# ESE 5310: Digital Signal Processing

Lecture 5: February 6, 2024

z-Transform & Inverse z-Transform

# Lecture Outline

- z-Transform
  - Regions of convergence (ROC) & properties
  - z-Transform properties
- Inverse z-transform
  - Inspection
  - Partial fraction
  - Power series expansion
- z-transform of difference equations

# z-Transform and ROC

# z-Transform and ROC

- Define the **forward z-transform** of x[n] as

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

- The core "basis functions" of the z-transform are the complex exponentials $z^n$ with arbitrary $z \in C$; these are the eigenfunctions of LTI systems for infinite-length signals

DEFINITION

Given a time signal $x[n]$, the **region of convergence** (ROC) of its $z$-transform $X(z)$ is the set of $z \in \mathbb{C}$ such that $X(z)$ converges, that is, the set of $z \in \mathbb{C}$ such that $x[n]\, z^{-n}$ is absolutely summable

$$\sum_{n=-\infty}^{\infty} |x[n]\, z^{-n}| \ < \ \infty$$

# ROC Example 1

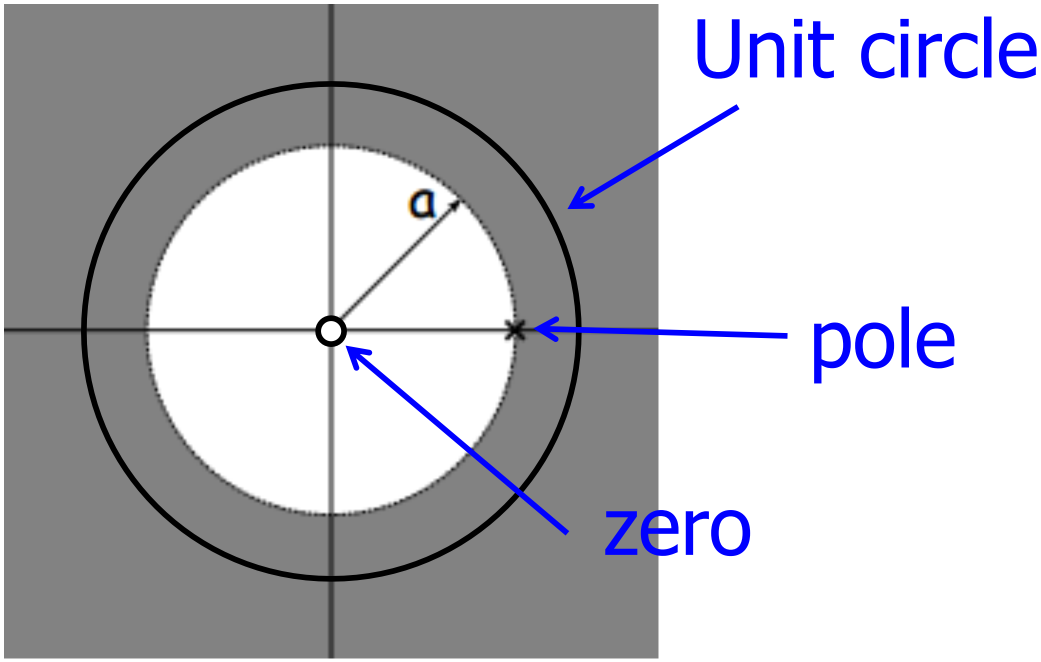
- What is the DTFT of $x_1[n]=a^n u[n]$?

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}, \qquad -\pi \le \omega < \pi$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega, \qquad -\infty \le n < \infty$$

$$X_1(z) = \frac{z}{z-a}$$

$$ROC = \{z: \ |z| > |a|\}$$



# ROC Example 2

- What is the z-transform of $x_2[n]$? ROC?

$$x_2[n] = \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n]$$

$$X_2(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{1 + \frac{1}{3}z^{-1}} \quad ROC = \left\{z: |z| > \frac{1}{2}\right\}$$

- Hint: $x_1[n] = a^n u[n] \xleftrightarrow{Z} \dfrac{1}{1 - az^{-1}} \quad ROC = \{z: \ |z| > |a|\}$

# ROC Example 3

- What is the z-transform of $x_3[n]$?  ROC?

$$x_3[n] = -a^n u[-n-1]$$

$$\boxed{X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}}$$

Left-sided sequence

# ROC Example 3

- What is the z-transform of $x_3[n]$? ROC?

$$x_3[n] = -a^n u[-n-1]$$

$$\boxed{X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}}$$

- The z-transform without ROC does not uniquely define a sequence!

# ROC Example 4

- What is the z-transform of $x_4[n]$? ROC?

$$x_4[n] = -\left(\frac{1}{2}\right)^n u[-n-1] + \left(-\frac{1}{3}\right)^n u[n]$$

# ROC Example 4

- What is the z-transform of $x_4[n]$? ROC?

$$x_4[n] = -\left(\frac{1}{2}\right)^n u[-n-1] + \left(-\frac{1}{3}\right)^n u[n]$$

two-sided sequence

- Hint:

$$x_1[n] = a^n u[n] \xleftrightarrow{Z} \frac{1}{1-az^{-1}}, \qquad ROC = \{z: \ |z| > |a|\}$$

$$x_3[n] = -a^n u[-n-1] \xleftrightarrow{Z} \frac{1}{1-az^{-1}}, \qquad ROC = \{z: \ |z| < |a|\}$$

# ROC Example 5

- What is the z-transform of $x_5[n]$? ROC?

$$x_5[n] = \left(\frac{1}{2}\right)^n u[n] - \left(-\frac{1}{3}\right)^n u[-n-1]$$

two-sided sequence

$$x_1[n] = a^n u[n] \xleftrightarrow{Z} \frac{1}{1-az^{-1}}, \qquad ROC = \{z : \quad |z| > |a|\}$$

$$x_3[n] = -a^n u[-n-1] \xleftrightarrow{Z} \frac{1}{1-az^{-1}}, \qquad ROC = \{z : \quad |z| < |a|\}$$

# ROC Example 6

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

- What is the z-transform of $x_6[n]$? ROC?

$$x_6[n] = a^n u[n] u[-n + M - 1]$$

# ROC Example 6

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

- What is the z-transform of $x_6[n]$? ROC?

$$x_6[n] = a^n u[n] u[-n+M-1]$$

finite length sequence

# ROC Example 6

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

- What is the z-transform of $x_6[n]$? ROC?

$$x_6[n] = a^n u[n] u[-n+M-1]$$

Zero cancels pole

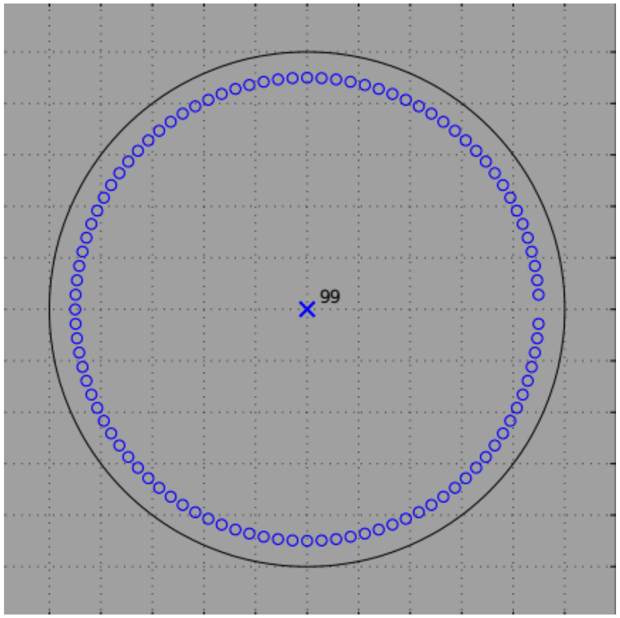
$$X_6(z) = \frac{1 - a^M z^{-M}}{1 - a z^{-1}}$$

$$= \prod_{k=1}^{M-1} (1 - a e^{j2\pi k/M} z^{-1})$$

# ROC Example 6

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

- What is the z-transform of $x_6[n]$? ROC?




$$X_6(z) = \frac{1 - a^M z^{-M}}{1 - az^{-1}}$$

M=100

Zero cancels pole

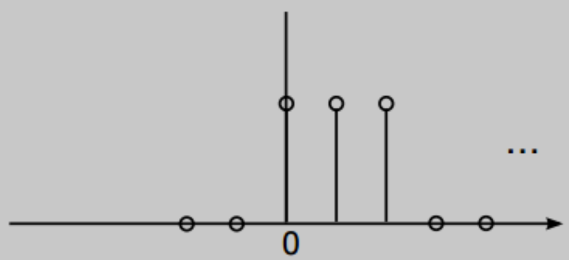
$$= \prod_{k=1}^{M-1} \left(1 - a e^{j2\pi k/M} z^{-1}\right)$$

