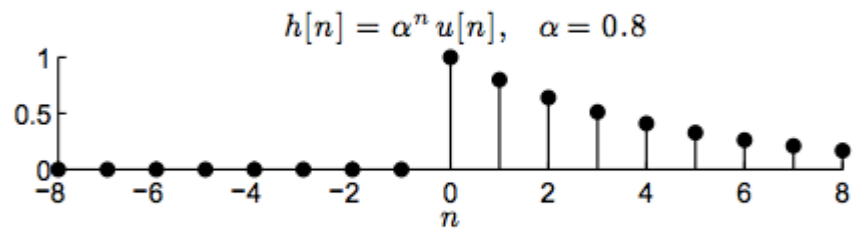
Causal System Revisited

DEFINITION

A system $\mathcal{H}$ is **causal** if the output $y[n]$ at time n depends only the input $x[m]$ for times $m \leq n$. In words, causal systems do not look into the future

- An LTI system is causal if its impulse response is causal:

$$h[n] = 0 \text{ for } n < 0$$



- To prove, note that the convolution does not look into the future if the impulse response is causal

$$y[n] = \sum_{m=-\infty}^{\infty} h[n-m]\,x[m] \qquad h[n-m] = 0 \text{ when } m > n;$$

Duration of Impulse

DEFINITION

An LTI system has a **finite impulse response** (FIR) if the duration of its impulse response h is finite

Duration of Impulse

DEFINITION

An LTI system has a **finite impulse response** (FIR) if the duration of its impulse response h is finite

- Example: Moving average

$$y[n] = \mathcal{H}\{x[n]\} = \frac{1}{2}(x[n] + x[n-1])$$

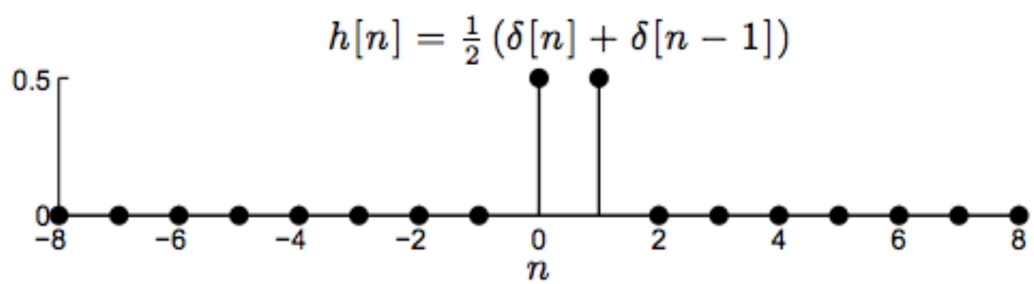
Duration of Impulse

DEFINITION

An LTI system has a **finite impulse response** (FIR) if the duration of its impulse response h is finite

- Example: Moving average

$$y[n] = \mathcal{H}\{x[n]\} = \frac{1}{2}\left(x[n] + x[n-1]\right)$$



Duration of Impulse

DEFINITION

An LTI system has an **infinite impulse response** (IIR) if the duration of its impulse response h is infinite

Duration of Impulse

DEFINITION

An LTI system has an **infinite impulse response** (IIR) if the duration of its impulse response h is infinite

- Example: Recursive average

$$y[n] \;=\; \mathcal{H}\{x[n]\} \;=\; x[n] + \alpha\, y[n-1]$$

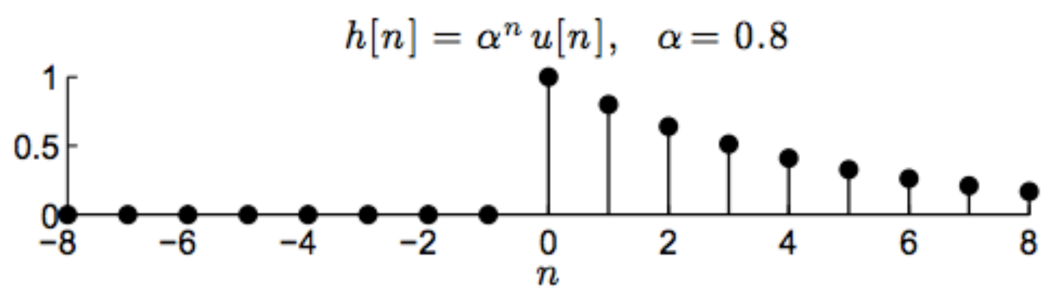
Duration of Impulse

DEFINITION

An LTI system has an **infinite impulse response** (IIR) if the duration of its impulse response h is infinite

