ESE 5310: Digital Signal Processing

Lecture 6: February 8, 2024

Sampling and Reconstruction

Power Series Expansion

- Expansion of the z-transform definition

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

$$= \cdots + x[-2]z^{2} + x[-1]z + x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots$$

Power Series Expansion w/ Long Division

$$X(z) = \frac{1}{1 - az^{-1}}, \qquad |z| > |a|$$

Reminder: Difference Equations

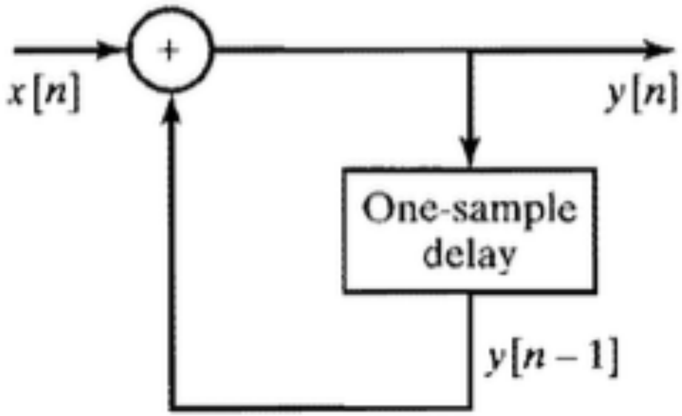
- Accumulator example

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

$$y[n] = x[n] + \sum_{k=-\infty}^{n-1} x[k]$$

$$y[n] = x[n] + y[n-1]$$

$$\boxed{y[n] - y[n-1] = x[n]}$$



$$\boxed{\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]}$$

Difference Equation to z-Transform

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

$$y[n] = -\sum_{k=1}^{N}\left(\frac{a_k}{a_0}\right) y[n-k] + \sum_{m=0}^{M}\left(\frac{b_m}{a_0}\right) x[n-m]$$

- Difference equations of this form behave as causal LTI systems
  - when the input is zero prior to n=0
  - Initial rest equations are imposed prior to the time when input becomes nonzero
    - i.e y[-N]=y[-N+1]=…=y[-1]=0

Example: 1st-Order System

$$y[n] = ay[n-1] + x[n]$$

Example: 1st-Order System

$$y[n] = ay[n-1] + x[n]$$

$$H(z) = \frac{1}{1 - az^{-1}}$$

$$h[n] = a^n u[n]$$

Why right sided?

Difference Equation to z-Transform

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

$$\sum_{k=0}^{N}\left(\frac{a_k}{a_0}\right) z^{-k} Y(z) = \sum_{m=0}^{M}\left(\frac{b_m}{a_0}\right) z^{-m} X(z)$$

$$\Rightarrow Y(z) = \frac{\sum_{m=0}^{M}(b_m) z^{-m}}{\sum_{k=0}^{N}(a_k) z^{-k}} X(z)$$

$$\boxed{H(z) = \frac{\sum_{m=0}^{M}(b_k) z^{-k}}{\sum_{k=0}^{N}(a_k) z^{-k}}}$$

Example: 1st-Order System

$$y[n] = ay[n-1] + x[n]$$

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

$$H(z) = \frac{\overset{b_0}{1}}{\underset{a_0}{1} - \underset{a_1}{a}z^{-1}}$$

$$H(z) = \frac{\sum_{m=0}^{M} (b_k) z^{-k}}{\sum_{k=0}^{N} (a_k) z^{-k}}$$

$$h[n] = a^n u[n]$$

Why right sided?

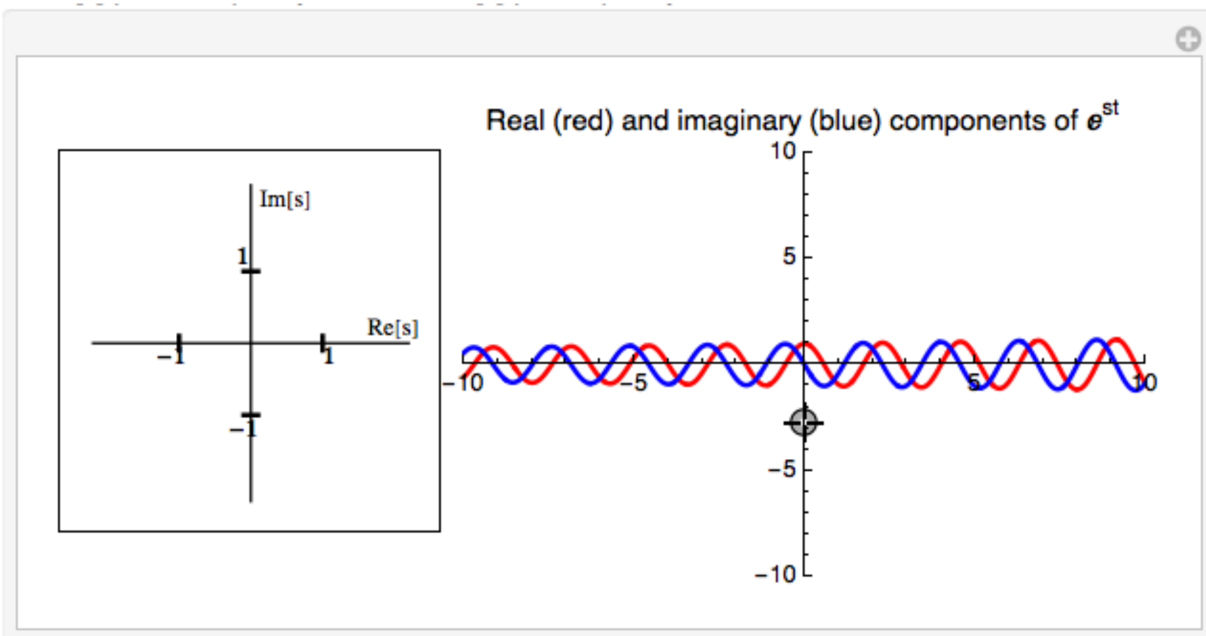
Reminder: Laplace Transform

- The Laplace transform takes a function of time, t, and transforms it to a function of a complex variable, s.
- Because the transform is invertible, no information is lost, and it is reasonable to think of a function f(t) and its Laplace transform F(s) as two views of the same phenomenon.
- Each view has its uses, and some features of the phenomenon are easier to understand in one view or the other.