# Properties of ROC

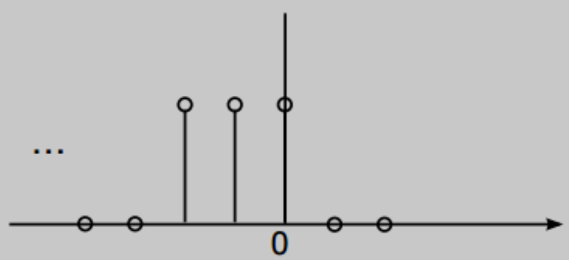
- For right-sided sequences: ROC extends outward from the outermost pole to infinity
  - Examples 1,2
- For left-sided: inwards from inner most pole to zero
  - Example 3
- For two-sided, ROC is a ring - or does not exist
  - Examples 4,5

# Properties of ROC

- For finite duration sequences, ROC is the entire z-plane, except possibly $z=0$, $z=\infty$ (Example 6)




$$X(z) = 1 + z^{-1} + z^{-2} \quad \text{ROC excludes } z = 0$$




$$X(z) = 1 + z^{1} + z^{2} \quad \text{ROC excludes } z = \infty$$

# Formal Properties of the ROC

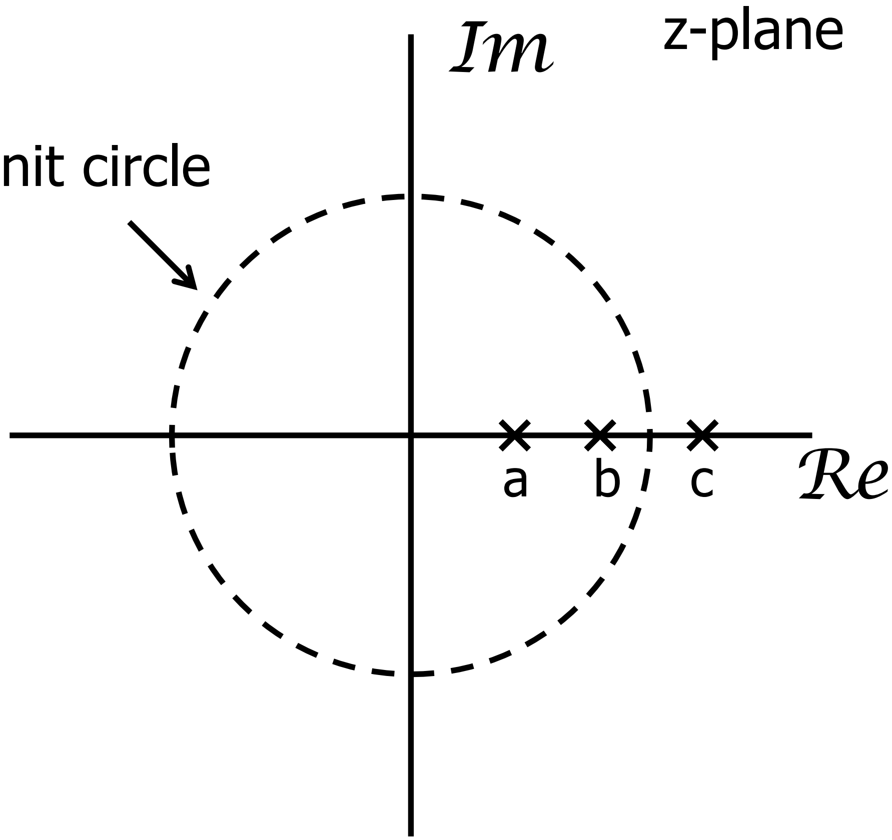
- PROPERTY 1:
  - The ROC will either be of the form $0 < r_R < |z|$, or $|z| < r_L < \infty$, or, in general the annulus, i.e., $0 < r_R < |z| < r_L < \infty$.
- PROPERTY 2:
  - The Fourier transform of x[n] converges absolutely if and only if the ROC of the z-transform of x[n] includes the unit circle.
- PROPERTY 3:
  - The ROC cannot contain any poles.
- PROPERTY 4:
  - If x[n] is *a finite-duration sequence*, i.e., a sequence that is zero except in a finite interval $-\infty < N_1 < n < N_2 < \infty$, then the ROC is the entire z-plane, except possibly $z = 0$ or $z = \infty$.

# Formal Properties of the ROC

- PROPERTY 5:
  - If x[n] is a *right-sided sequence*, the ROC extends outward from the *outermost* finite pole in X(z) to (and possibly including) $z = \infty$.
- PROPERTY 6:
  - If x[n] is a *left-sided sequence*, the ROC extends inward from the *innermost* nonzero pole in X(z) to (and possibly including) z=0.
- PROPERTY *7:*
  - A *two-sided sequence* is an infinite-duration sequence that is neither right sided nor left sided. If x[n] is a two-sided sequence, the ROC will consist of a ring in the z-plane, bounded on the interior and exterior by a pole and, consistent with Property 3, not containing any poles.
- PROPERTY 8:
  - The ROC must be a connected region.

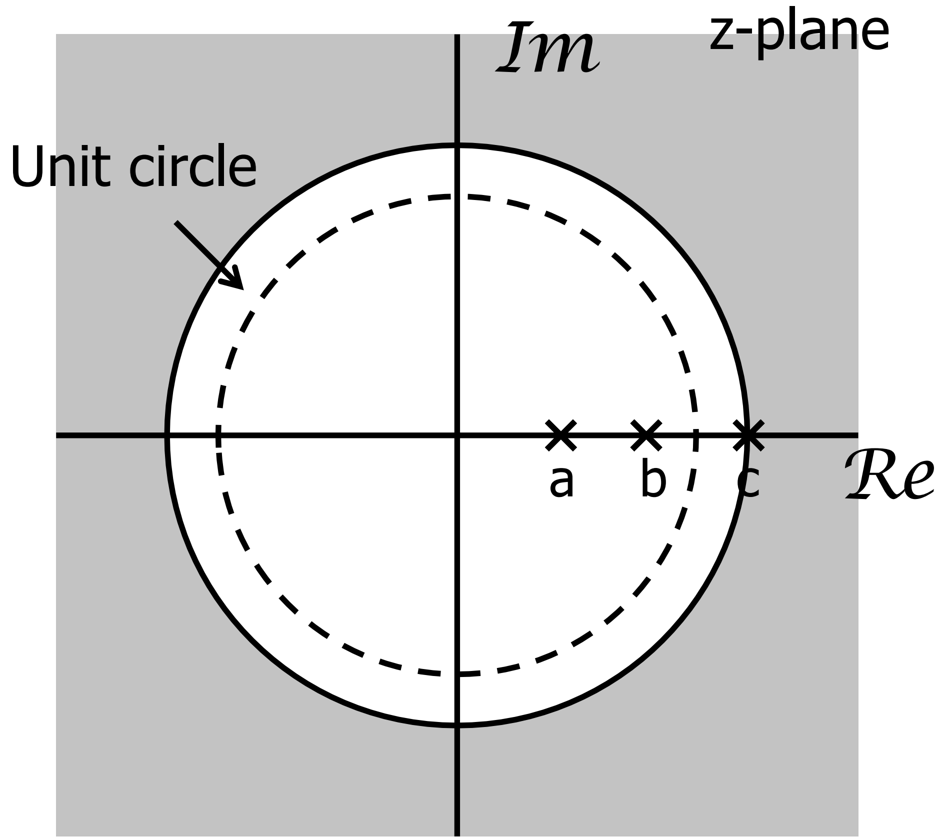
# Example: ROC from Pole-Zero Plot

- How many possible ROCs?