- Example: Recursive average

$$y[n] = \mathcal{H}\{x[n]\} = x[n] + \alpha\, y[n-1]$$



BIBO Stability Revisited

DEFINITION

An LTI system is **bounded-input bounded-output (BIBO) stable** if a bounded input x always produces a bounded output y

$$\text{bounded } x \longrightarrow \boxed{h} \longrightarrow \text{bounded } y$$

BIBO Stability Revisited

DEFINITION

An LTI system is **bounded-input bounded-output (BIBO) stable** if a bounded input x always produces a bounded output y

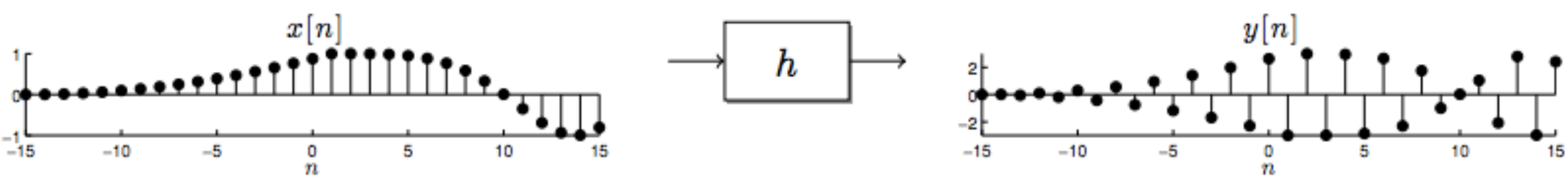
$$\text{bounded } x \longrightarrow \boxed{h} \longrightarrow \text{bounded } y$$

- Bounded input and output:

$$\|x\|_\infty < \infty \quad \text{and} \quad \|y\|_\infty < \infty$$

- Where

$$\|x\|_\infty = \max|x[n]|$$



BIBO Stability Revisited

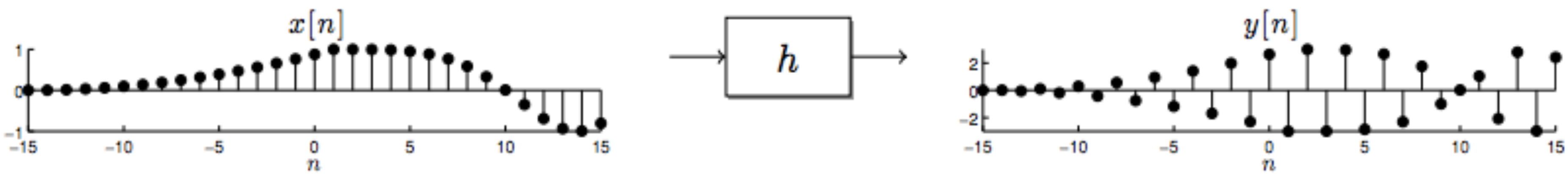
DEFINITION

An LTI system is **bounded-input bounded-output (BIBO) stable** if a bounded input x always produces a bounded output y

$$\text{bounded } x \longrightarrow \boxed{h} \longrightarrow \text{bounded } y$$

- Bounded input and output:

$$\|x\|_\infty < \infty \quad \text{and} \quad \|y\|_\infty < \infty$$

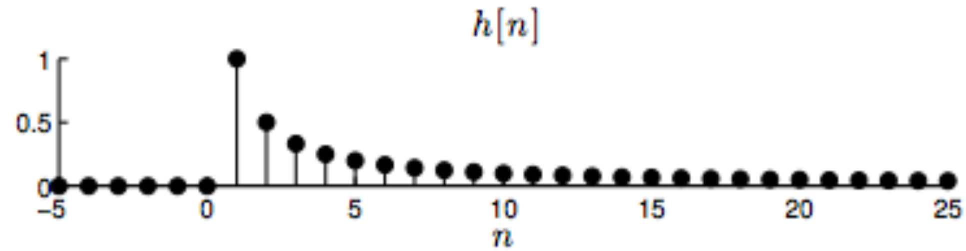


- An LTI system is BIBO stable if and only if

$$\|h\|_1 = \sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

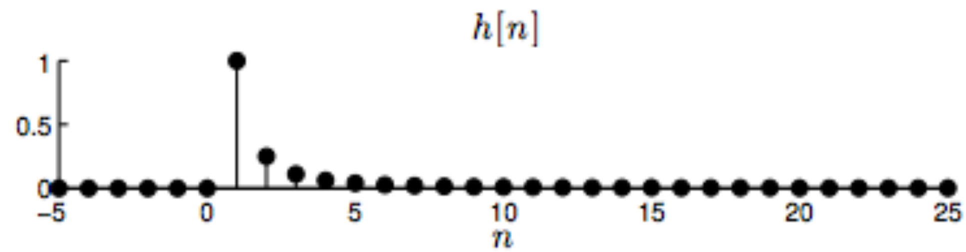
Examples

Example: $h[n] = \begin{cases} \frac{1}{n} & n \geq 1 \\ 0 & \text{otherwise} \end{cases}$



$\|h\|_1 = \sum_{n=1}^{\infty} \left|\frac{1}{n}\right| = \infty \;\Rightarrow\;$ not BIBO

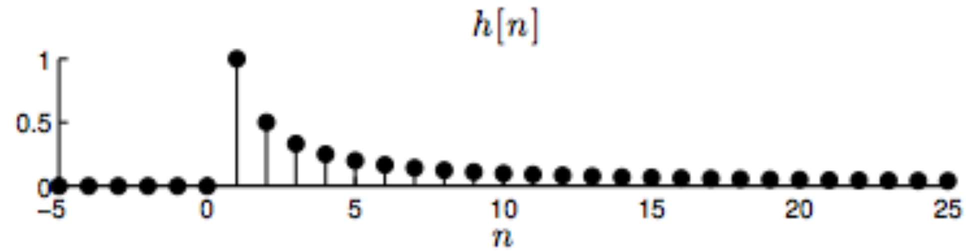
Example: $h[n] = \begin{cases} \frac{1}{n^2} & n \geq 1 \\ 0 & \text{otherwise} \end{cases}$



$\|h\|_1 = \sum_{n=1}^{\infty} \left|\frac{1}{n^2}\right| = \frac{\pi^2}{6} \;\Rightarrow\;$ BIBO

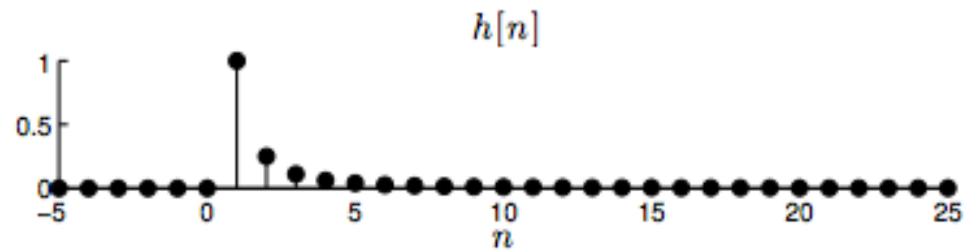
Examples

Example: $h[n] = \begin{cases} \frac{1}{n} & n \geq 1 \\ 0 & \text{otherwise} \end{cases}$



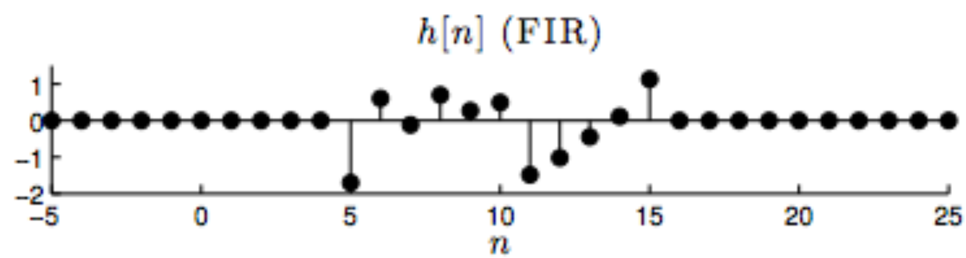
$\|h\|_1 = \sum_{n=1}^{\infty} \left|\frac{1}{n}\right| = \infty \;\Rightarrow$ not BIBO

Example: $h[n] = \begin{cases} \frac{1}{n^2} & n \geq 1 \\ 0 & \text{otherwise} \end{cases}$



$\|h\|_1 = \sum_{n=1}^{\infty} \left|\frac{1}{n^2}\right| = \frac{\pi^2}{6} \;\Rightarrow$ BIBO