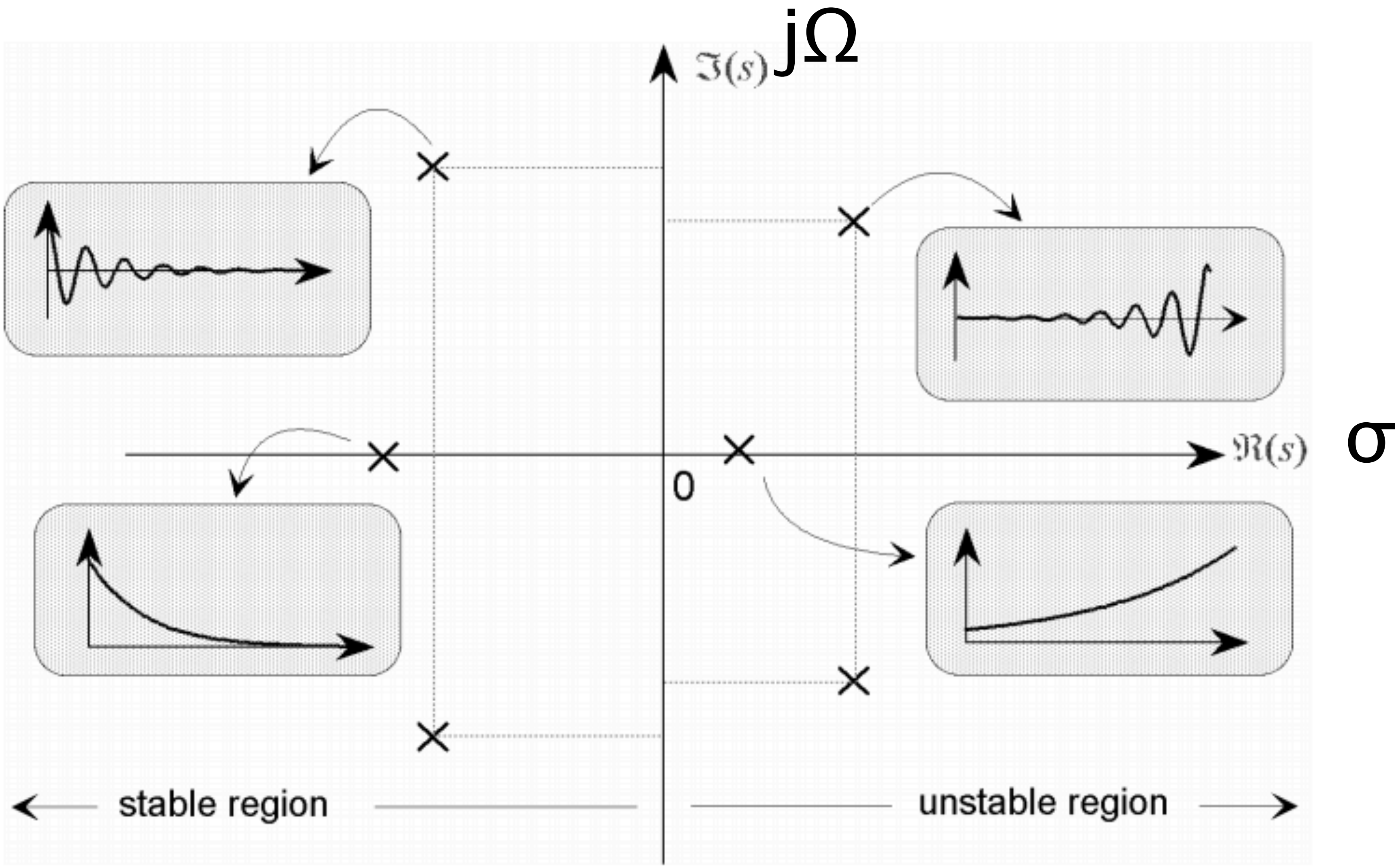
S-Plane

- $s=\sigma+j\Omega$

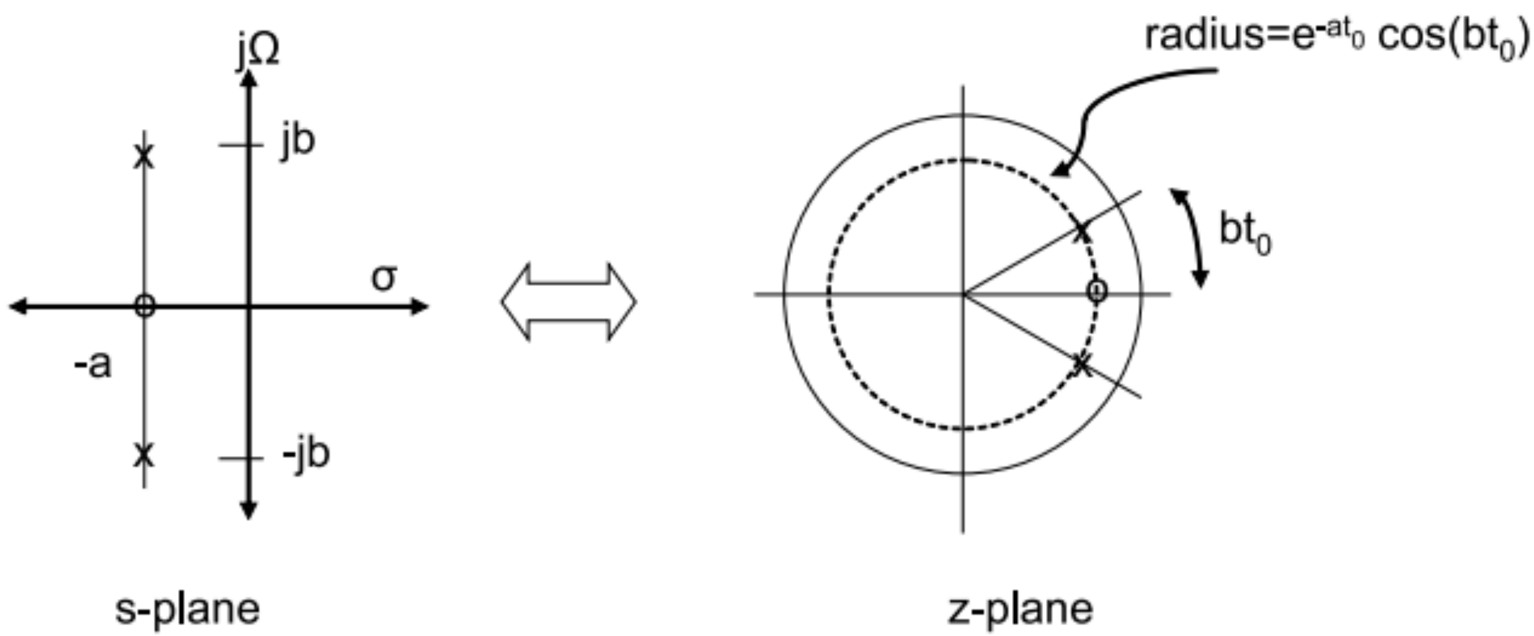
- Wolfram Demo



S-plane and stability

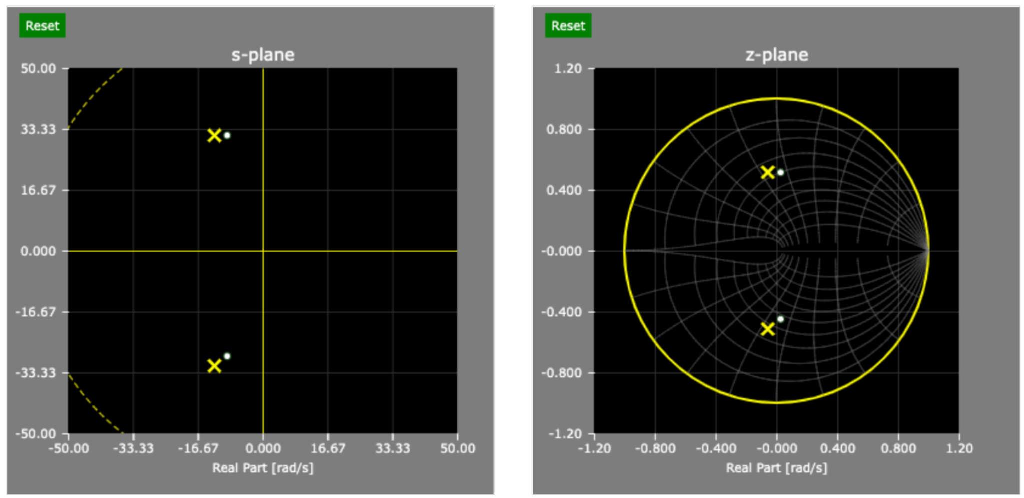


S-Plane Mapping to Z-Plane



S-Plane Mapping to Z-Plane

- Interactive Demo



The scope shows the system response to a step. The scope is clickable & draggable - interactive demo is here.

- https://controlsystemsacademy.com/0003/0003.html

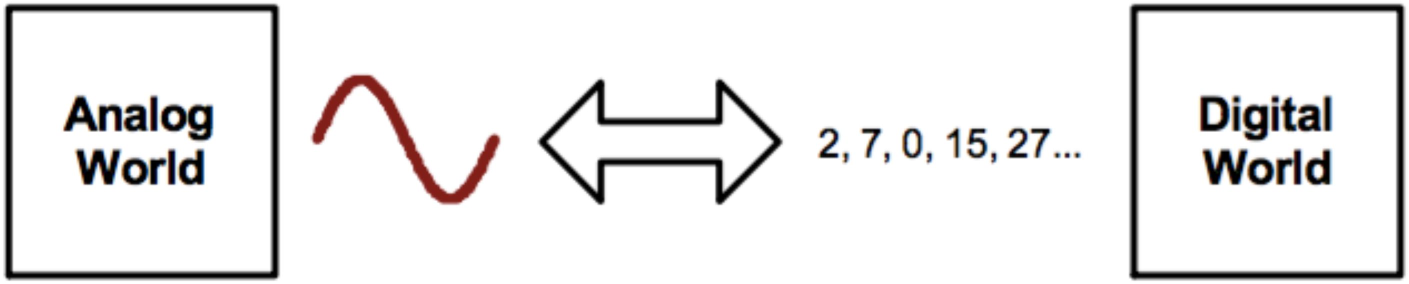
Lecture Outline

- Sampling
  - Frequency Response of Sampled Signal
- Reconstruction
- Anti-aliasing Filter

Video Example

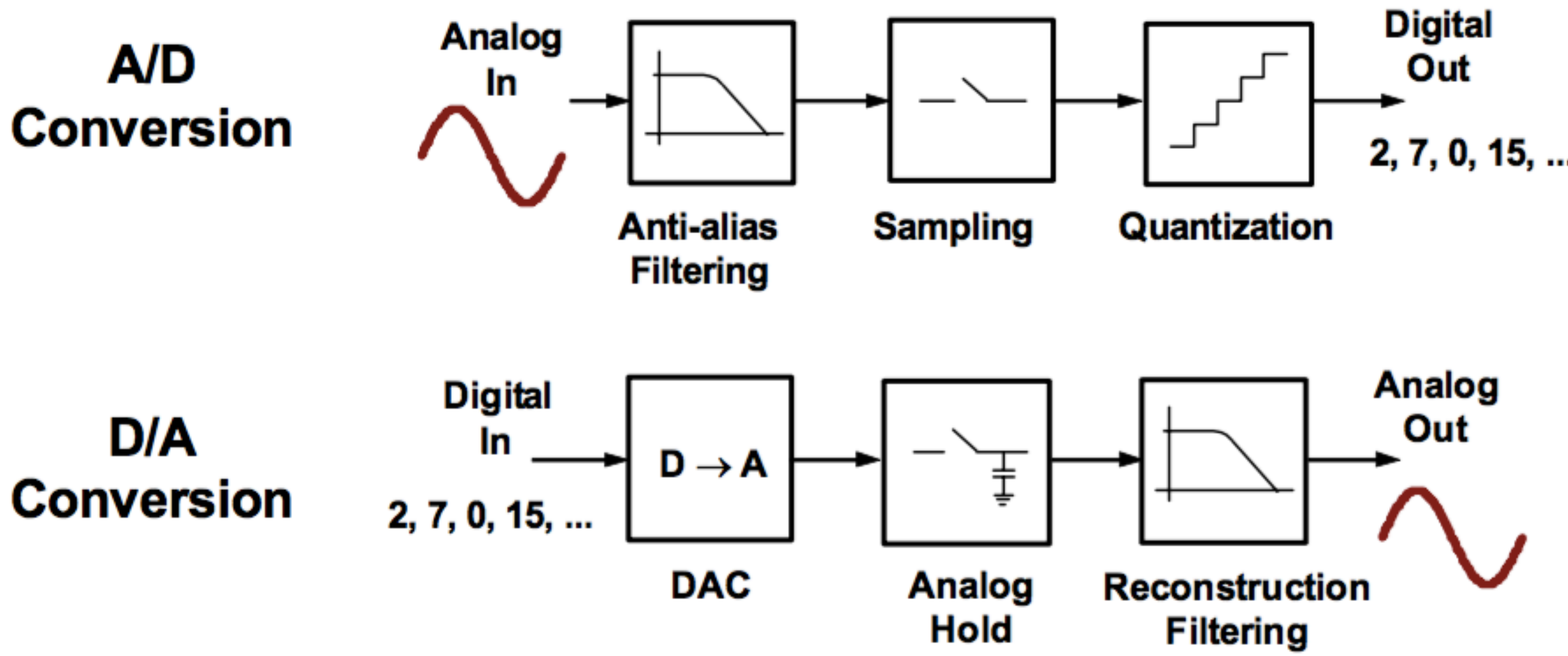
- https://www.youtube.com/watch?v=ByTsISFXUoY

The Data Conversion Problem



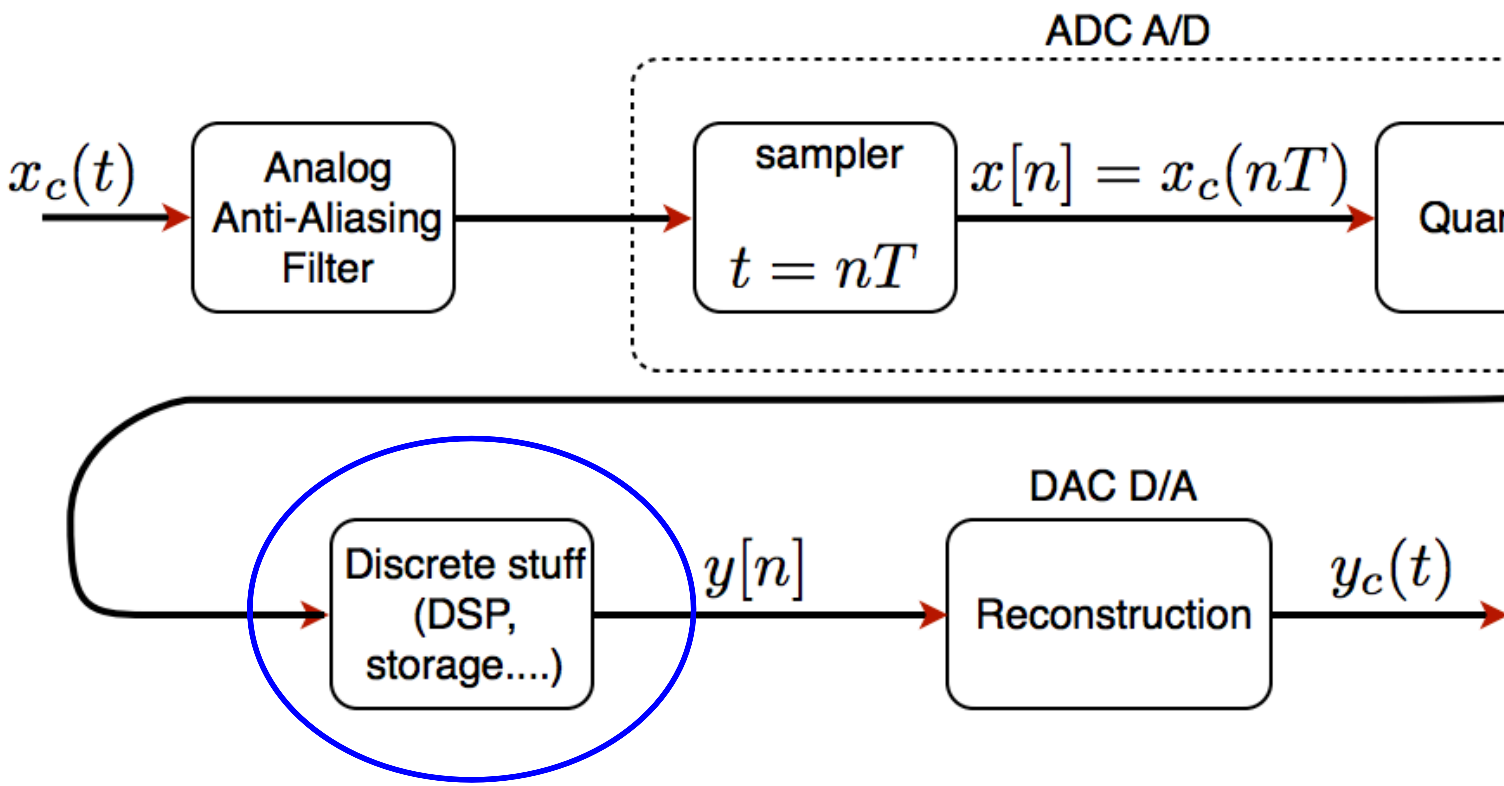
- Real world signals
  - Continuous time, continuous amplitude
- Digital abstraction
  - Discrete time, discrete amplitude
- Two problems
  - How to go discretize in time and amplitude
    - A/D conversion
  - How to "undescretize" in time and amplitude
    - D/A conversion

Overview



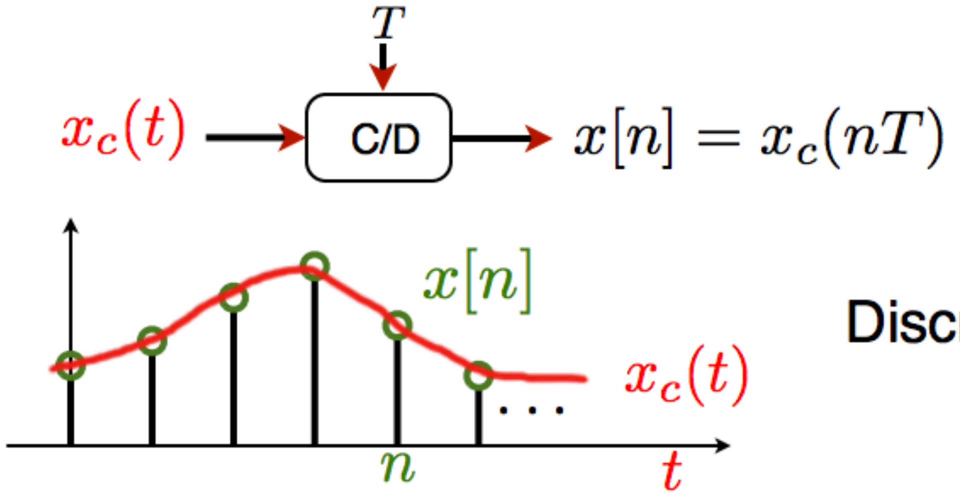
- We'll first look at these building blocks from a functional, "black box" perspective

DSP System



ADC A/D
$x_c(t)$
Analog Anti-Aliasing Filter
sampler
$t = nT$
$x[n] = x_c(nT)$
Quantizer
DAC D/A
Discrete stuff (DSP, storage....)
$y[n]$
Reconstruction
$y_c(t)$