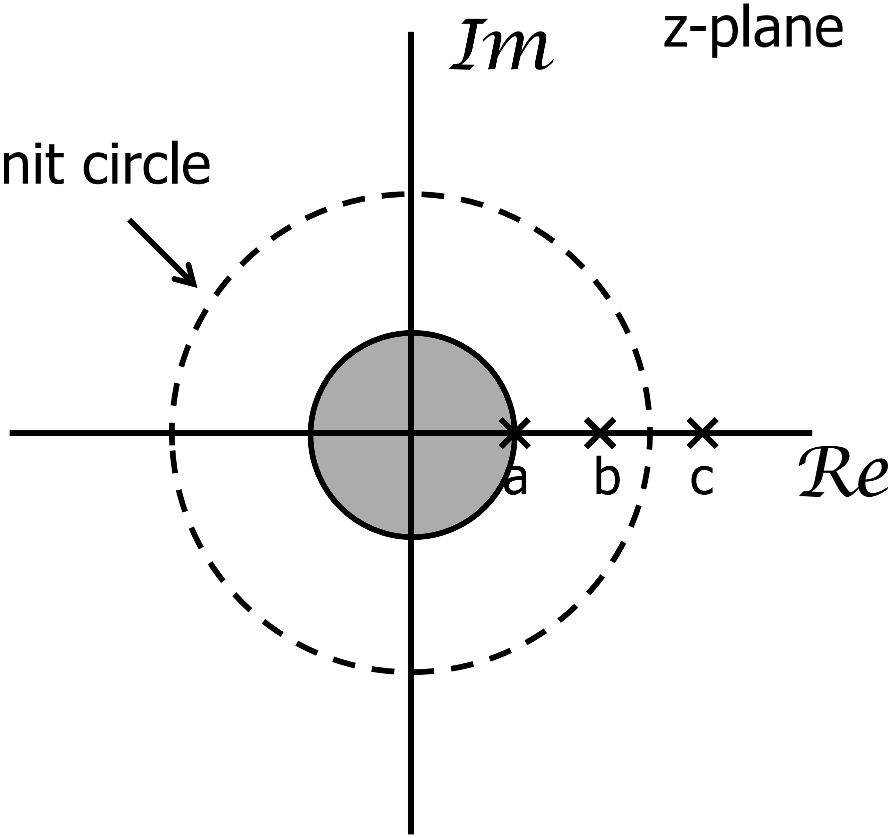
# Example: ROC from Pole-Zero Plot

## ROC 1: right-sided



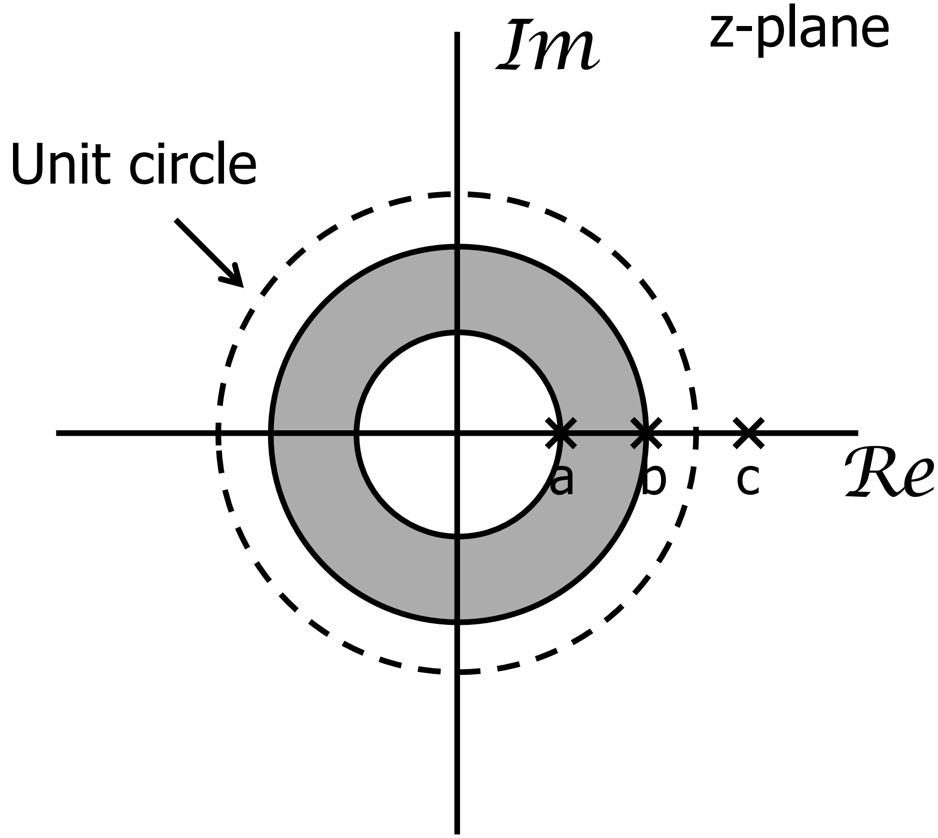
# Example: ROC from Pole-Zero Plot

## ROC 2: left-sided



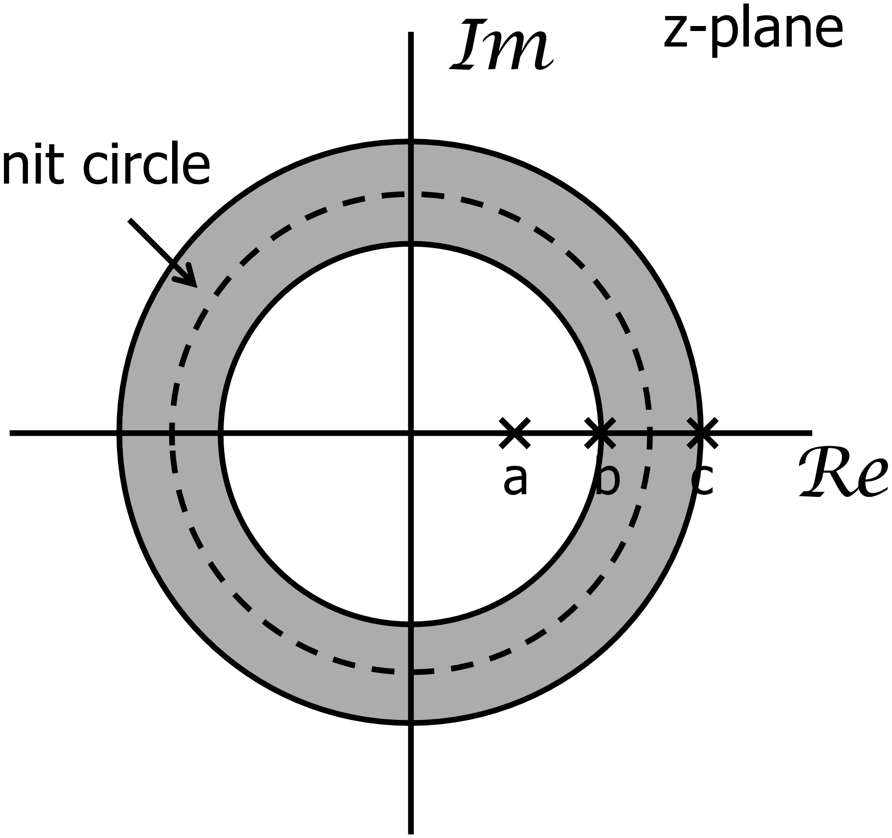
# Example: ROC from Pole-Zero Plot

## ROC 3: two-sided



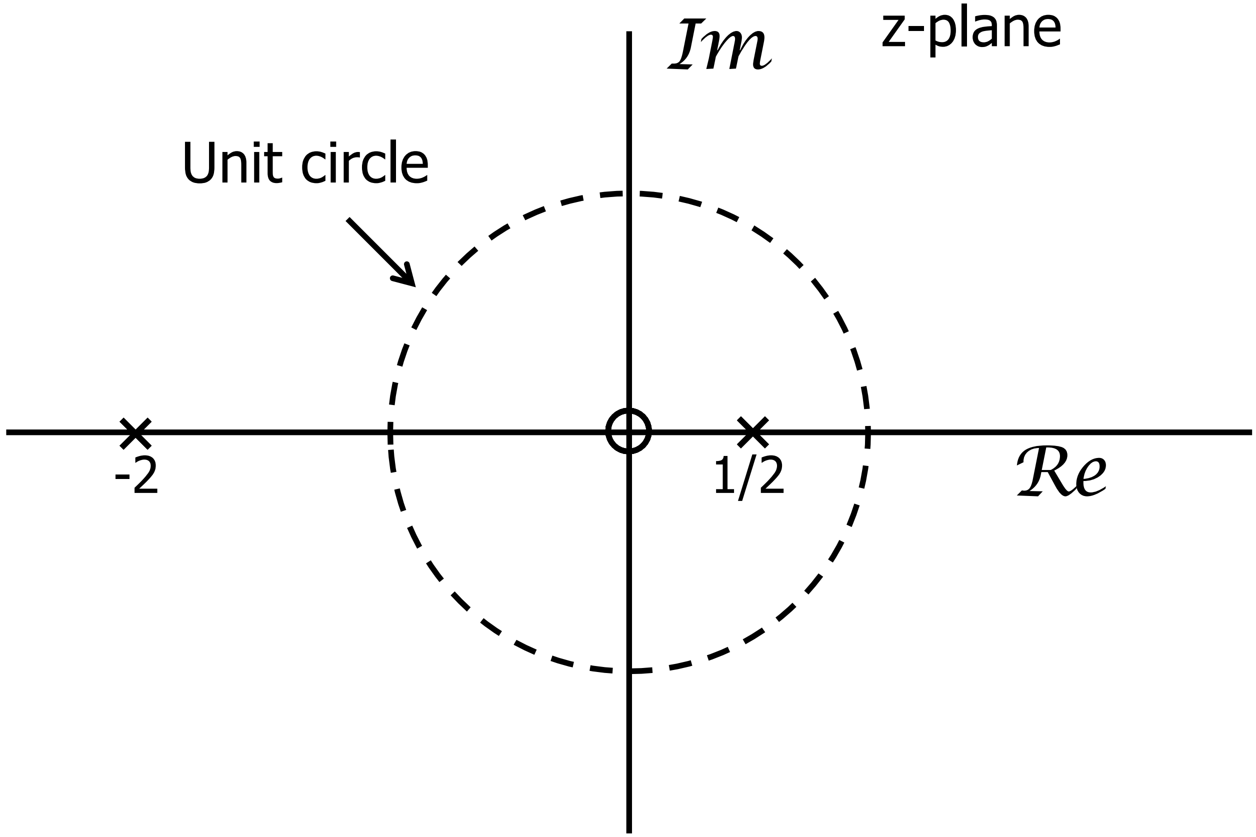
# Example: ROC from Pole-Zero Plot

## ROC 4: two-sided



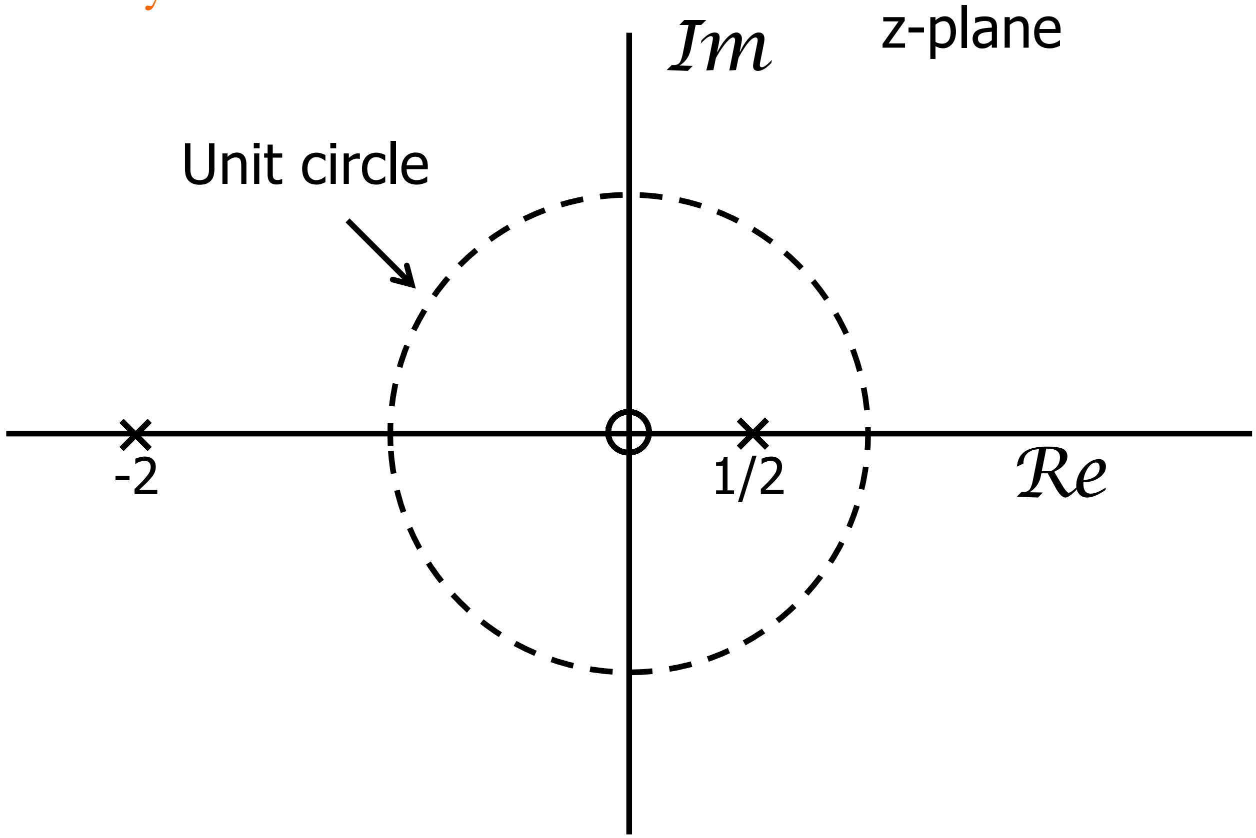
# Example: Pole-Zero Plot

- H(z) for an LTI System
  - How many possible ROCs?



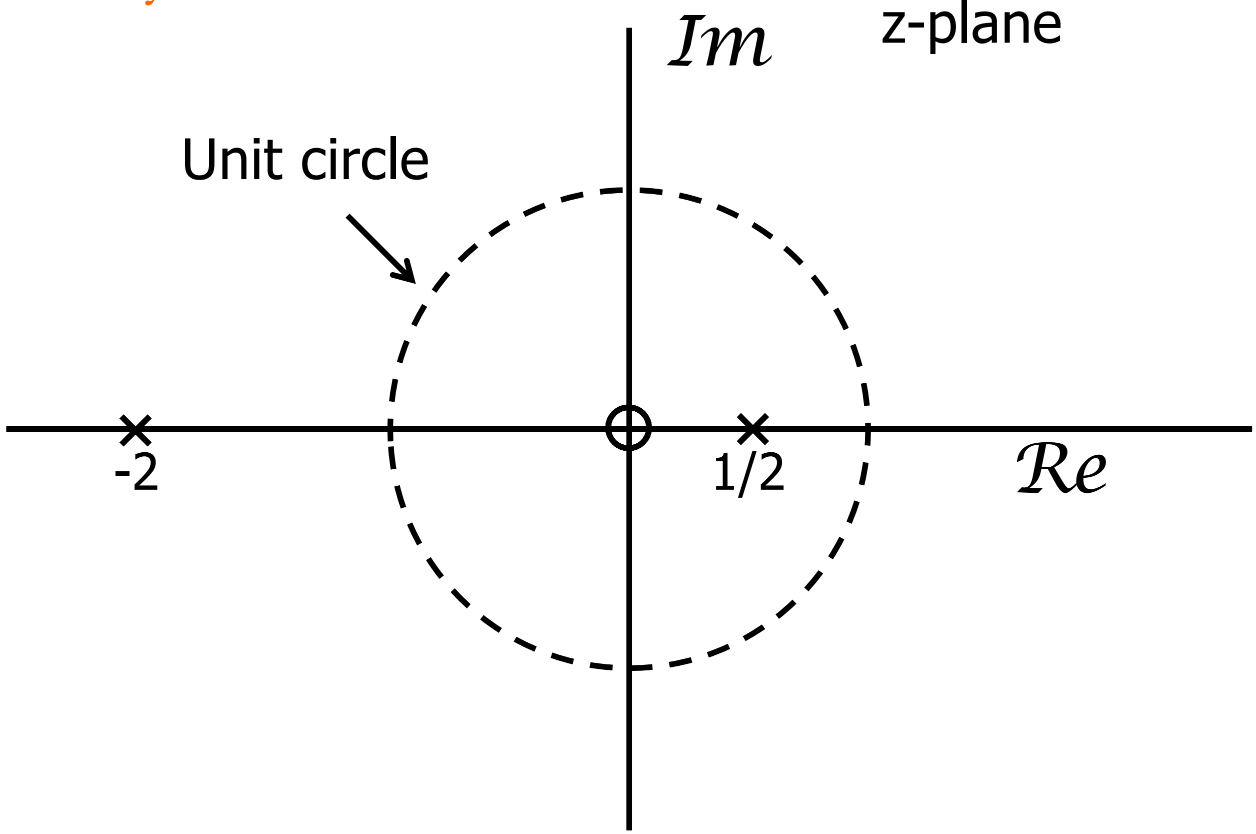
# Example: Pole-Zero Plot

- H(z) for an LTI System
  - How many possible ROCs?
  - What if system is causal?



# Example: Pole-Zero Plot

- H(z) for an LTI System
  - How many possible ROCs?