Example: h FIR $\Rightarrow$ BIBO



Example

- Example: Recall the recursive average system $y[n] = \mathcal{H}\{x[n]\} = x[n] + \alpha\, y[n-1]$

- Impulse response: $h[n] = \alpha^n\, u[n]$

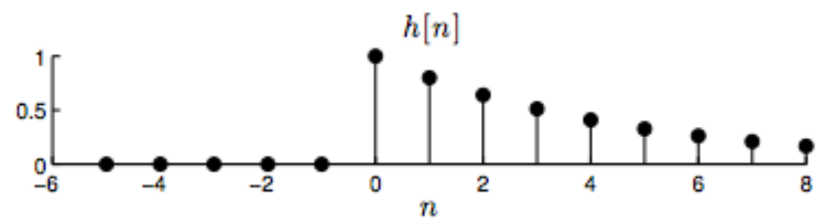
Example

- Example: Recall the recursive average system $y[n] = \mathcal{H}\{x[n]\} = x[n] + \alpha\, y[n-1]$

- Impulse response: $h[n] = \alpha^n\, u[n]$

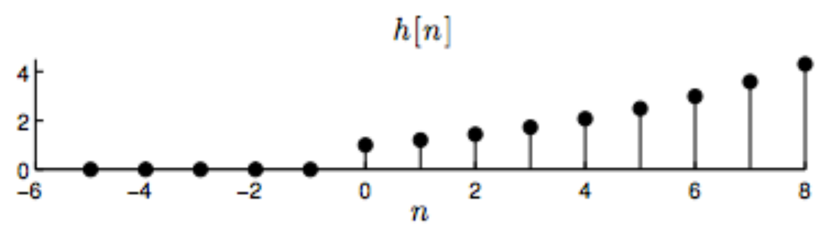
For $|\alpha| < 1$

$\|h\|_1 = \sum_{n=0}^{\infty} |\alpha|^n = \frac{1}{1-|\alpha|} < \infty \;\Rightarrow$ BIBO



For $|\alpha| > 1$

$\|h\|_1 = \sum_{n=0}^{\infty} |\alpha|^n = \infty \;\Rightarrow$ not BIBO



Difference Equations

- Accumulator example

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

Difference Equations

- Accumulator example

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

$$y[n] = x[n] + \sum_{k=-\infty}^{n-1} x[k]$$

$$y[n] = x[n] + y[n-1]$$

$$y[n] - y[n-1] = x[n]$$

Difference Equations

- Accumulator example

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

$$y[n] = x[n] + \sum_{k=-\infty}^{n-1} x[k]$$

$$y[n] = x[n] + y[n-1]$$

$$\boxed{y[n] - y[n-1] = x[n]}$$

$$\boxed{\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]}$$

Difference Equations

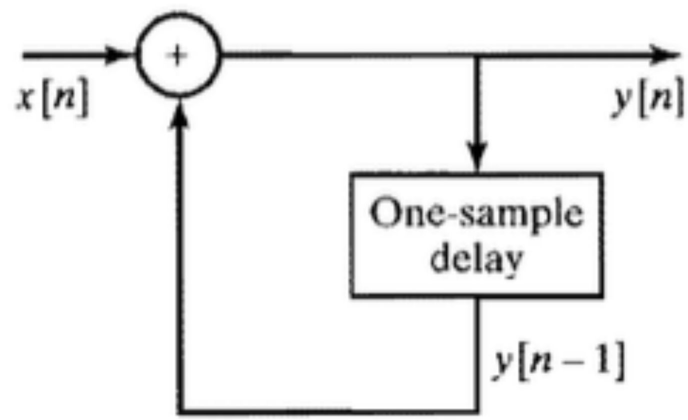
- Accumulator example

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

$$y[n] = x[n] + \sum_{k=-\infty}^{n-1} x[k]$$

$$y[n] = x[n] + y[n-1]$$

$$\boxed{y[n] - y[n-1] = x[n]}$$



$$\boxed{\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]}$$

Example: Difference Equation

- Moving Average System

$$y[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x[n-k]$$

- Causal?