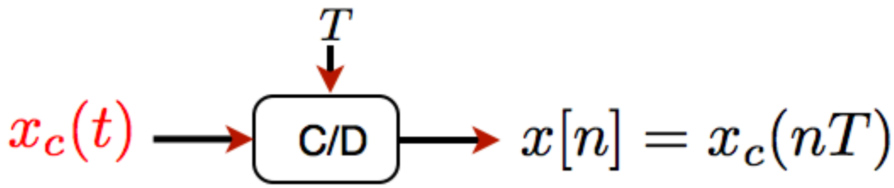
Ideal Sampling Model

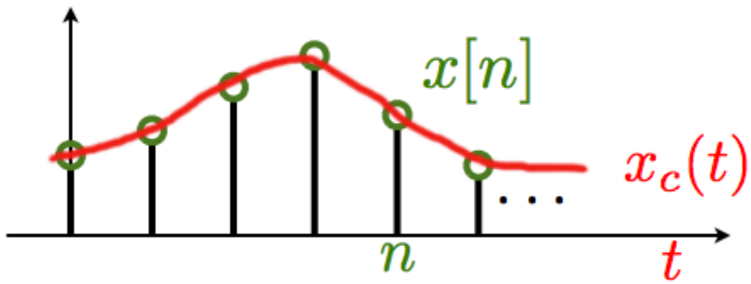


Discrete and Continuous

- Ideal continuous-to-discrete time (C/D) converter
  - T is the sampling period
  - f_s=1/T is the sampling frequency
  - Ω_s=2π/T

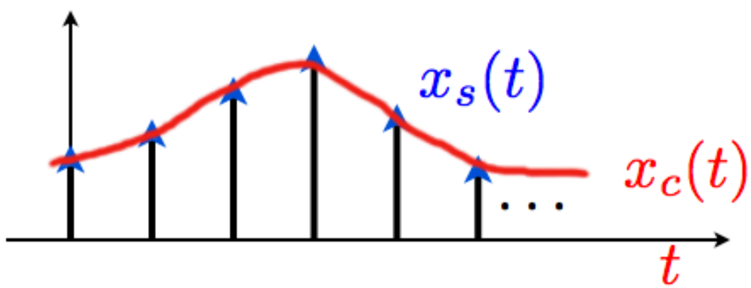
Ideal Sampling Model





Discrete and Continuous

define impulsive sampling:



Continuous

$$x_s(t) = \cdots + x_c(0)\delta(t) + x_c(T)\delta(t-T) + \cdots$$

$$x_s(t) = x_c \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Ideal Sampling Model

$$x_s(t) = x_c \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

- Dirac delta function, δ(t)
  - Infinitely high and thin, area of 1
  - Not physical—for modeling

$$x[n] \leftrightarrow x_s(t) \leftrightarrow x_c(t)$$

- Three signals. How are they related? In time? In frequency?

Frequency Domain Analysis

- How is x[n] related to $x_s(t)$ in frequency domain?

$$x[n] = x_c(nT)$$

$$x_s(t) = x_c \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Frequency Domain Analysis

- How is x[n] related to $x_s(t)$ in frequency domain?

$$x[n] = x_c(nT)$$

$$x_s(t) = x_c \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$x_s(t)$:C.T

$$X_s(j\Omega) = \sum_n x_c(nT) e^{-j\Omega nT}$$

$x[n]$:D.T

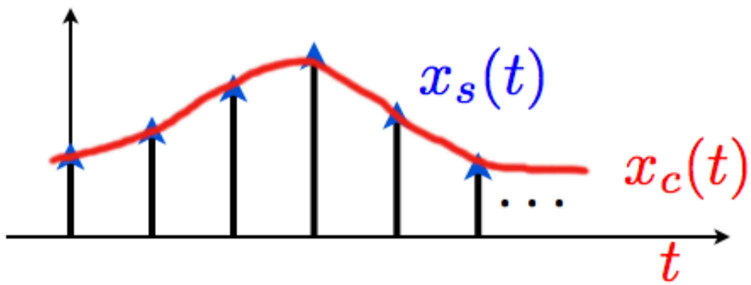
$$X(e^{j\omega}) = \sum_n x[n] e^{-j\omega n}$$

$$\omega = \Omega T$$

$$\boxed{X(e^{j\omega}) = X_s(j\Omega)\big|_{\Omega=\omega/T}}$$

$$\boxed{X_s(j\Omega) = X(e^{j\omega})\big|_{\omega=\Omega T}}$$

Frequency Domain Analysis



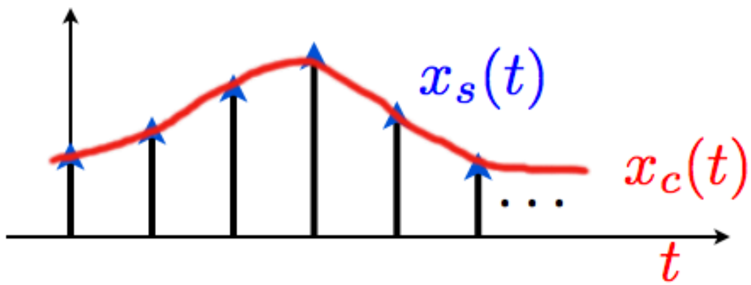
Continuous

$$x_s(t) = \cdots + x_c(0)\delta(t) + x_c(T)\delta(t-T) + \cdots$$

$$x_s(t) = x_c \underbrace{\sum_{n=-\infty}^{\infty} \delta(t-nT)}_{\triangleq s(t)}$$

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)$$

Frequency Domain Analysis



Continuous

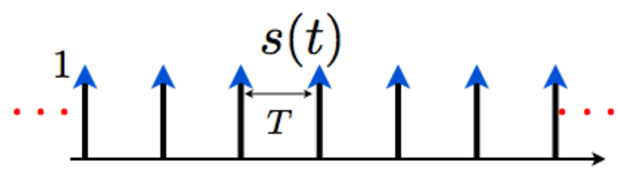
$$x_s(t) = \cdots + x_c(0)\delta(t) + x_c(T)\delta(t-T) + \cdots$$

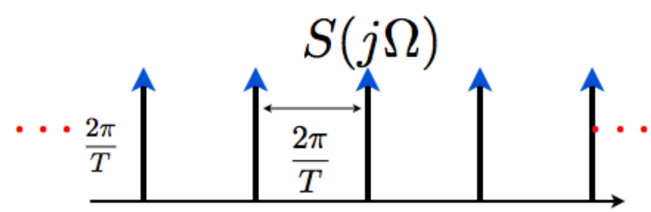
$$x_s(t) = x_c \underbrace{\sum_{n=-\infty}^{\infty} \delta(t-nT)}_{\triangleq s(t)}$$

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)$$

$$s(t) \quad \leftrightarrow \quad S(j\Omega)$$

$$S(j\Omega) \quad = \quad \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\Omega - \underbrace{\frac{2\pi}{T}}_{=\Omega_s} k)$$





Frequency Domain Analysis

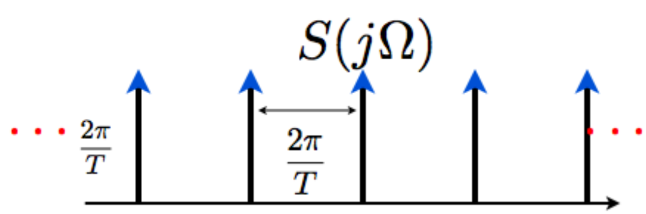
$$x_s(t) = s(t) \cdot x_c(t)$$

Frequency Domain Analysis

$$x_s(t) = s(t) \cdot x_c(t)$$

$$X_s(j\Omega) = \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega)$$

$$S(j\Omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\Omega - \underbrace{\frac{2\pi}{T}}_{=\Omega_s} k)$$

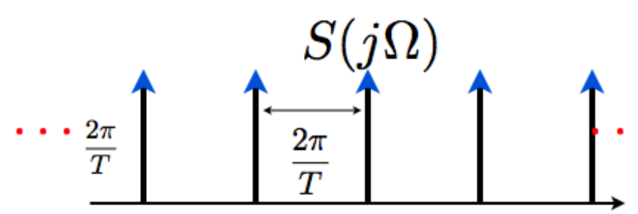


Frequency Domain Analysis

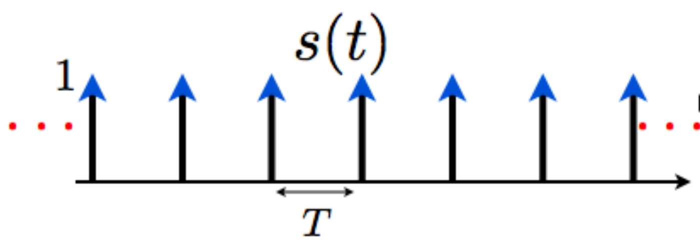
$$x_s(t) = s(t) \cdot x_c(t)$$

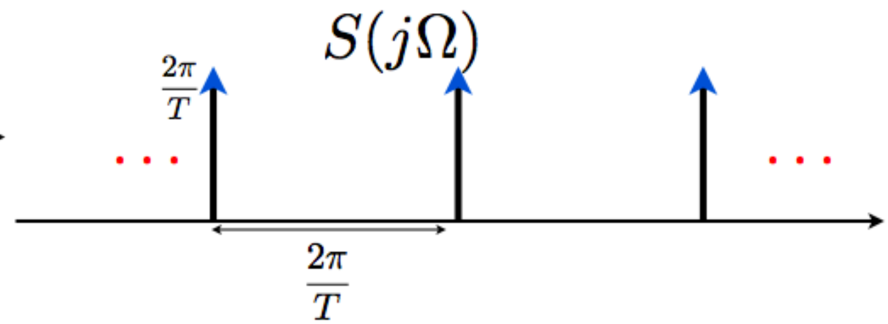
$$\boxed{\begin{aligned} X_s(j\Omega) &= \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega) \\ &= \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s)) \quad \Big| \quad \Omega_s = \frac{2\pi}{T} \end{aligned}}$$

$$S(j\Omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\Omega - \underbrace{\frac{2\pi}{T}}_{=\Omega_s} k)$$