  - What if system is stable?



# Region of Convergence (ROC)

DEFINITION

Given a time signal $x[n]$, the **region of convergence** (ROC) of its $z$-transform $X(z)$ is the set of $z \in \mathbb{C}$ such that $X(z)$ converges, that is, the set of $z \in \mathbb{C}$ such that $x[n]\,z^{-n}$ is absolutely summable

$$\sum_{n=-\infty}^{\infty} |x[n]\,z^{-n}| < \infty$$

# BIBO Stability Revisited

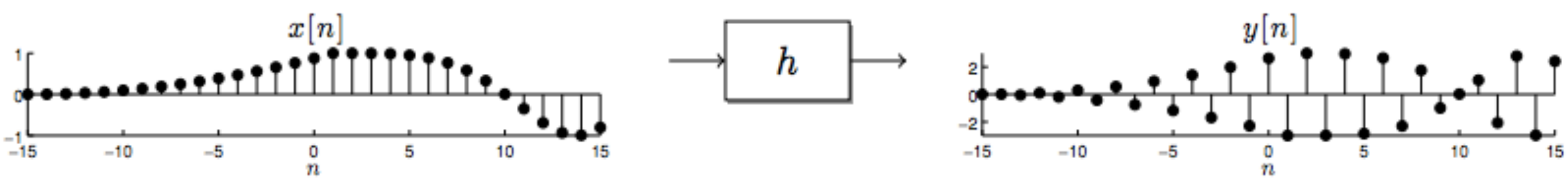
$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k}$$

DEFINITION

An LTI system is **bounded-input bounded-output (**
input $x$ always produces a bounded output $y$

bounded $x$ $\longrightarrow$ $\boxed{h}$ $\longrightarrow$ bounded $y$

- Bounded input and output means $\|x\|_\infty < \infty$ and $\|y\|_\infty < \infty$