Example: Difference Equation

- Moving Average System

$$y[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x[n-k]$$

- Let $M_1=0$ (i.e. system is causal)

$$y[n] = \frac{1}{M_2 + 1} \sum_{k=0}^{M_2} x[n-k]$$

Eigenfunctions

Eigenfunction

- $x[n]=e^{j\omega n}$

- Eigenfunction: function when acted upon by a linear operator is scaled by an eigenvalue
  - $\boldsymbol{D}f = \lambda f$

Eigenvalue (frequency response)

- $x[n]=e^{j\omega n}$

$$y[n]=\sum_{k=-\infty}^{\infty} x[n-k]h[k]$$

$$=\sum_{k=-\infty}^{\infty} e^{j\omega(n-k)}h[k]$$

$$=e^{j\omega n}\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$

$$=H(e^{j\omega})e^{j\omega n}$$

$$\boxed{H(e^{j\omega})=\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}}$$

- Describes the change in amplitude and phase of signal at frequency ω
- Frequency response
- Complex value
  - Re and Im
  - Mag and Phase

DT Frequency Response

- $H(e^{j(\omega+2\pi)})$?

$$\boxed{H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k}}$$

DT Frequency Response

- $H(e^{j(\omega+2\pi)})$?

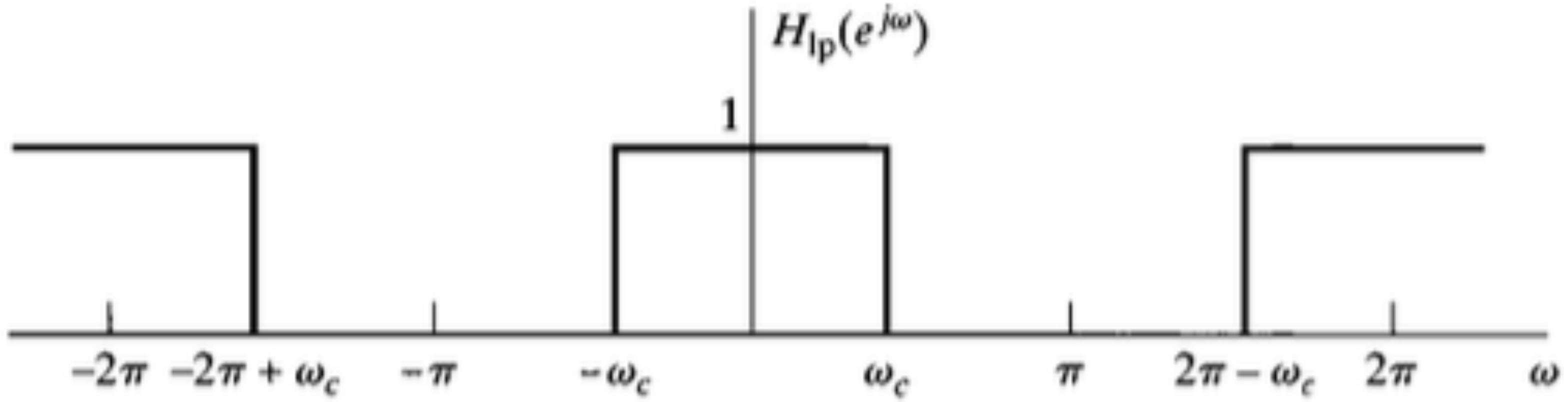
$$H\left(e^{j(\omega+2\pi)}\right) = \sum_{k=-\infty}^{\infty} h[k]e^{-j(\omega+2\pi)k}$$

$$= \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}e^{-j2\pi k}$$

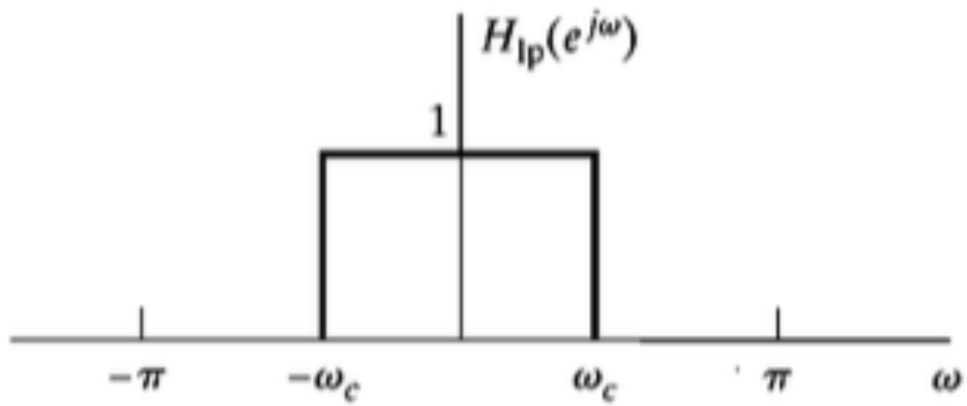
$$= \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$