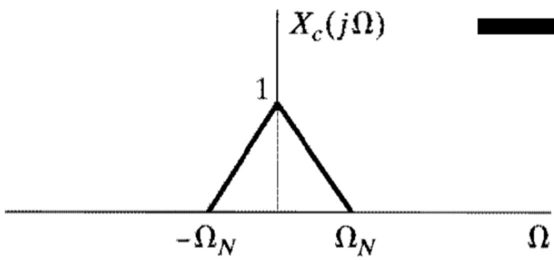


Frequency Domain Analysis



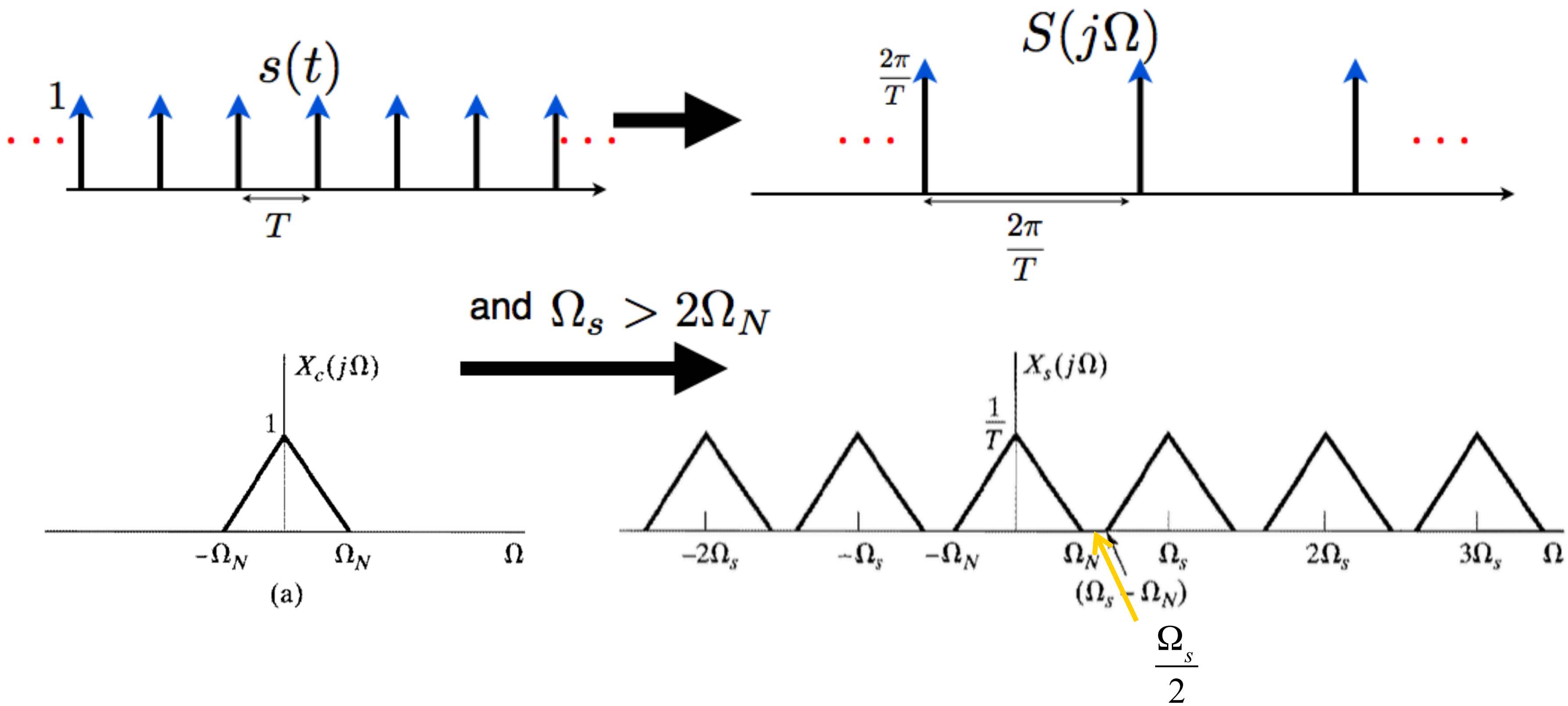


and $\Omega_s > 2\Omega_N$

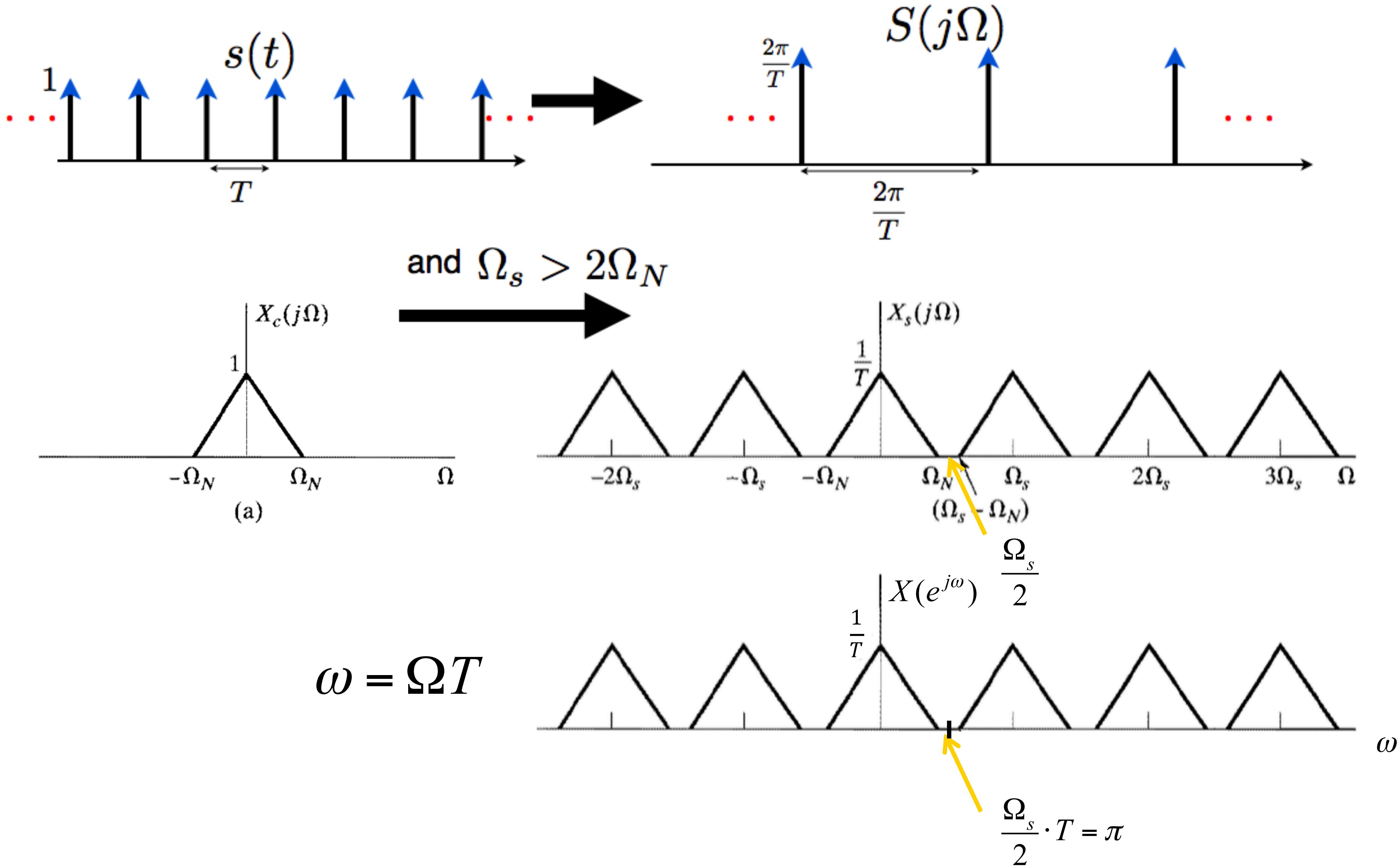


(a)