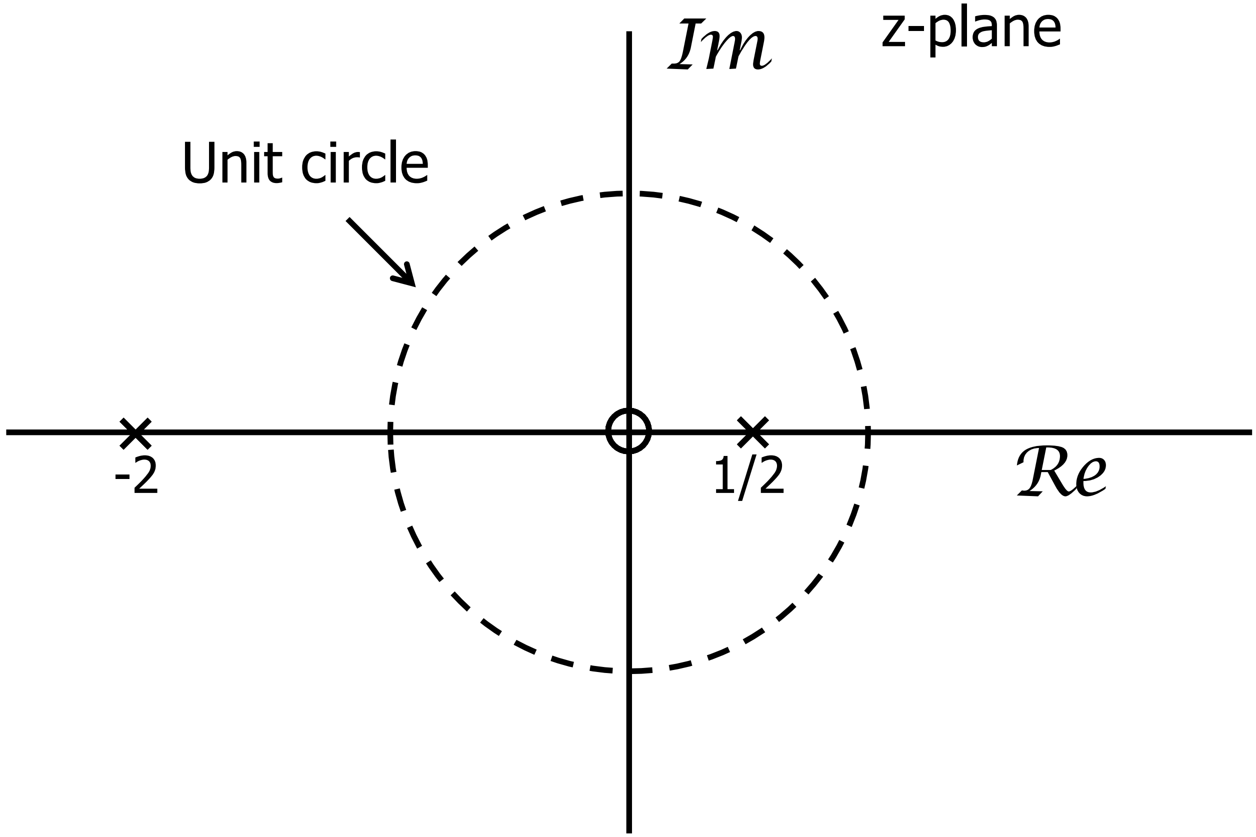



- **Fact:** An LTI system with impulse response $h$ is BIBO stable <u>if and only if</u>

$$\|h\|_1 = \sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

# Example: Pole-Zero Plot

- H(z) for an LTI System
  - How many possible ROCs?
  - Stable and causal?



# z-transform Pairs

**TABLE 3.1** SOME COMMON $z$-TRANSFORM PAIRS

| Sequence | Transform | ROC |
|---|---|---|
| 1. $\delta[n]$ | $1$ | All $z$ |
| 2. $u[n]$ | $\dfrac{1}{1-z^{-1}}$ | $\lvert z\rvert > 1$ |
| 3. $-u[-n-1]$ | $\dfrac{1}{1-z^{-1}}$ | $\lvert z\rvert < 1$ |
| 4. $\delta[n-m]$ | $z^{-m}$ | All $z$ except 0 (if $m > 0$) or $\infty$ (if $m < 0$) |
| 5. $a^n u[n]$ | $\dfrac{1}{1-az^{-1}}$ | $\lvert z\rvert > \lvert a\rvert$ |
| 6. $-a^n u[-n-1]$ | $\dfrac{1}{1-az^{-1}}$ | $\lvert z\rvert < \lvert a\rvert$ |
| 7. $na^n u[n]$ | $\dfrac{az^{-1}}{(1-az^{-1})^2}$ | $\lvert z\rvert > \lvert a\rvert$ |
| 8. $-na^n u[-n-1]$ | $\dfrac{az^{-1}}{(1-az^{-1})^2}$ | $\lvert z\rvert < \lvert a\rvert$ |
| 9. $\cos(\omega_0 n)u[n]$ | $\dfrac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$ | $\lvert z\rvert > 1$ |
| 10. $\sin(\omega_0 n)u[n]$ | $\dfrac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$ | $\lvert z\rvert > 1$ |
| 11. $r^n\cos(\omega_0 n)u[n]$ | $\dfrac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$ | $\lvert z\rvert > r$ |
| 12. $r^n\sin(\omega_0 n)u[n]$ | $\dfrac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$ | $\lvert z\rvert > r$ |
| 13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$ | $\dfrac{1-a^Nz^{-N}}{1-az^{-1}}$ | $\lvert z\rvert > 0$ |

# Properties of z-Transform

- Linearity:

$$ax_1[n]+bx_2[n] \leftrightarrow aX_1(z)+bX_2(z)$$

- Time shifting:

$$x[n] \leftrightarrow X(z)$$

$$x[n-n_d] \leftrightarrow z^{-n_d}X(z)$$

- Multiplication by exponential sequence

$$x[n] \leftrightarrow X(z)$$

$$z_0^n x[n] \leftrightarrow X\left(\frac{z}{z_0}\right)$$

# Properties of z-Transform

- Time Reversal:

$$x[n] \leftrightarrow X(z)$$

$$x[-n] \leftrightarrow X(z^{-1})$$

- Differentiation of transform:

$$x[n] \leftrightarrow X(z)$$

$$nx[n] \leftrightarrow -z\frac{dX(z)}{dz}$$

- Convolution in Time:

$$y[n] = x[n] * h[n]$$

$$Y(z) = X(z)H(z)$$

$ROC_Y$ at least $ROC_X \wedge ROC_H$

# z-transform Properties

**TABLE 3.2** SOME $z$-TRANSFORM PROPERTIES

| Property Number | Section Reference | Sequence | Transform | ROC |
|---|---|---|---|---|
| | | $x[n]$ | $X(z)$ | $R_x$ |
| | | $x_1[n]$ | $X_1(z)$ | $R_{x_1}$ |
| | | $x_2[n]$ | $X_2(z)$ | $R_{x_2}$ |
| 1 | 3.4.1 | $ax_1[n] + bx_2[n]$ | $aX_1(z) + bX_2(z)$ | Contains $R_{x_1} \cap R_{x_2}$ |
| 2 | 3.4.2 | $x[n - n_0]$ | $z^{-n_0}X(z)$ | $R_x$, except for the possible addition or deletion of the origin or $\infty$ |
| 3 | 3.4.3 | $z_0^n x[n]$ | $X(z/z_0)$ | $\lvert z_0 \rvert R_x$ |
| 4 | 3.4.4 | $nx[n]$ | $-z\dfrac{dX(z)}{dz}$ | $R_x$ |
| 5 | 3.4.5 | $x^*[n]$ | $X^*(z^*)$ | $R_x$ |
| 6 | | $\mathcal{R}e\{x[n]\}$ | $\frac{1}{2}[X(z) + X^*(z^*)]$ | Contains $R_x$ |
| 7 | | $\mathcal{I}m\{x[n]\}$ | $\frac{1}{2j}[X(z) - X^*(z^*)]$ | Contains $R_x$ |
| 8 | 3.4.6 | $x^*[-n]$ | $X^*(1/z^*)$ | $1/R_x$ |
| 9 | 3.4.7 | $x_1[n] * x_2[n]$ | $X_1(z)X_2(z)$ | Contains $R_{x_1} \cap R_{x_2}$ |

# Inverse z-Transform

# Inverse z-Transform

- Recall the inverse DTFT

$$x[n] = \int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, \frac{d\omega}{2\pi}$$

# Inverse z-Transform

- Recall the inverse DTFT

$$x[n] = \int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, \frac{d\omega}{2\pi}$$

- There is a similar formula for the inverse z-transform using a contour integral

$$x[n] = \oint_C X(z)\, z^n\, \frac{dz}{j2\pi z}$$

- Contour integrals are fun but beyond the scope of this course!