$$= H(e^{j\omega})$$

Periodicity of Low Pass Freq Response



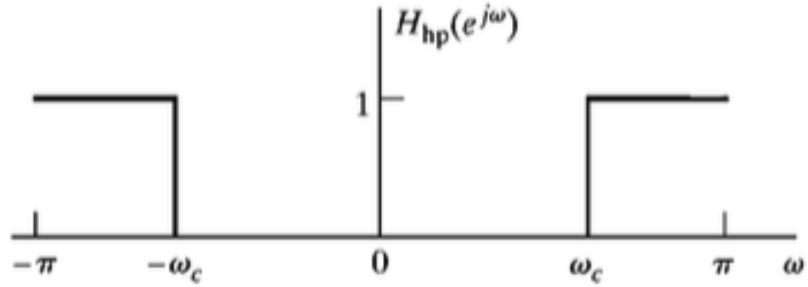
(a)



(b)

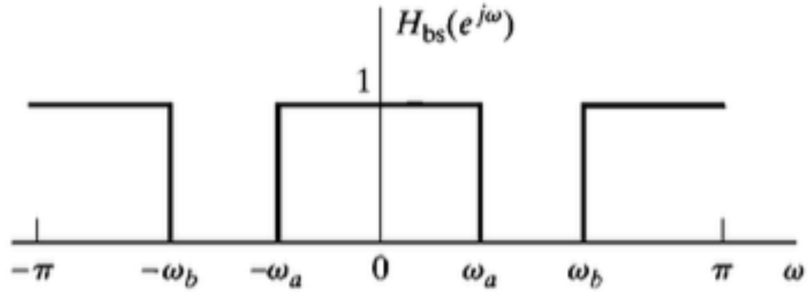
Other Filters

High-pass



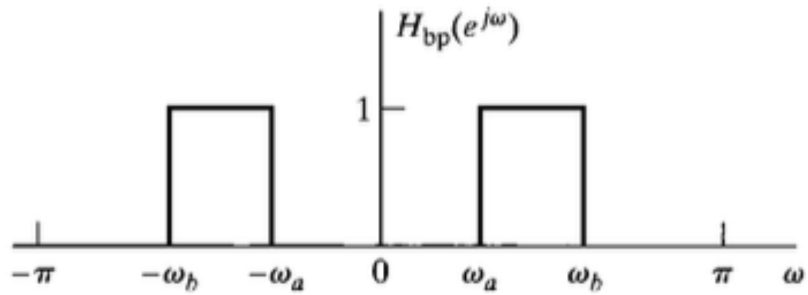
(a)

Band-stop



(b)

Band-pass



(c)

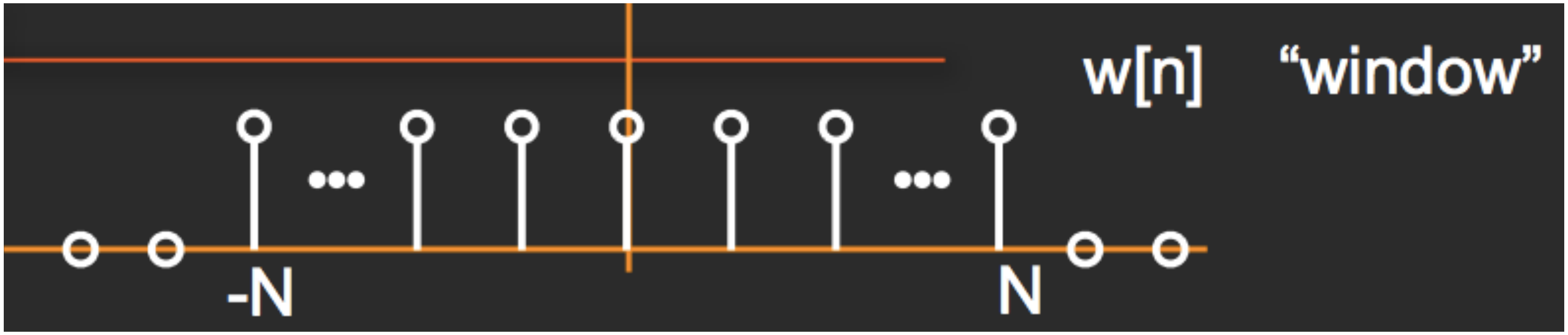
Discrete-Time Fourier Transform (DTFT)