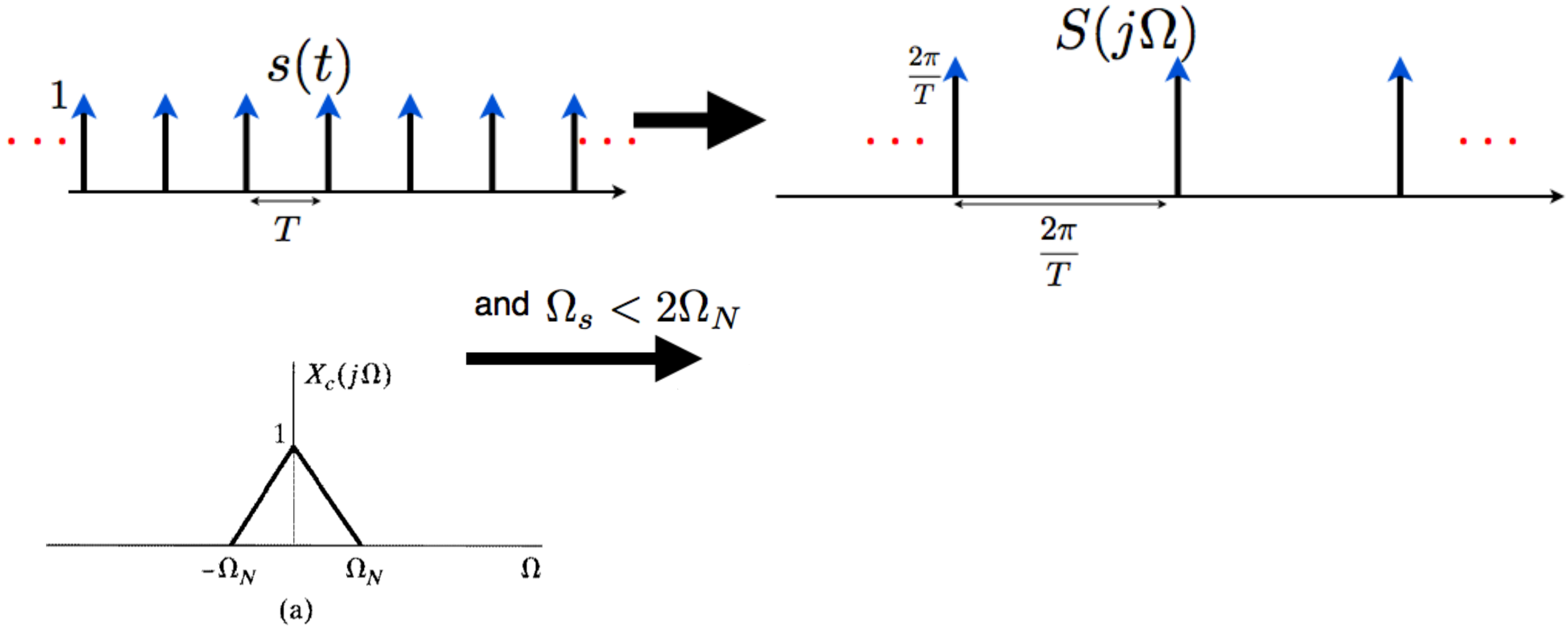
Frequency Domain Analysis



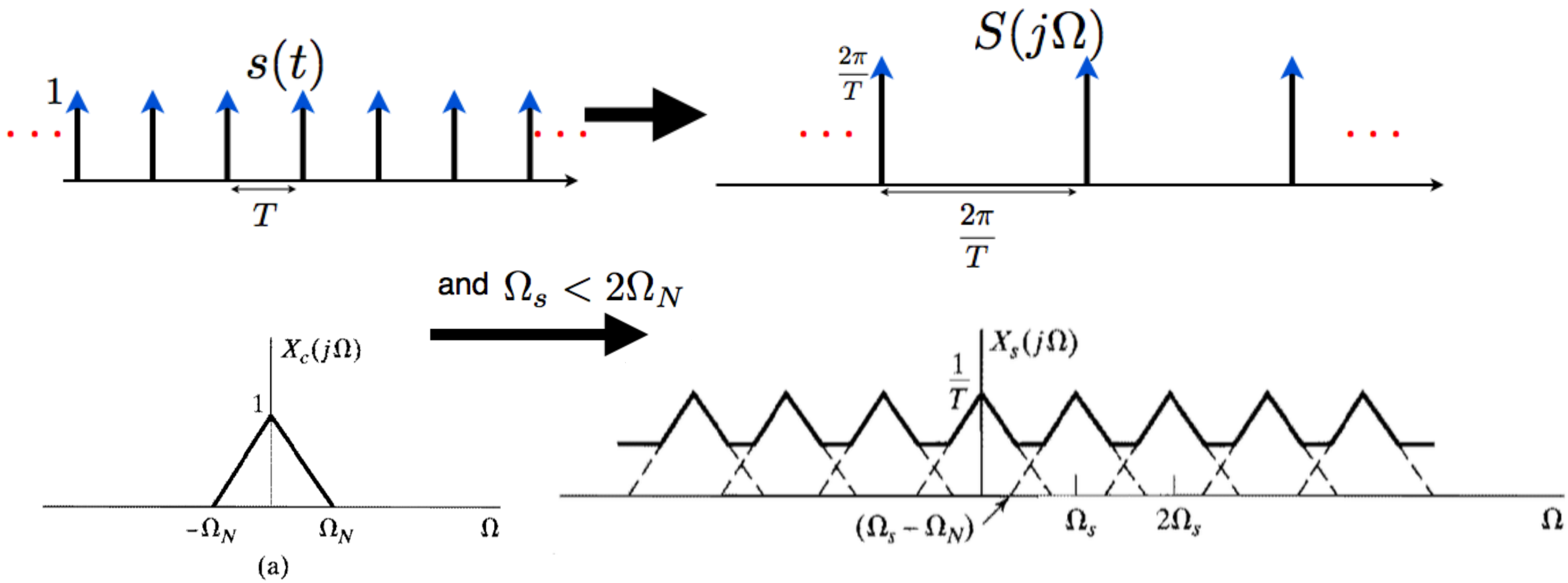
Frequency Domain Analysis



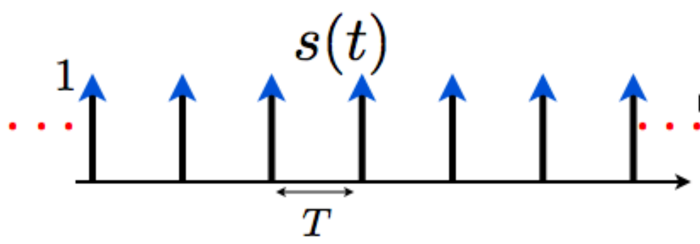
Frequency Domain Analysis w/ Aliasing

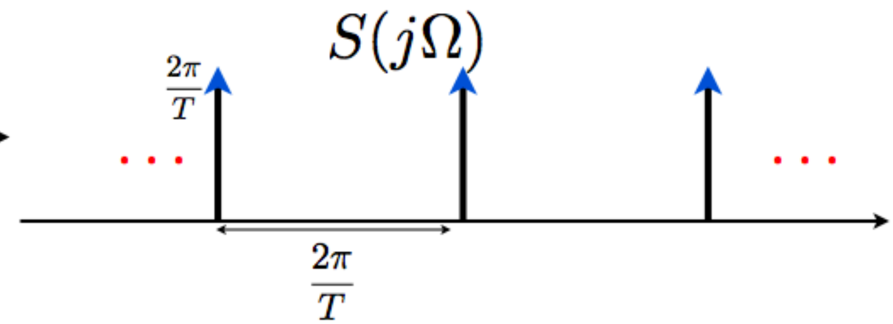


Frequency Domain Analysis w/Aliasing

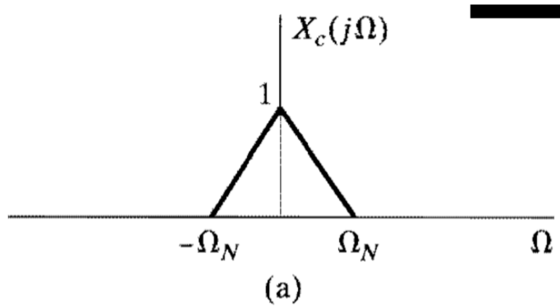


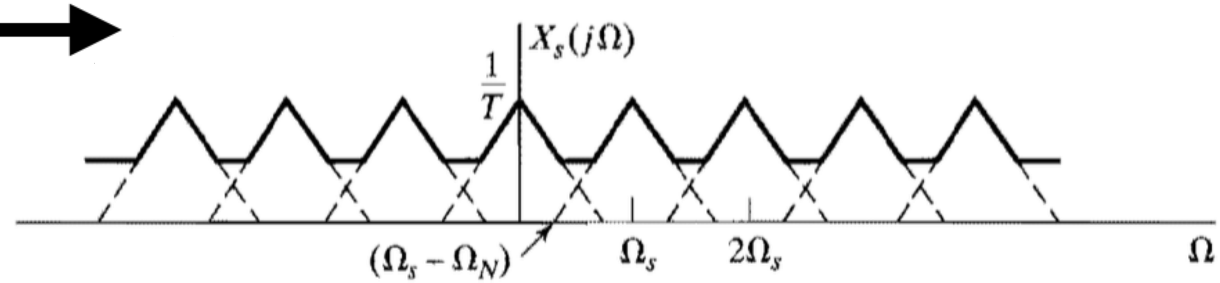
Frequency Domain Analysis w/Aliasing



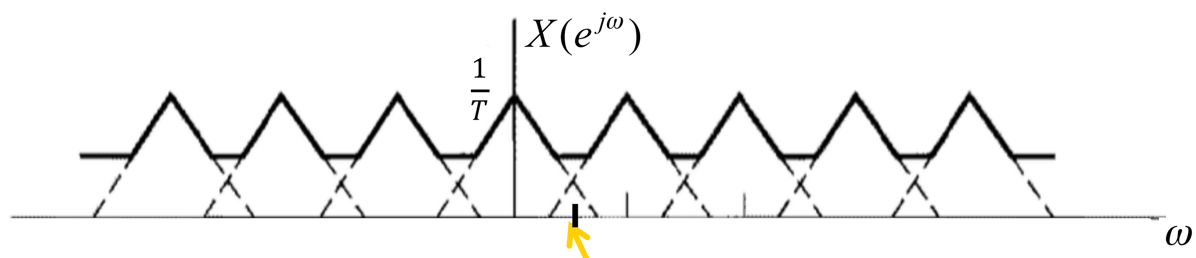


and $\Omega_s < 2\Omega_N$





$$\omega = \Omega T$$



$$\frac{\Omega_s}{2} \cdot T = \pi$$

Example: Cosine Input

- Sample the continuous-time signal $x_c(t) = \cos(4000\pi t)$ with sampling period $T = \frac{1}{6000}s \;\; (f_s = 6kHz)$

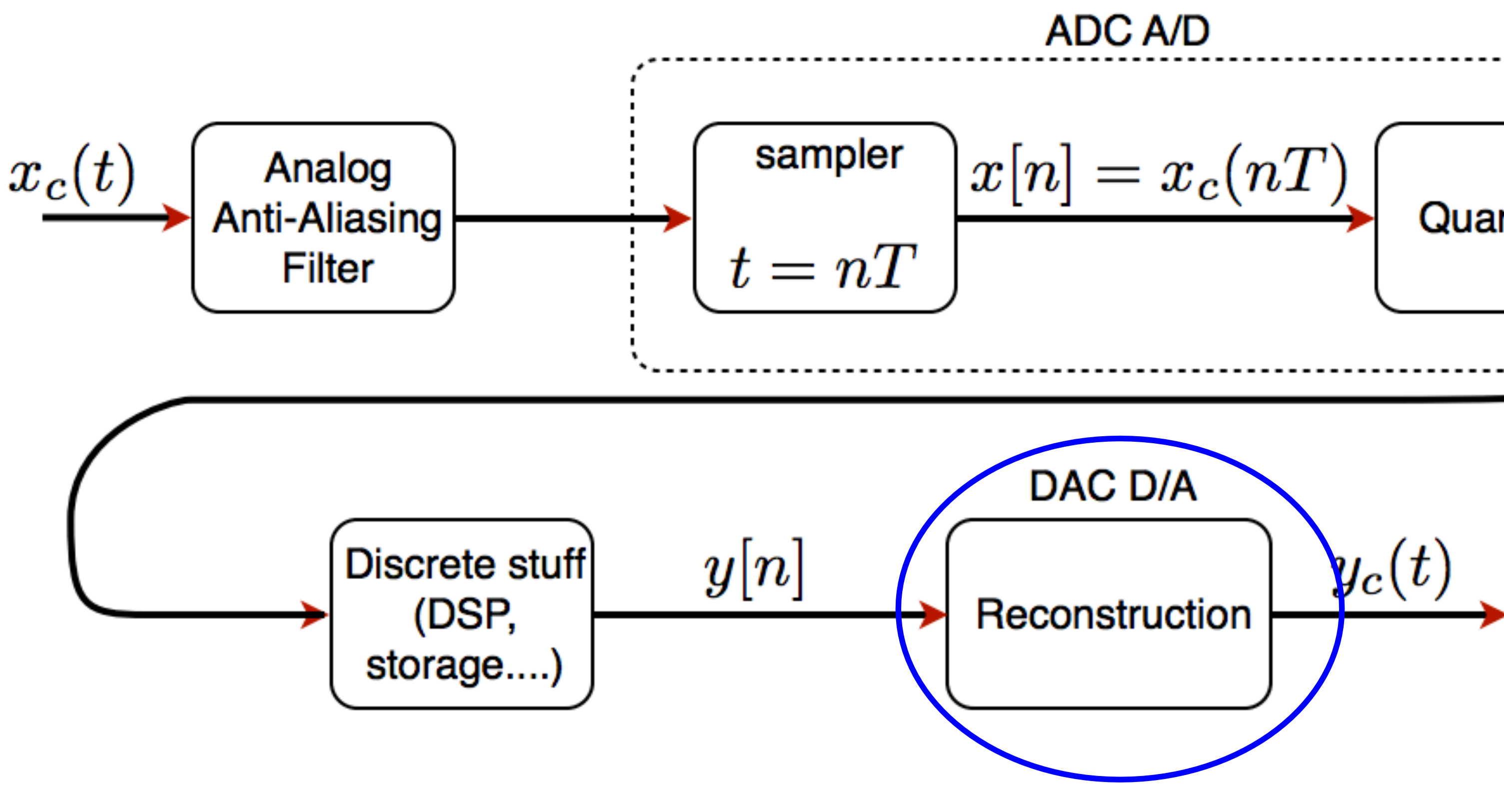
$$
\begin{aligned}
X_s(j\Omega) &= \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega) \\
&= \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s)) \quad \Big| \quad \Omega_s = \frac{2\pi}{T}
\end{aligned}
$$

Example: Cosine Input

- Sample the continuous-time signal $x_c(t) = \cos(16000\pi t)$ with sampling period $T = \frac{1}{6000}s$ $(f_s = 6kHz)$

$$\boxed{\begin{aligned} X_s(j\Omega) &= \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega) \\ &= \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s)) \quad \Big| \quad \Omega_s = \frac{2\pi}{T} \end{aligned}}$$

DSP System



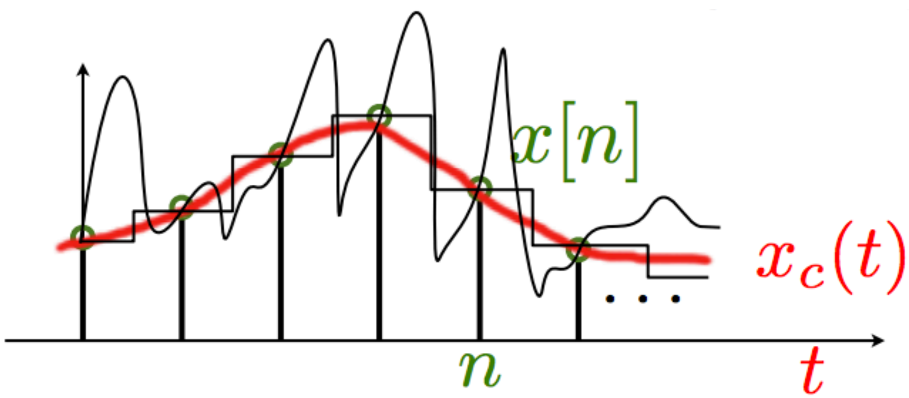
ADC A/D
$x_c(t)$
Analog
Anti-Aliasing
Filter
sampler
$t = nT$
$x[n] = x_c(nT)$
Quantizer
DAC D/A
Discrete stuff
(DSP,
storage....)
$y[n]$
Reconstruction
$y_c(t)$

Reconstruction of Bandlimited Signals

- Nyquist Sampling Theorem: Suppose $x_c(t)$ is bandlimited. I.e.

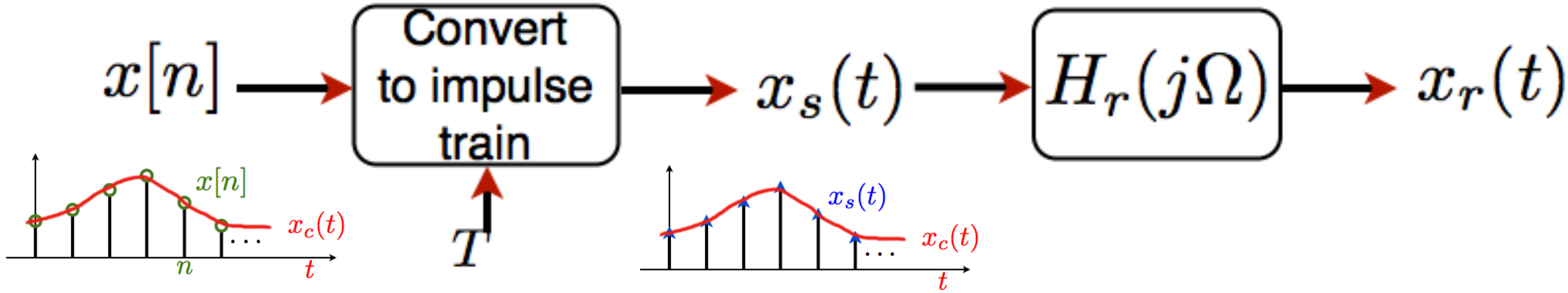
$$X_c(j\Omega) = 0 \;\; \forall \quad |\Omega| \geq \Omega_N$$

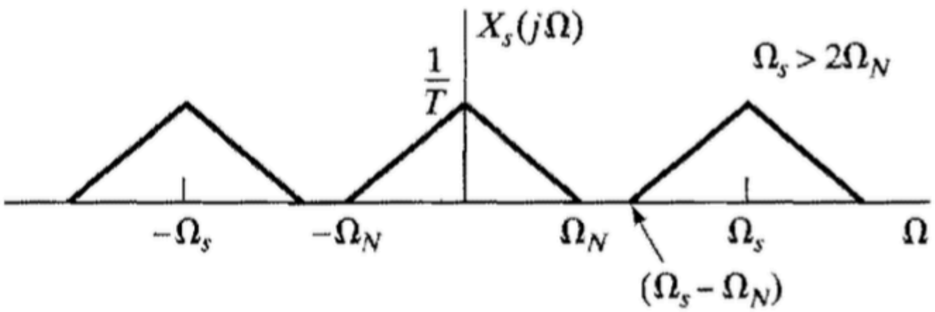
- If $\Omega_s \geq 2\Omega_N$, then $x_c(t)$ can be uniquely determined from its samples $x[n]=x_c(nT)$
- Bandlimitedness is the key to uniqueness



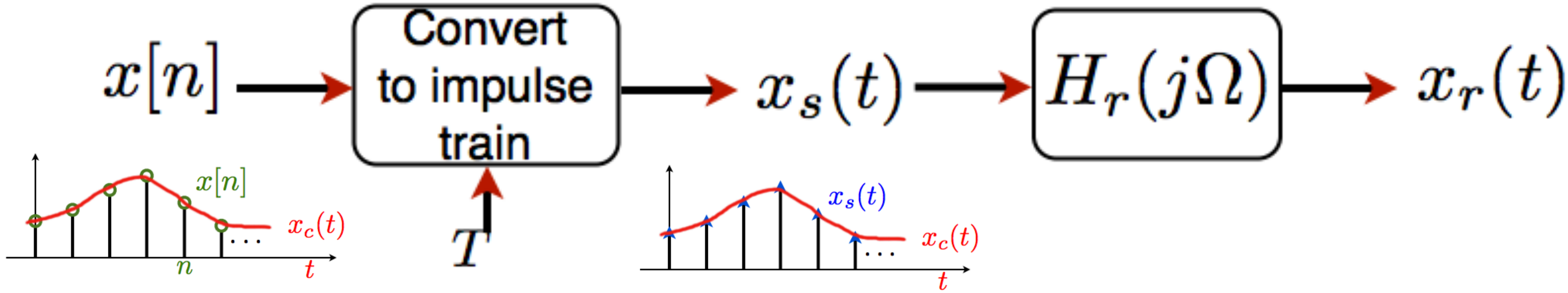
Mulitiple signals go through the samples, but only one is bandlimited within our sampling band