# Inverse z-Transform

- Ways to avoid it:
  - Inspection (known transforms)
  - Properties of the z-transform
  - Partial fraction expansion
  - Power series expansion
    - w/ long division

# Z-Transform Pairs

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# Z-Transform Properties

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| 3 | 3.4.3 | $z_0^n x[n]$ | $X(z/z_0)$ | $\|z_0\|R_x$ |
| 4 | 3.4.4 | $nx[n]$ | $-z\dfrac{dX(z)}{dz}$ | $R_x$ |
| 5 | 3.4.5 | $x^*[n]$ | $X^*(z^*)$ | $R_x$ |
| 6 | | $\mathcal{R}e\{x[n]\}$ | $\frac{1}{2}[X(z)+X^*(z^*)]$ | Contains $R_x$ |
| 7 | | $\mathcal{I}m\{x[n]\}$ | $\frac{1}{2j}[X(z)-X^*(z^*)]$ | Contains $R_x$ |
| 8 | 3.4.6 | $x^*[-n]$ | $X^*(1/z^*)$ | $1/R_x$ |
| 9 | 3.4.7 | $x_1[n]*x_2[n]$ | $X_1(z)X_2(z)$ | Contains $R_{x_1} \cap R_{x_2}$ |

# Partial Fraction Expansion

- Let

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{z^N \sum_{k=0}^{M} b_k z^{M-k}}{z^M \sum_{k=0}^{N} a_k z^{N-k}}$$

- M zeros and N poles at nonzero locations

# Partial Fraction Expansion

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{z^N \sum_{k=0}^{M} b_k z^{M-k}}{z^M \sum_{k=0}^{N} a_k z^{N-k}}$$

- Factored numerator/denominator

$$X(z) = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M} (1 - c_k z^{-1})}{\prod_{k=1}^{N} (1 - d_k z^{-1})}$$

# Partial Fraction Expansion

- If M<N and the poles are 1st order

$$X(z) = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M} (1 - c_k z^{-1})}{\prod_{k=1}^{N} (1 - d_k z^{-1})} = \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}$$

- where

$$A_k = (1 - d_k z^{-1}) X(z) \Big|_{z = d_k}$$

# Example: 2nd-Order z-Transform

- 2nd-order = two poles

$$X(z) = \frac{1}{\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)}, \qquad ROC = \left\{ z : \ \frac{1}{2} < |z| \right\}$$

# Example: 2nd-Order z-Transform

- $2^{nd}$-order = two poles

$$X(z) = \frac{1}{\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)}, \qquad ROC = \left\{ z : \ \frac{1}{2} < |z| \right\}$$

$$X(z) = \frac{A_1}{\left(1 - \frac{1}{4}z^{-1}\right)} + \frac{A_2}{\left(1 - \frac{1}{2}z^{-1}\right)}$$

# Example: 2nd-Order z-Transform

- 2nd-order = two poles

$$A_k = (1 - d_k z^{-1}) X(z)\Big|_{z=d_k}$$

$$X(z) = \frac{1}{\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)}, \qquad ROC = \left\{z : \frac{1}{2} < |z|\right\}$$

$$A_1 = (1 - \frac{1}{4}z^{-1}) X(z)\Big|_{z=1/4} = \frac{(1 - \frac{1}{4}z^{-1})}{(1 - \frac{1}{4}z^{-1})(1 - \frac{1}{2}z^{-1})}\Bigg|_{z=1/4} = -1$$

$$A_2 = (1 - \frac{1}{2}z^{-1}) X(z)\Big|_{z=1/2} = \frac{(1 - \frac{1}{2}z^{-1})}{(1 - \frac{1}{4}z^{-1})(1 - \frac{1}{2}z^{-1})}\Bigg|_{z=1/2} = 2$$

# Example: 2nd-Order z-Transform

- 2nd-order = two poles

$$X(z)=\frac{-1}{\left(1-\frac{1}{4}z^{-1}\right)}+\frac{2}{\left(1-\frac{1}{2}z^{-1}\right)}, \qquad ROC=\left\{z:\ \frac{1}{2}<|z|\right\}$$

# Example: 2nd-Order z-Transform

- 2nd-order = two poles

Right sided

$$X(z) = \frac{-1}{\left(1 - \frac{1}{4}z^{-1}\right)} + \frac{2}{\left(1 - \frac{1}{2}z^{-1}\right)}, \qquad ROC = \left\{ z : \frac{1}{2} < |z| \right\}$$

| | | |
|---|---|---|
| 5. $a^n u[n]$ | $\frac{1}{1 - az^{-1}}$ | $\lvert z \rvert > \lvert a \rvert$ |

# Example: 2nd-Order z-Transform

- 2nd-order = two poles

Right sided

$$X(z)=\frac{-1}{\left(1-\frac{1}{4}z^{-1}\right)}+\frac{2}{\left(1-\frac{1}{2}z^{-1}\right)}, \qquad ROC=\left\{z:\ \frac{1}{2}<|z|\right\}$$

5. $a^n u[n]$ $\qquad \frac{1}{1-az^{-1}} \qquad |z|>|a|$

$$x[n]=-\left(\frac{1}{4}\right)^n u[n]+2\left(\frac{1}{2}\right)^n u[n]$$

# Partial Fraction Expansion

- If M≥N and the poles are 1st order

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}$$

- Where $B_k$ is found by long division and

$$A_k = (1 - d_k z^{-1}) X(z) \Big|_{z=d_k}$$

# Example: Partial Fractions

- M=N=2 and poles are first order

$$X(z)=\frac{1+2z^{-1}+z^{-2}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}, \qquad ROC=\left\{z:\ 1<|z|\right\}$$

# Example: Partial Fractions

- M=N=2 and poles are first order

$$X(z)=\frac{1+2z^{-1}+z^{-2}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}, \qquad ROC=\{z:\ 1<|z|\}$$

$$=\frac{1+2z^{-1}+z^{-2}}{(1-\frac{1}{2}z^{-1})(1-z^{-1})}$$

$$X(z)=B_0+\frac{A_1}{1-\frac{1}{2}z^{-1}}+\frac{A_2}{1-z^{-1}}$$

# Example: Partial Fractions

- M=N=2 and poles are first order

$$X(z) = B_0 + \frac{A_1}{1-\frac{1}{2}z^{-1}} + \frac{A_2}{1-z^{-1}}, \qquad ROC = \left\{ z:\ 1 < |z| \right\}$$

$$\begin{array}{r}
2 \\
\frac{1}{2}z^{-2} - \frac{3}{2}z^{-1} + 1 \enclose{longdiv}{z^{-2} + 2z^{-1} + 1} \\
\underline{z^{-2} - 3z^{-1} + 2} \\
5z^{-1} - 1
\end{array}$$

$$X(z) = 2 + \frac{-1 + 5z^{-1}}{(1-\frac{1}{2}z^{-1})(1-z^{-1})}$$

# Example: Partial Fractions

- M=N=2 and poles are first order

$$X(z) = B_0 + \frac{A_1}{1-\frac{1}{2}z^{-1}} + \frac{A_2}{1-z^{-1}}, \qquad ROC = \left\{z:\ 1<|z|\right\}$$

$$\frac{-1+5z^{-1}}{(1-\frac{1}{2}z^{-1})(1-z^{-1})} = \frac{A_1}{1-\frac{1}{2}z^{-1}} + \frac{A_2}{1-z^{-1}}$$

# Example: Partial Fractions

- M=N=2 and poles are first order

$$X(z) = 2 - \frac{9}{1 - \frac{1}{2}z^{-1}} + \frac{8}{1 - z^{-1}}, \qquad ROC = \{z:\ 1 < |z|\}$$

$$x[n] = 2\delta[n] - 9\left(\frac{1}{2}\right)^n u[n] + 8u[n]$$

# Power Series Expansion

- Expansion of the z-transform definition

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= \cdots + x[-2]z^{2} + x[-1]z + x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots$$

# Example: Finite-Length Sequence

- Poles and zeros?

$$X(z) = z^2\left(1 - \frac{1}{2}z^{-1}\right)(1 + z^{-1})(1 - z^{-1})$$

# Example: Finite-Length Sequence

- Poles and zeros?

$$X(z) = z^2\left(1-\frac{1}{2}z^{-1}\right)(1+z^{-1})(1-z^{-1})$$

$$= z^2 - \frac{1}{2}z - 1 + \frac{1}{2}z^{-1}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

$$= \cdots + x[-2]z^2 + x[-1]z + x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots$$

# Example: Finite-Length Sequence

- Poles and zeros?

$$X(z) = z^2\left(1-\frac{1}{2}z^{-1}\right)(1+z^{-1})(1-z^{-1})$$

$$= z^2 - \frac{1}{2}z - 1 + \frac{1}{2}z^{-1}$$

$$x[n] = \begin{cases} 1, & n=-2 \\ -1/2, & n=-1 \\ -1, & n=0 \\ 1/2, & n=1 \\ 0, & else \end{cases} = \delta[n+2] - \frac{1}{2}\delta[n+1] - \delta[n] + \frac{1}{2}\delta[n-1]$$

# Example: Finite-Length Sequence

- Poles and zeros?