DTFT Definition

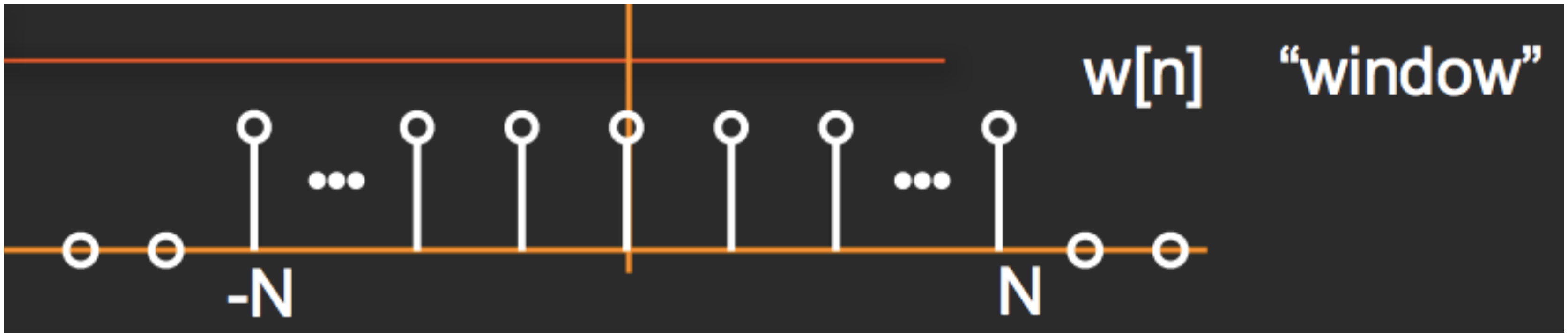
$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

Example: Window DTFT



Example: Window DTFT



$$W(e^{j\omega}) = \sum_{k=-\infty}^{\infty} w[k] e^{-j\omega k}$$

$$= \sum_{k=-N}^{N} e^{-j\omega k}$$

Example: Window DTFT

$$W(e^{j\omega}) = \sum_{k=-N}^{N} e^{-j\omega k}$$

$$= e^{j\omega N} + e^{j\omega(N-1)} + \ldots + e^{j\omega 0} + \ldots e^{-j\omega(N-1)} + e^{-j\omega N}$$

$$= e^{-j\omega N}(1 + e^{j\omega} + \ldots + e^{j\omega N} + \ldots + e^{j\omega(2N-1)} + e^{j\omega 2N})$$

Useful sum: $1 + p + p^2 + \ldots + p^M = \dfrac{1 - p^{M+1}}{1 - p}$

Example: Window DTFT

$$W(e^{j\omega}) = \sum_{k=-N}^{N} e^{-j\omega k}$$

$$= e^{j\omega N} + e^{j\omega(N-1)} + \ldots + e^{j\omega 0} + \ldots e^{-j\omega(N-1)} + e^{-j\omega N}$$

$$= e^{-j\omega N}(1 + e^{j\omega} + \ldots + e^{j\omega N} + \ldots + e^{j\omega(2N-1)} + e^{j\omega 2N})$$

Useful sum: $1 + p + p^2 + \ldots + p^M = \dfrac{1-p^{M+1}}{1-p}$

$$p = e^{j\omega} \qquad M = 2N$$

$$W(e^{j\omega}) = e^{-j\omega N}\frac{1-e^{j\omega(2N+1)}}{1-e^{j\omega}}$$

Example: Window DTFT

$$W(e^{j\omega}) = e^{-j\omega N} \frac{1 - e^{j\omega(2N+1)}}{1 - e^{j\omega}}$$

Example: Window DTFT

$$W(e^{j\omega}) = e^{-j\omega N}\frac{1-e^{j\omega(2N+1)}}{1-e^{j\omega}}$$

$$= \frac{e^{-j\omega N} - e^{j\omega(N+1)}}{1-e^{j\omega}}$$

$$= \frac{e^{-j\omega N} - e^{j\omega(N+1)}}{1-e^{j\omega}} \times \frac{e^{-j\omega/2}}{e^{-j\omega/2}}$$