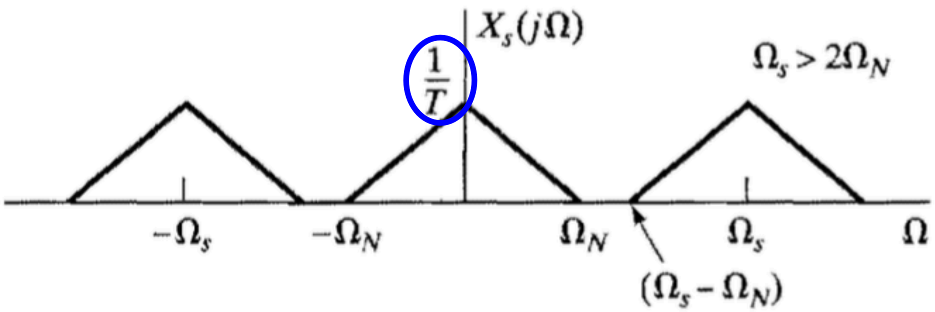
Reconstruction in Frequency Domain

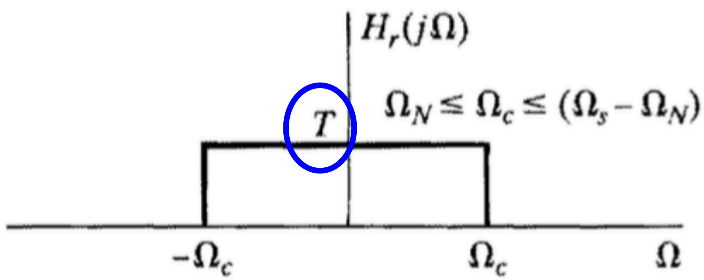




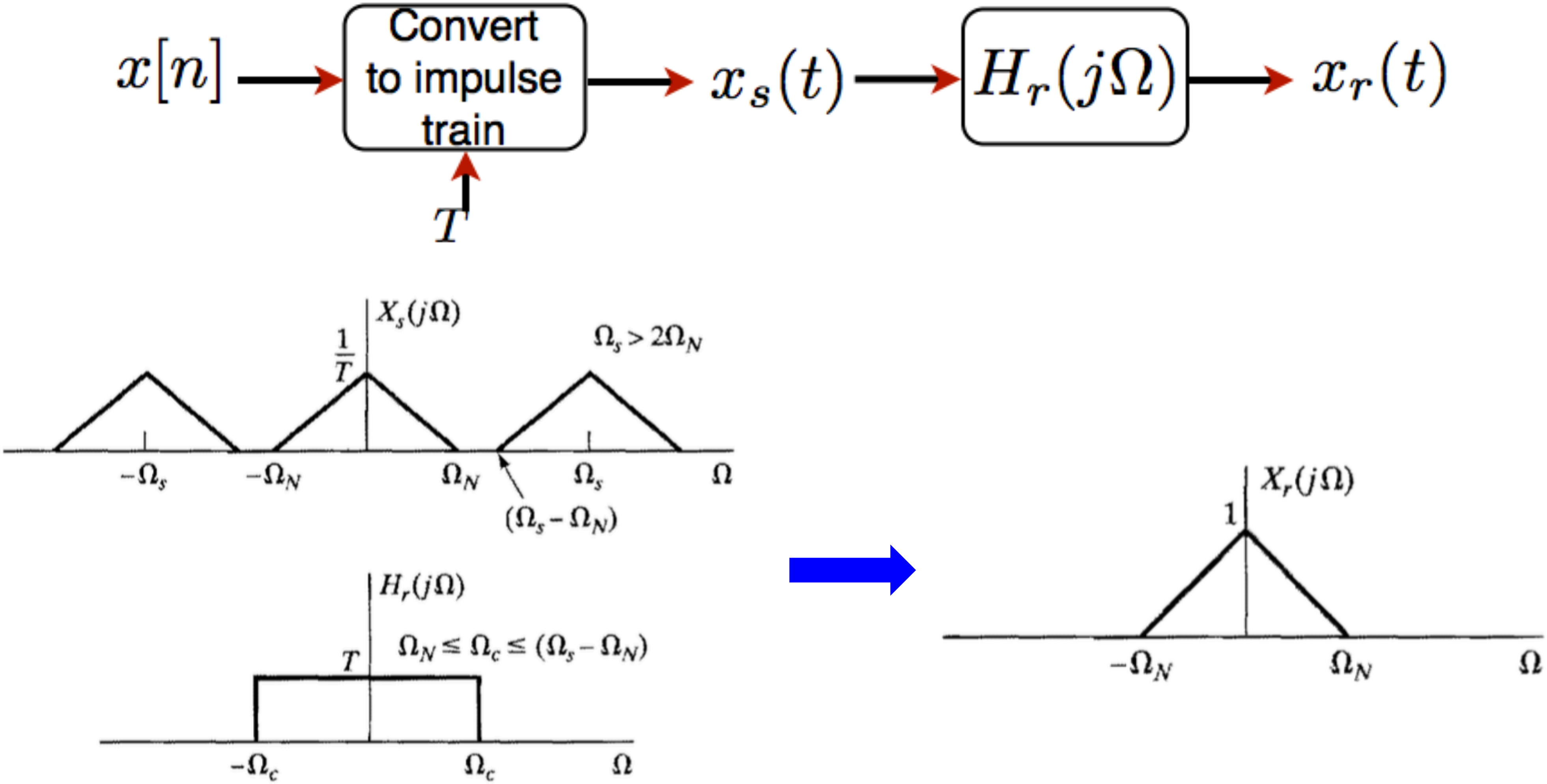
Reconstruction in Frequency Domain



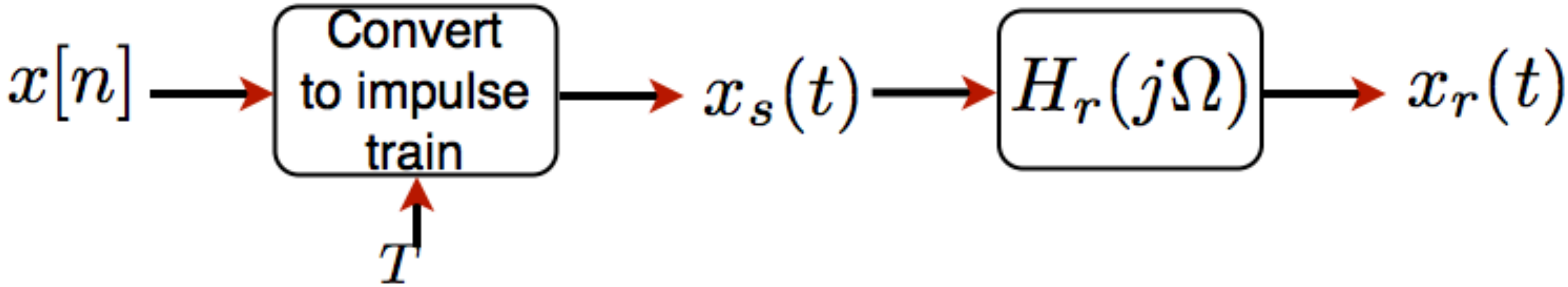


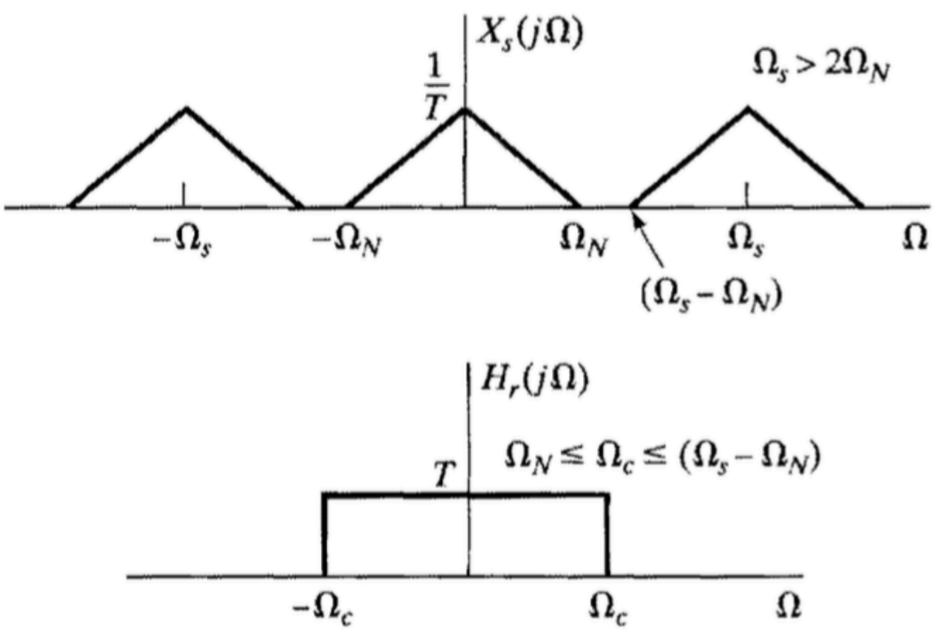


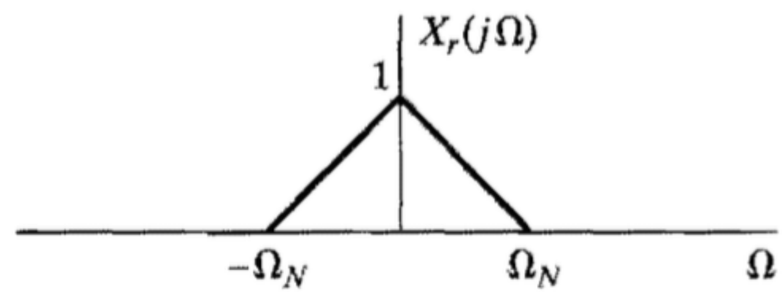
Reconstruction in Frequency Domain



Reconstruction in Frequency Domain







$2\Omega_N = \Omega_s$

$\Omega_N = \Omega_C = \Omega_s/2$

Sample at Nyquist and filter at signal badnwidth

Example: Cosine Input

- Sample and reconstruct the continuous-time signal $x_c(t) = \cos(4000\pi t)$ with sampling period $T = \frac{1}{6000} s \ \ (f_s = 6kHz)$

$$
\begin{aligned}
X_s(j\Omega) &= \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega) \\
&= \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s)) \quad \Big| \quad \Omega_s = \frac{2\pi}{T}
\end{aligned}
$$

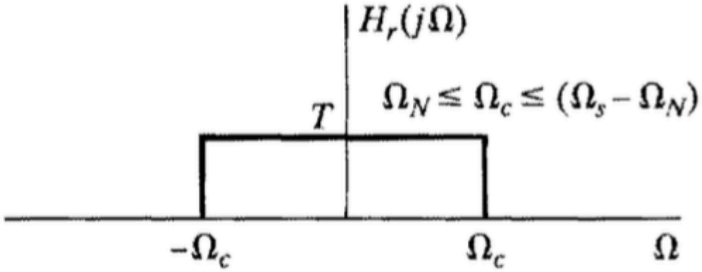
Example: Cosine Input

- Sample and reconstruct the continuous-time signal $x_c(t) = \cos(16000\pi t)$ with sampling period $T = \frac{1}{6000}s$ $(f_s = 6kHz)$

$$
\begin{aligned}
X_s(j\Omega) &= \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega) \\
&= \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s)) \quad \Big| \quad \Omega_s = \frac{2\pi}{T}
\end{aligned}
$$

Reconstruction in Time Domain

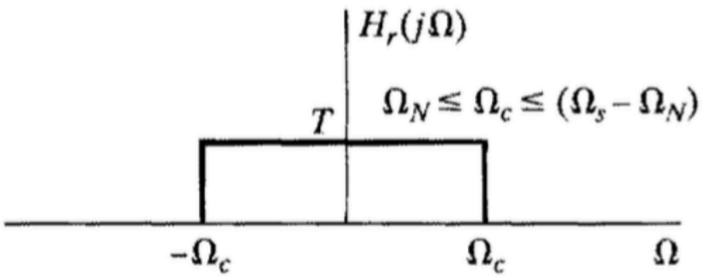
$\Omega_N = \Omega_C = \Omega_s/2$



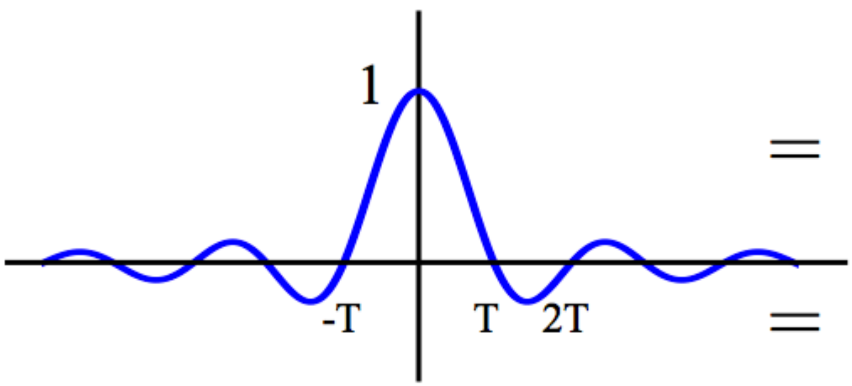
$$h_r(t) = \frac{1}{2\pi}\int_{-\Omega_s/2}^{\Omega_s/2} T e^{j\Omega t} d\Omega$$