$$X(z) = z^2\left(1-\frac{1}{2}z^{-1}\right)(1+z^{-1})(1-z^{-1})$$

$$= z^2 - \frac{1}{2}z - 1 + \frac{1}{2}z^{-1}$$

$$x[n] = \begin{cases} 1, & n=-2 \\ -1/2, & n=-1 \\ -1, & n=0 \\ 1/2, & n=1 \\ 0, & else \end{cases} = \delta[n+2] - \frac{1}{2}\delta[n+1] - \delta[n] + \frac{1}{2}\delta[n-1]$$

4. $\delta[n-m]$ $\quad z^{-m}$

# Power Series Expansion w/ Long Division

$$X(z) = \frac{1}{1 - az^{-1}}, \qquad |z| > |a|$$

# Reminder: Difference Equations

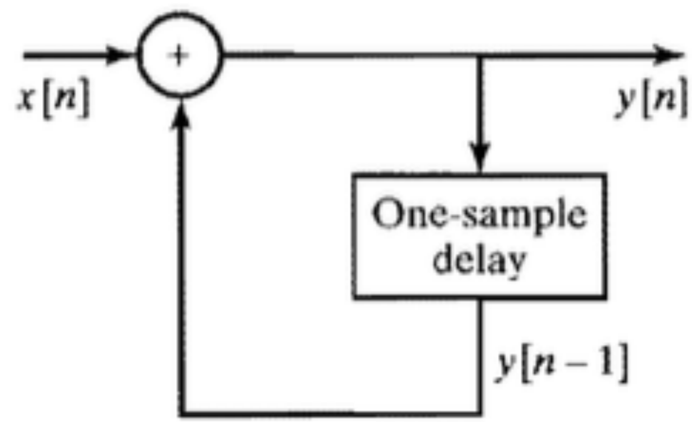
- Accumulator example

$$y[n]=\sum_{k=-\infty}^{n}x[k]$$

$$y[n]=x[n]+\sum_{k=-\infty}^{n-1}x[k]$$

$$y[n]=x[n]+y[n-1]$$

$$\boxed{y[n]-y[n-1]=x[n]}$$




$$\boxed{\sum_{k=0}^{N}a_k y[n-k]=\sum_{m=0}^{M}b_m x[n-m]}$$

# Difference Equation to z-Transform

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

$$y[n] = -\sum_{k=1}^{N} \left(\frac{a_k}{a_0}\right) y[n-k] + \sum_{m=0}^{M} \left(\frac{b_m}{a_0}\right) x[n-m]$$

- Difference equations of this form behave as causal LTI systems
  - when the input is zero prior to n=0
  - Initial rest equations are imposed prior to the time when input becomes nonzero
    - i.e y[-N]=y[-N+1]=…=y[-1]=0

# Difference Equation to z-Transform

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

$$\sum_{k=0}^{N} \left(\frac{a_k}{a_0}\right) z^{-k} Y(z) = \sum_{m=0}^{M} \left(\frac{b_m}{a_0}\right) z^{-m} X(z)$$

# Difference Equation to z-Transform

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

$$\sum_{k=0}^{N} \left(\frac{a_k}{a_0}\right) z^{-k} Y(z) = \sum_{m=0}^{M} \left(\frac{b_m}{a_0}\right) z^{-m} X(z)$$

$$\Rightarrow Y(z) = \frac{\sum_{m=0}^{M} (b_m) z^{-m}}{\sum_{k=0}^{N} (a_k) z^{-k}} X(z)$$

$$\boxed{H(z) = \frac{\sum_{m=0}^{M} (b_k) z^{-k}}{\sum_{k=0}^{N} (a_k) z^{-k}}}$$

# Example: 1st-Order System

$$y[n] = ay[n-1] + x[n]$$

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

$$H(z) = \frac{\sum_{m=0}^{M} (b_k) z^{-k}}{\sum_{k=0}^{N} (a_k) z^{-k}}$$

# Example: $1^{st}$-Order System

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

$$y[n] = ay[n-1] + x[n]$$

$$H(z) = \frac{\sum_{m=0}^{M} (b_k) z^{-k}}{\sum_{k=0}^{N} (a_k) z^{-k}}$$

$$H(z) = \frac{\overbrace{1}^{b_0}}{\underbrace{1}_{a_0} - \underbrace{a}_{a_1} z^{-1}}$$

# Example: 1st-Order System

$$y[n] = ay[n-1] + x[n]$$

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

$$H(z) = \frac{\overset{b_0}{1}}{\underset{a_0}{1} - \underset{a_1}{a}z^{-1}}$$

$$H(z) = \frac{\sum_{m=0}^{M} (b_k) z^{-k}}{\sum_{k=0}^{N} (a_k) z^{-k}}$$

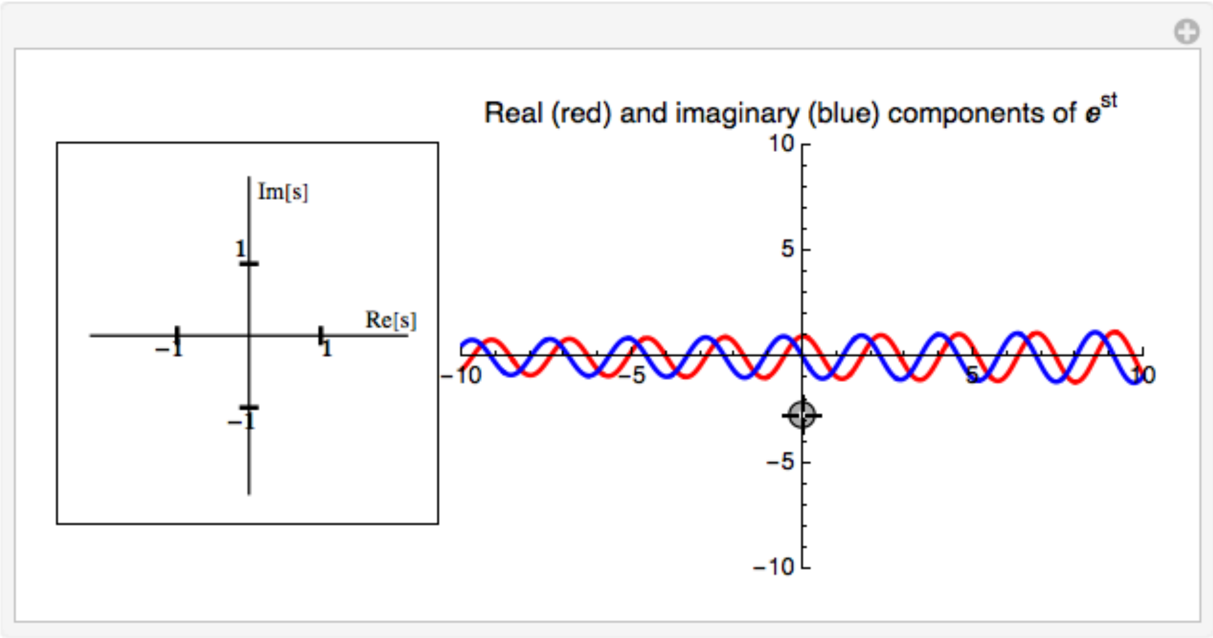
$$h[n] = a^n u[n]$$

Why right sided?