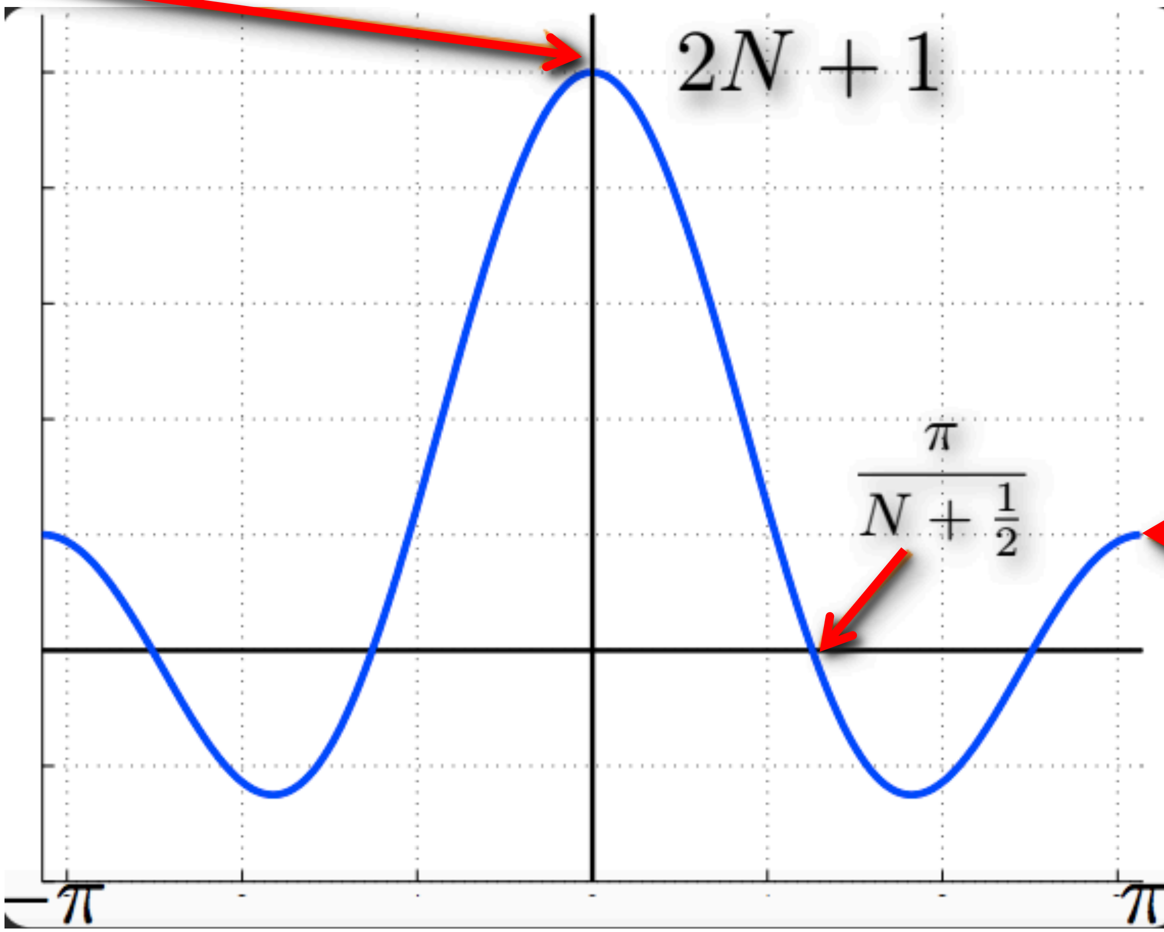
$$= \frac{e^{-j\omega(N+1/2)} - e^{j\omega(N+1/2)}}{e^{-j\omega/2} - e^{j\omega/2}} = \frac{\sin\left((N+1/2)\omega\right)}{\sin\left(\omega/2\right)}$$

Periodic sinc

Example: Window DTFT

$$W(e^{j\omega}) = \frac{\sin\left((N+1/2)\omega\right)}{\sin\left(\omega/2\right)}$$

Also, Σw[n]



=1 why?

Plot for N=2

Big Ideas

- LTI Systems are a special class of systems with significant signal processing applications
  - Can be characterized by the impulse response
- LTI System Properties
  - Causality and stability can be determined from impulse response
- Difference equations suggest implementation of systems
  - Give insight into complexity of system

Admin

- Complete Diagnostic Quiz by midnight tomorrow
  - Answers posted after
- HW 0: Brush up on background and Matlab tutorial
- HW 1 due 2/4
- First recitation posted
  - Basic Matlab usage
  - (in general recitations will get posted Th or F every week)