Reconstruction in Time Domain

$\Omega_N=\Omega_C=\Omega_s/2$

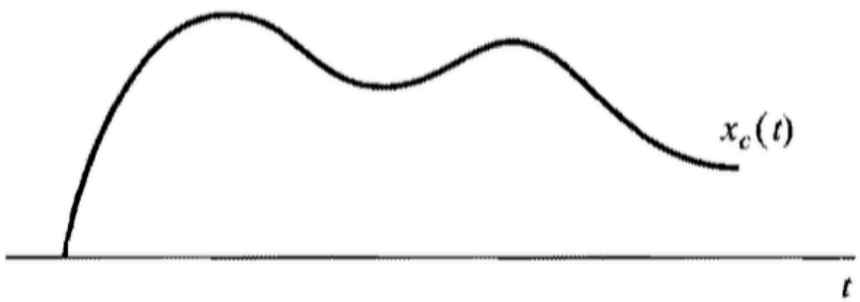


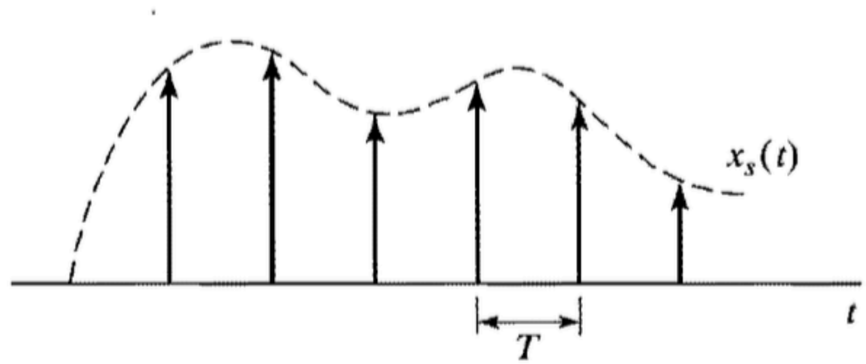
$$
\begin{aligned}
h_r(t) &= \frac{1}{2\pi}\int_{-\Omega_s/2}^{\Omega_s/2} T e^{j\Omega t} d\Omega \\
&= \frac{T}{2\pi}\frac{1}{jt} e^{j\Omega t}\Big|_{-\Omega_s/2}^{\Omega_s/2} \\
&= \frac{T}{\pi t}\frac{e^{j\frac{\Omega_s}{2}t} - e^{-j\frac{\Omega_s}{2}t}}{2j} \\
&= \frac{T}{\pi t}\sin(\frac{\Omega_s}{2}t) = \frac{T}{\pi t}\sin(\frac{\pi}{T}t) \\
&= \mathrm{sinc}(\frac{t}{T})
\end{aligned}
$$

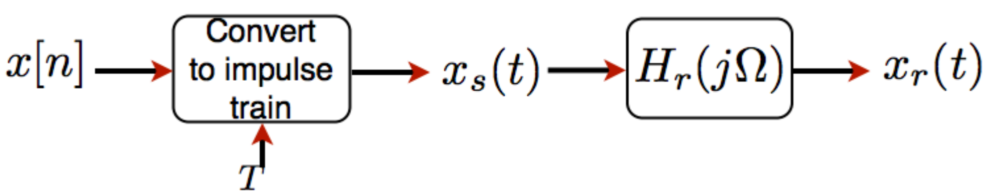


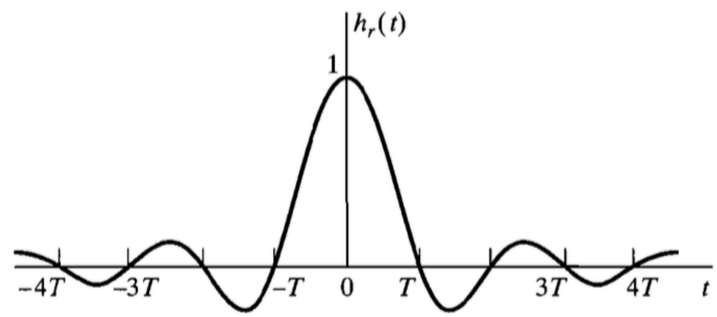
Reconstruction in Time Domain

$$x_r(t) = x_s(t) * h_r(t) = \left(\sum_n x[n]\delta(t - nT) \right) * h_r(t)$$
$$= \sum_n x[n] h_r(t - nT)$$



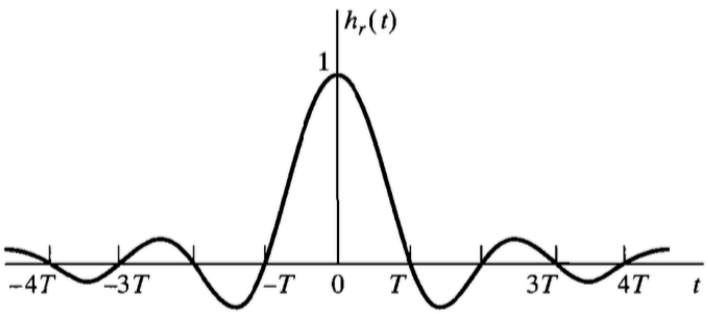




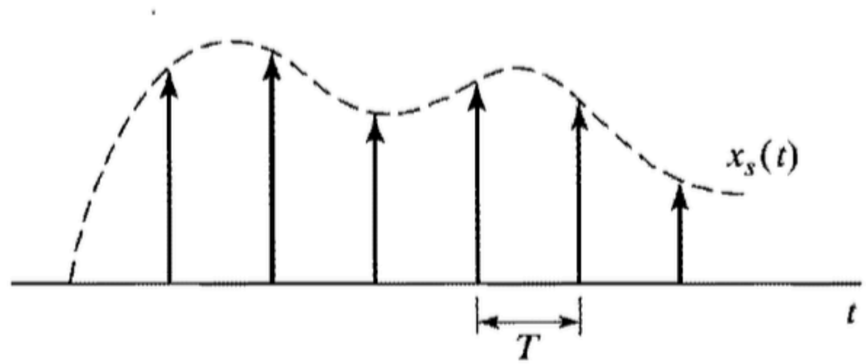


Reconstruction in Time Domain

$$x_r(t) = x_s(t) * h_r(t) = \left(\sum_n x[n]\delta(t - nT)\right) * h_r(t)$$
$$= \sum_n x[n]h_r(t - nT)$$



$*$

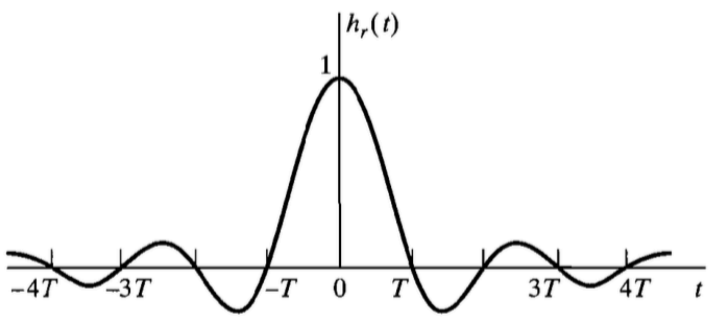


$=$

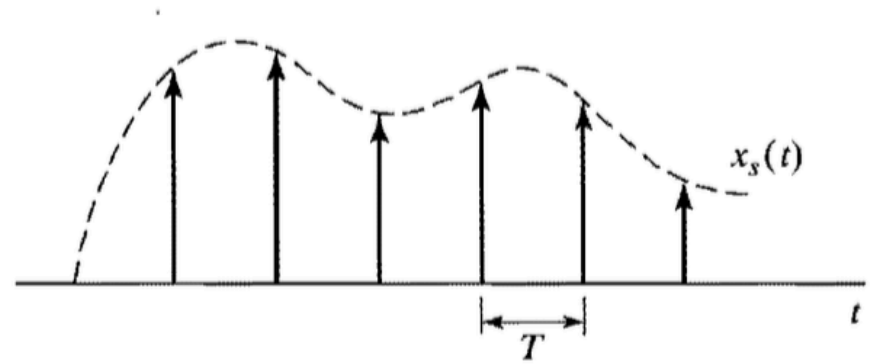
Reconstruction in Time Domain

$$x_r(t) = x_s(t) * h_r(t) = \left(\sum_n x[n]\delta(t-nT)\right) * h_r(t)$$

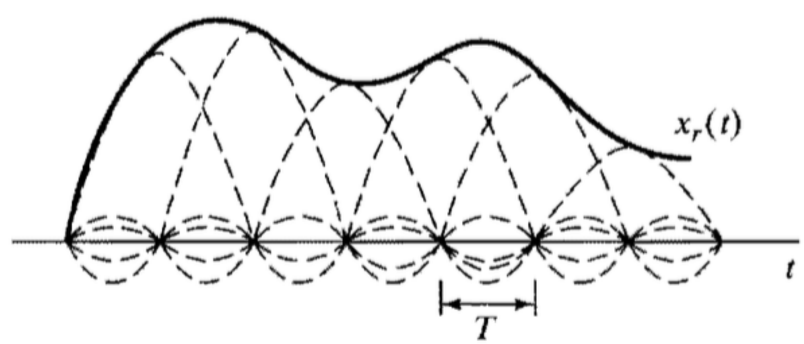
$$= \sum_n x[n]h_r(t-nT)$$



$*$



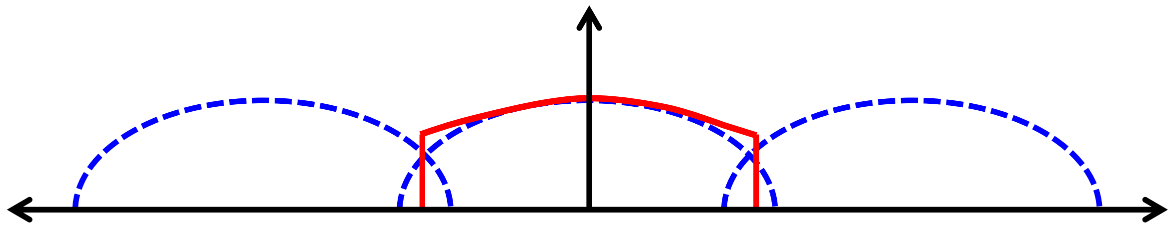
$=$



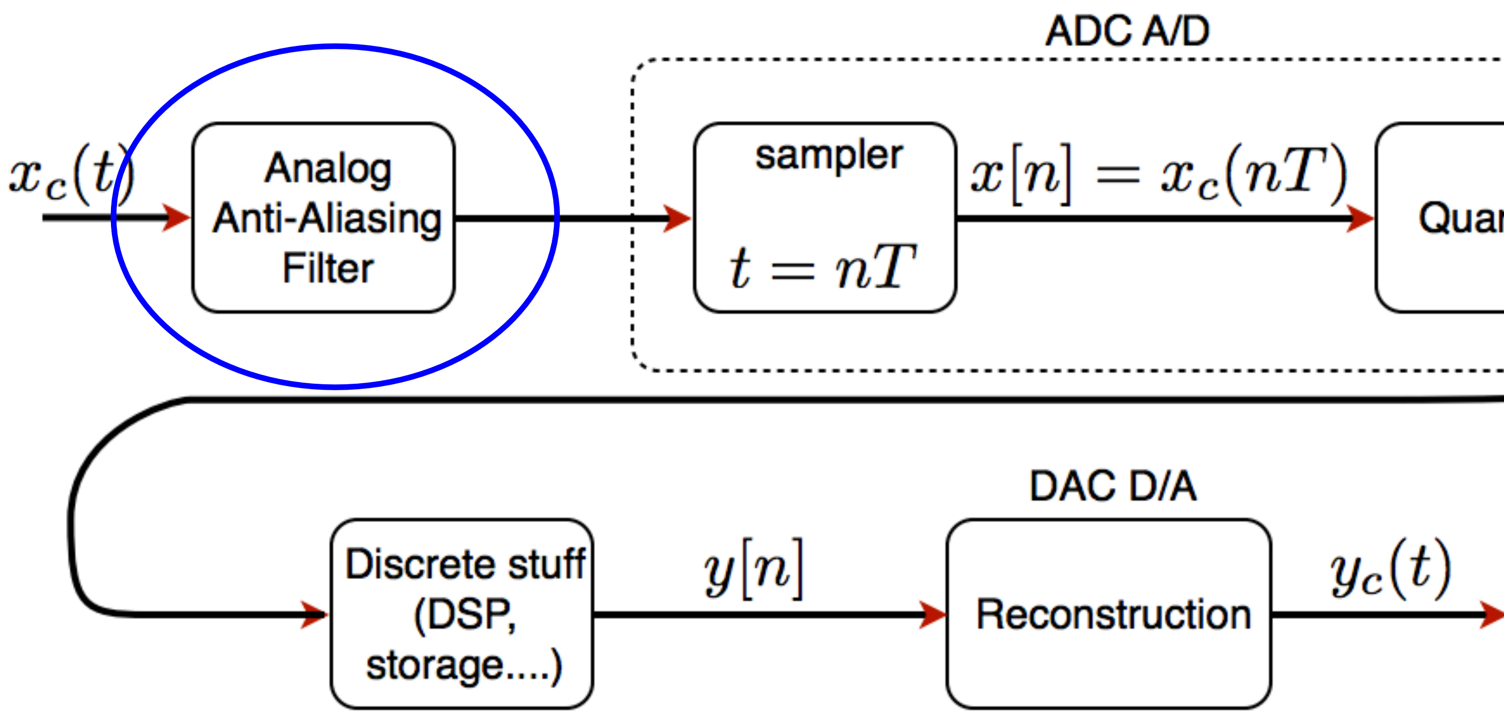
The sum of "sincs" gives $x_r(t)$ → unique signal that is bandlimited by sampling bandwidth