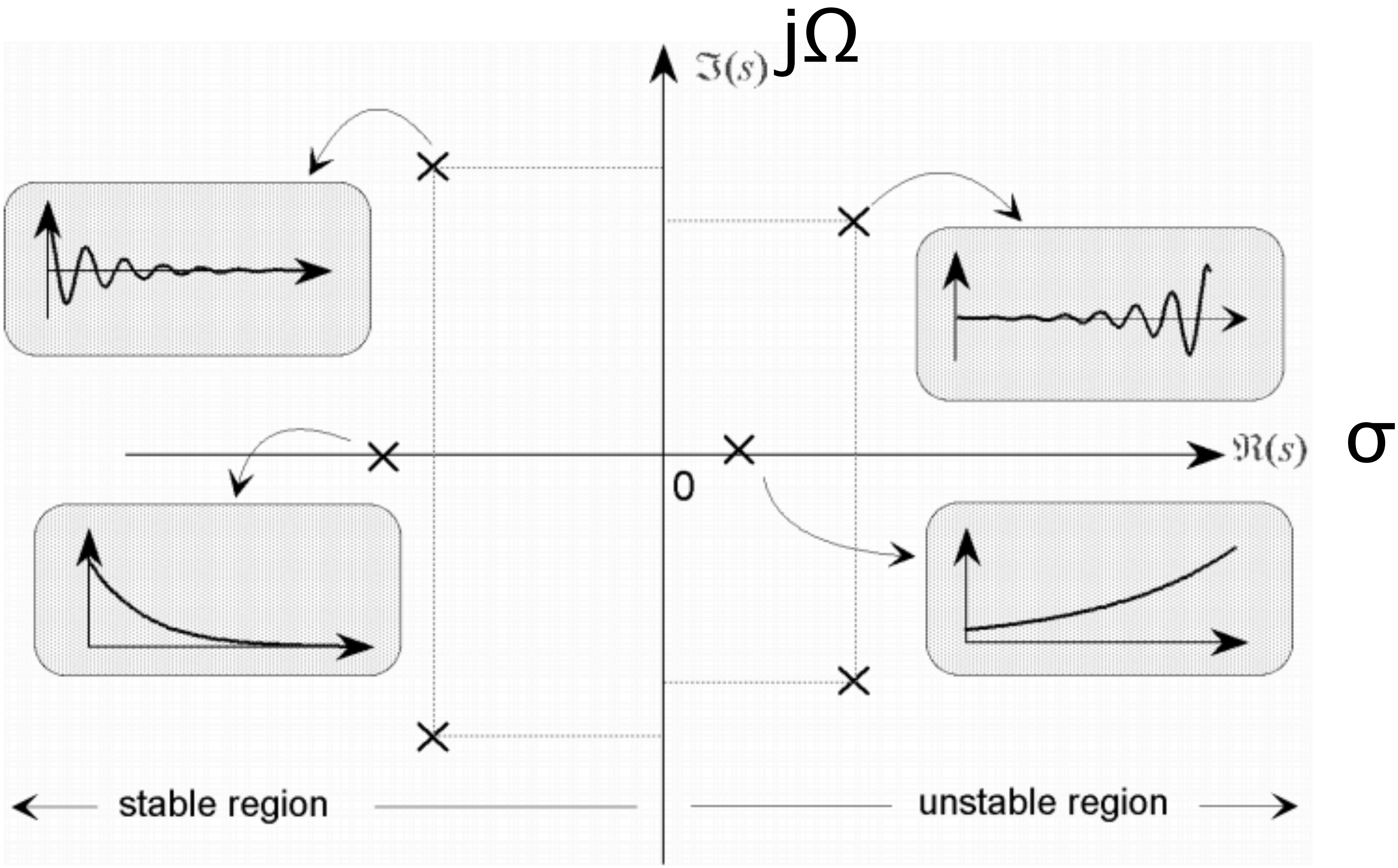
# S-Plane

- $s=\sigma+j\Omega$

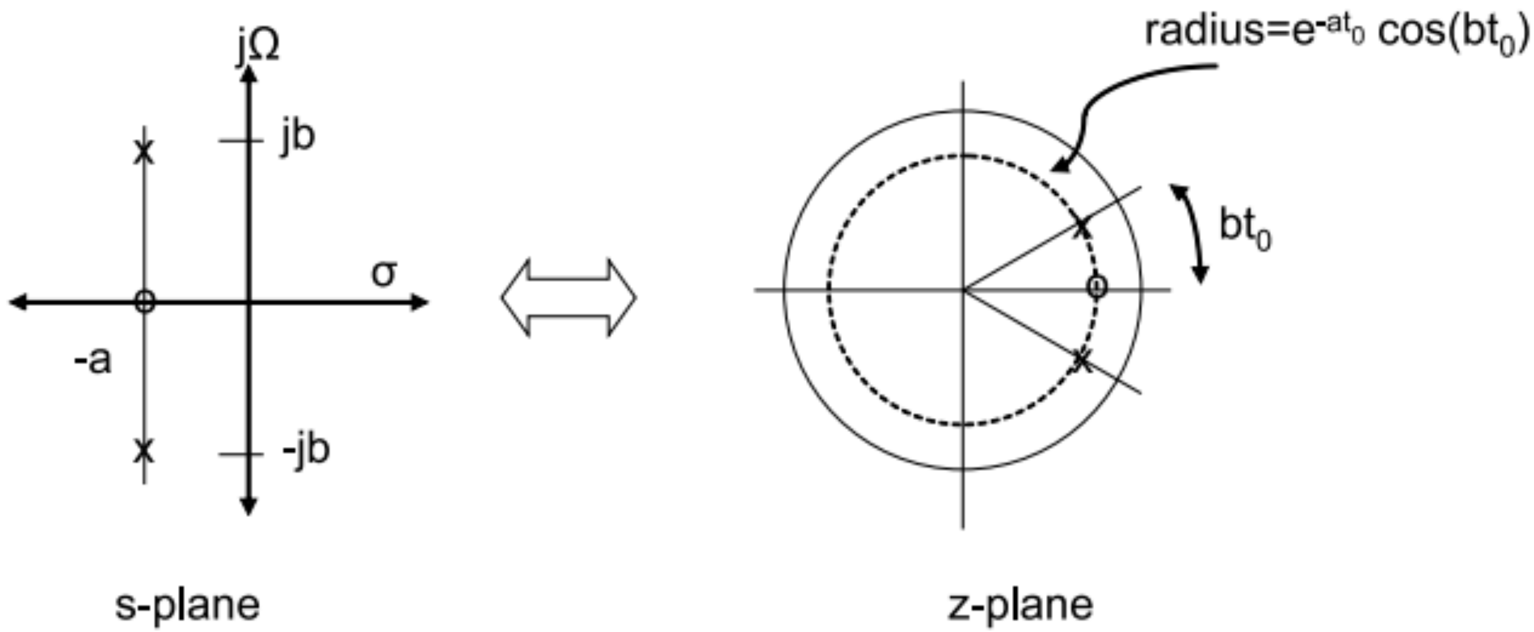
- Wolfram Demo



# S-plane and stability

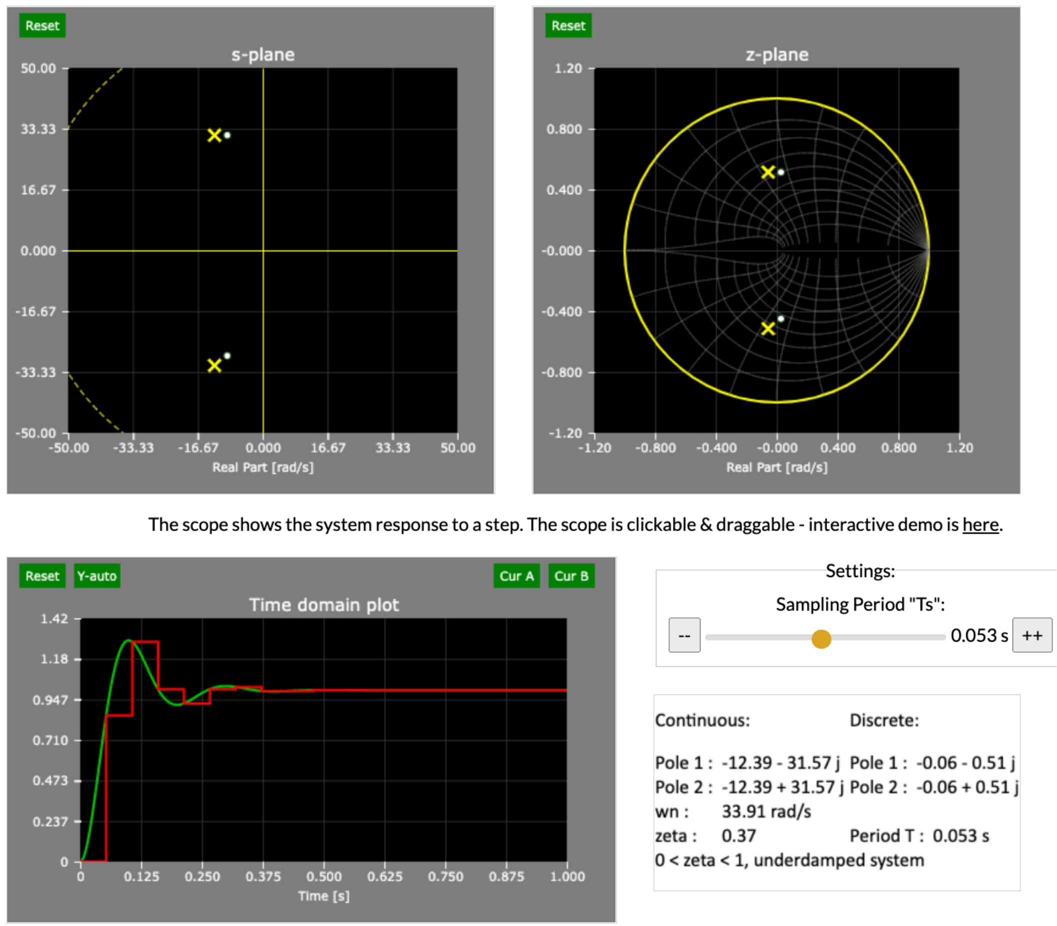


# S-Plane Mapping to Z-Plane



# S-Plane Mapping to Z-Plane

- Interactive Demo




- https://controlsystemsacademy.com/0003/0003.html

# Big Ideas

- z-Transform
  - Draw pole-zero plots
  - Must specify region of convergence (ROC)
- z-Transform properties
  - Similar to DTFT
- Inverse z-transform
  - Inspection, properties, partial fractions, power series
- Difference equations easy to transform
- S-plane to Z-plane mapping
  - Poles map directly. Zeros don't! More later…

# Admin

- HW 2 out now
  - Due 2/11 at midnight
  - Submit in Canvas/Gradescope
    - Make sure scans of HW are legible
- HW 3 posted on Sunday

- New recitation/example videos in Canvas
- Poll for Noah's OH in Ed