Aliasing

- If $\Omega_N > \Omega_s/2$, $x_r(t)$ is an aliased version of $x_c(t)$

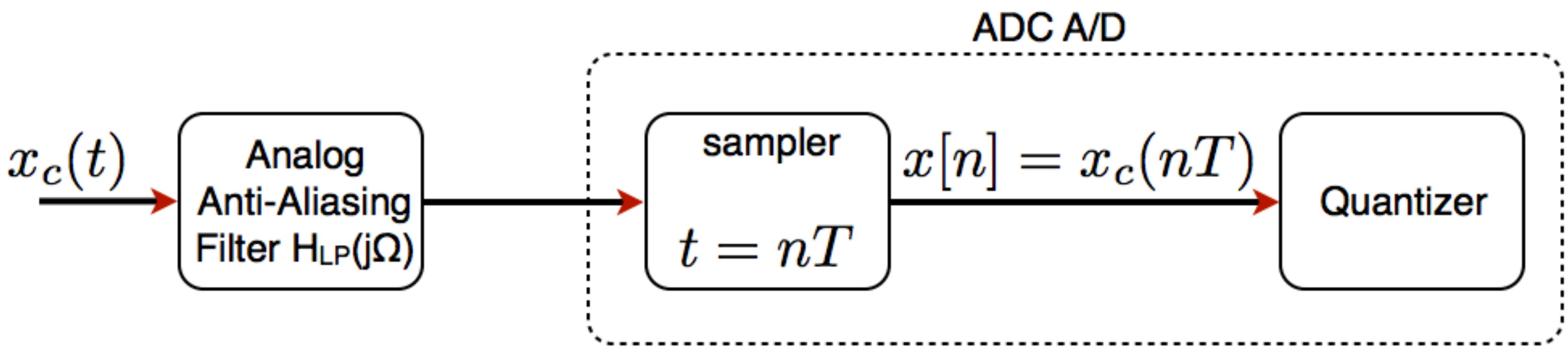


DSP System

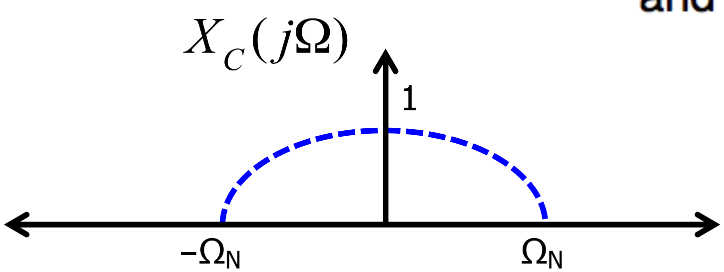


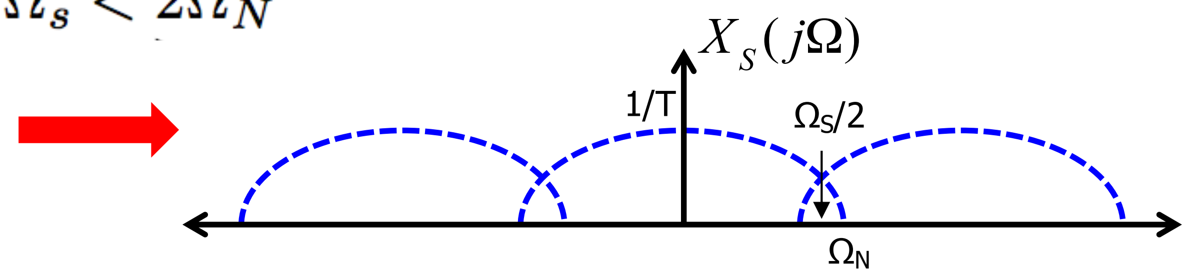
ADC A/D
$x_c(t)$
Analog Anti-Aliasing Filter
sampler
$t = nT$
$x[n] = x_c(nT)$
Quantizer
DAC D/A
Discrete stuff (DSP, storage....)
$y[n]$
Reconstruction
$y_c(t)$

Anti-Aliasing Filter

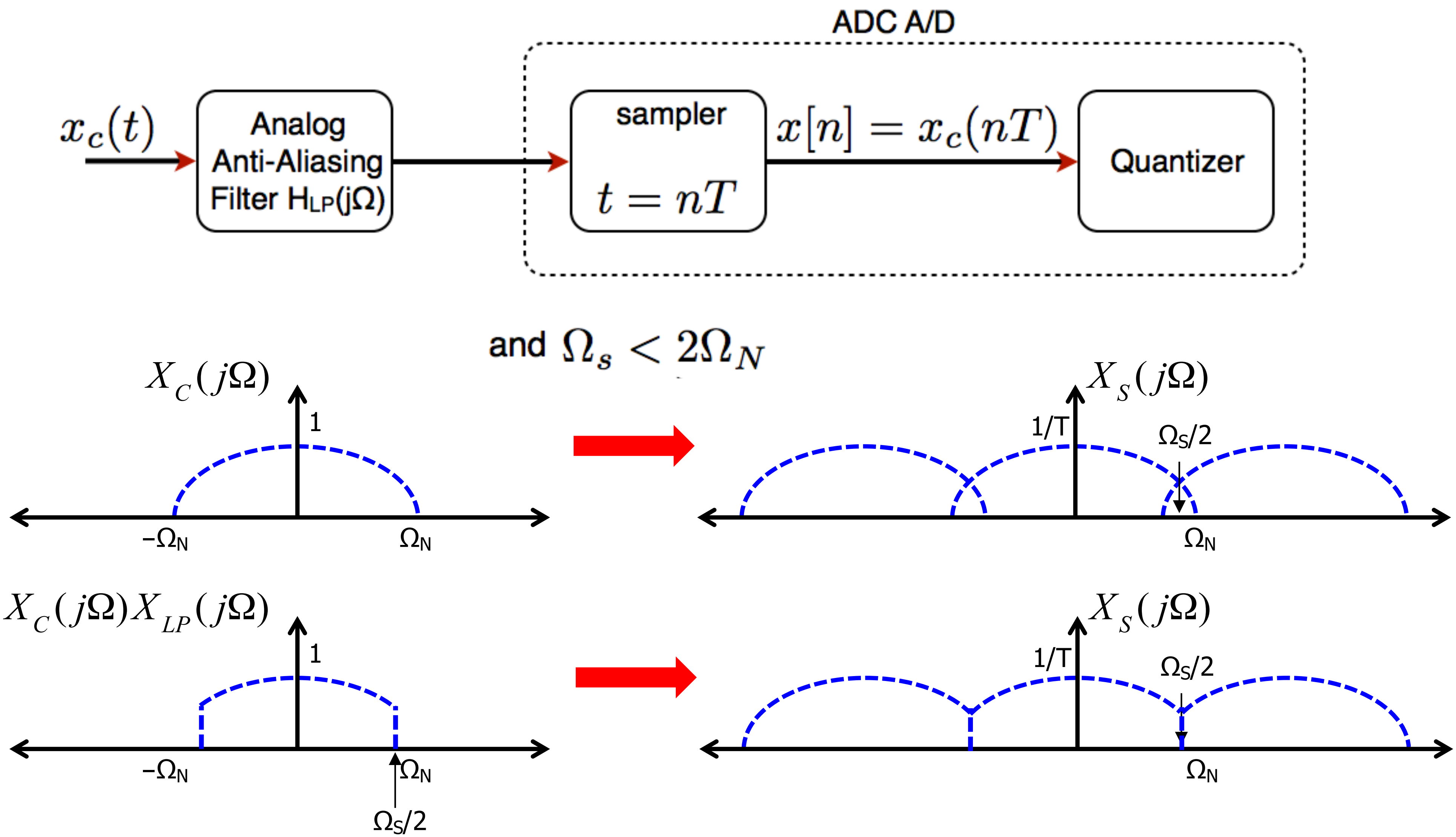


and $\Omega_s < 2\Omega_N$

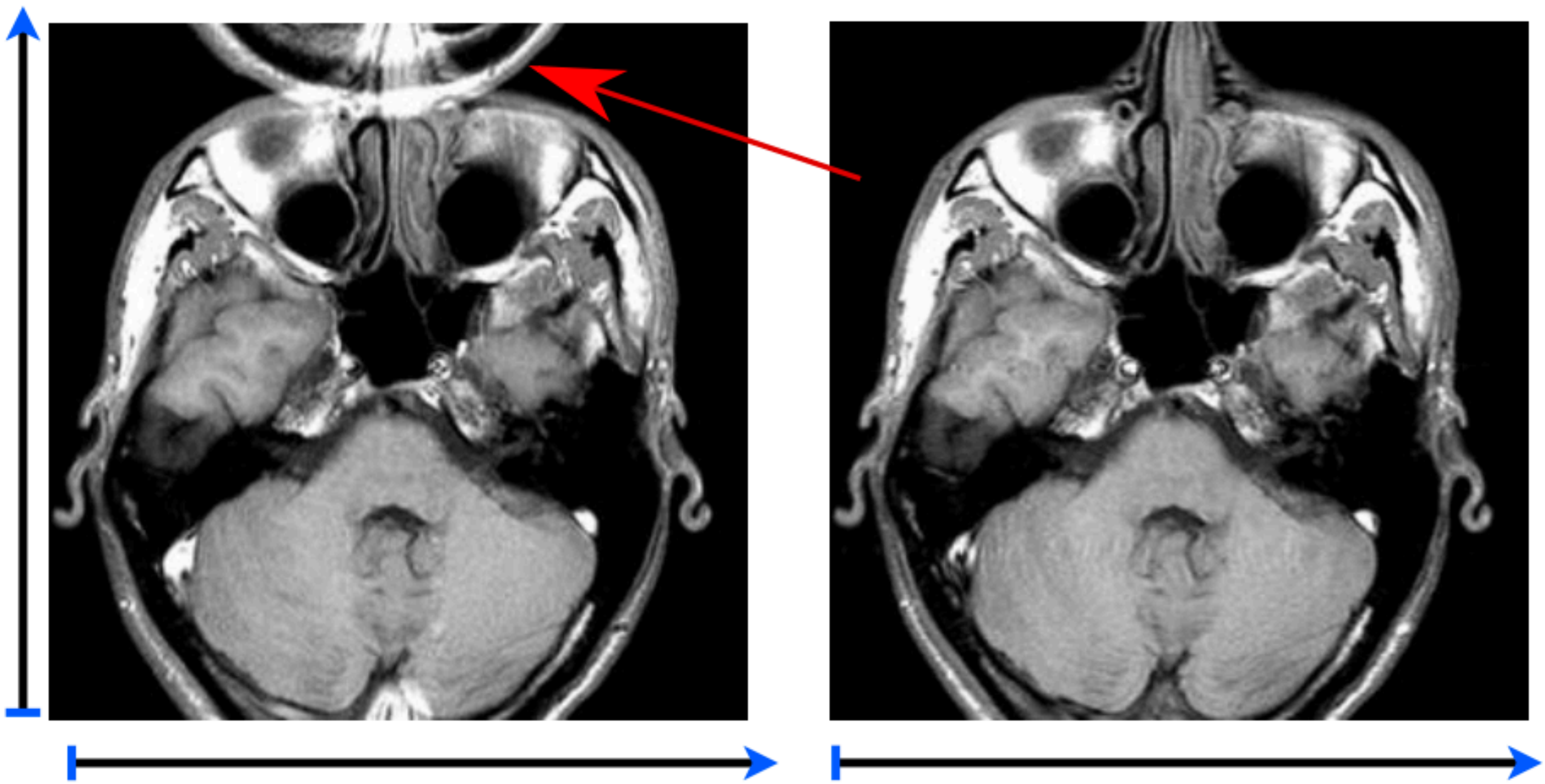




Anti-Aliasing Filter



MRI aliasing example



MRI anti-aliasing example

MRI anti-aliasing example

Big Ideas

- Sampling
  - Ideal sampling modeled as impulsive sampling
  - Sample at Nyquist rate for recovery of unique bandlimited signal (i.e. avoid aliasing)
- Frequency Response of Sampled Signal
  - Sampled signal is period replicated input CT signal
- Reconstruction
  - Low pass/reconstruction filter results in sum of sincs
- Anti-aliasing filtering
  - Force input signal to be bandlimited

Admin

- HW 2 out due Sunday 2/11 at midnight
- HW 3 posted 2/11