University of Pennsylvania
Department of Electrical and System Engineering
Digital Signal Processing

Midterm Thursday, March 21

- 4 Problems with point weightings shown. All 4 problems must be completed.
- Calculators allowed. (non cell phone)
- Closed book = No text allowed. One two-sided 8.5x11 cheat sheet allowed.

Name: Answers

Grade:

Q1	
Q2	
Q3	
Q4	
Total	Mean: 67.1, Stdev: 23.4

TABLE 2.3 FOURIER TRANSFORM PAIRS

Sequence	Fourier Transform
1. $\delta[n]$	1
2. $\delta[n-n_0]$	$e^{-j\omega n_0}$
3. $1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega+2\pi k)$
4. $a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{1-ae^{-j\omega}}$
5. $u[n]$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega+2\pi k)$
6. $(n+1)a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{(1-ae^{-j\omega})^2}$
7. $\frac{r^n \sin\omega_p(n+1)}{\sin\omega_p} u[n] \quad (\lvert r\rvert < 1)$	$\frac{1}{1-2r\cos\omega_p e^{-j\omega}+r^2e^{-j2\omega}}$
8. $\frac{\sin\omega_c n}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1, & \lvert\omega\rvert < \omega_c, \\ 0, & \omega_c < \lvert\omega\rvert \le \pi \end{cases}$
9. $x[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
10. $e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega-\omega_0+2\pi k)$
11. $\cos(\omega_0 n+\phi)$	$\sum_{k=-\infty}^{\infty} [\pi e^{j\phi}\delta(\omega-\omega_0+2\pi k)+\pi e^{-j\phi}\delta(\omega+\omega_0+2\pi k)]$

TABLE 2.2 FOURIER TRANSFORM THEOREMS

Sequence	Fourier Transform
$x[n]$	$X(e^{j\omega})$
$y[n]$	$Y(e^{j\omega})$
1. $ax[n]+by[n]$	$aX(e^{j\omega})+bY(e^{j\omega})$
2. $x[n-n_d] \quad (n_d$ an integer$)$	$e^{-j\omega n_d}X(e^{j\omega})$
3. $e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
4. $x[-n]$	$X(e^{-j\omega})$ $X^*(e^{j\omega})$ if $x[n]$ real.
5. $nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$
6. $x[n]*y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
7. $x[n]y[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$

Parseval's theorem:

8. $\sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \lvert X(e^{j\omega})\rvert^2 d\omega$

9. $\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$

TABLE 3.1 SOME COMMON z-TRANSFORM PAIRS

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
6. $-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
7. $na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
8. $-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
9. $\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $\sin(\omega_0 n)u[n]$	$\frac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $r^n\cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $r^n\sin(\omega_0 n)u[n]$	$\frac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\frac{1-a^Nz^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$

TABLE 3.2 SOME z-TRANSFORM PROPERTIES

Property Number	Section Reference	Sequence	Transform	ROC
		$x[n]$	$X(z)$	R_x
		$x_1[n]$	$X_1(z)$	R_{x_1}
		$x_2[n]$	$X_2(z)$	R_{x_2}
1	3.4.1	$ax_1[n]+bx_2[n]$	$aX_1(z)+bX_2(z)$	Contains $R_{x_1}\cap R_{x_2}$
2	3.4.2	$x[n-n_0]$	$z^{-n_0}X(z)$	R_x, except for the possible addition or deletion of the origin or ∞
3	3.4.3	$z_0^n x[n]$	$X(z/z_0)$	$\lvert z_0\rvert R_x$
4	3.4.4	$nx[n]$	$-z\frac{dX(z)}{dz}$	R_x
5	3.4.5	$x^*[n]$	$X^*(z^*)$	R_x
6		$\mathcal{R}e\{x[n]\}$	$\frac{1}{2}[X(z)+X^*(z^*)]$	Contains R_x
7		$\mathcal{I}m\{x[n]\}$	$\frac{1}{2j}[X(z)-X^*(z^*)]$	Contains R_x
8	3.4.6	$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
9	3.4.7	$x_1[n]*x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1}\cap R_{x_2}$

Trigonometric Identity:

$$e^{j\Theta} = cos(\Theta) + jsin(\Theta)$$

Geometric Series:

$$\sum_{n=0}^{N} r^n = \frac{1-r^{N+1}}{1-r}$$
$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}, \lvert r\rvert < 1$$

DTFT Equations:

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}$$
$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

Z-Transform Equations:

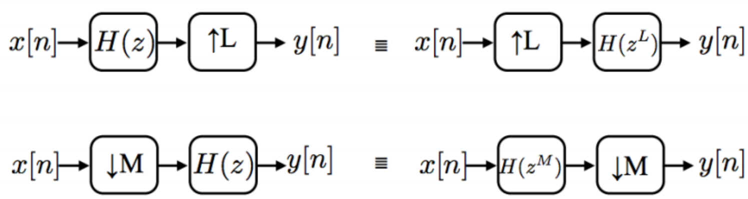
$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$
$$x[n] = \frac{1}{2\pi j} \oint_C X(z)z^{n-1}dz$$

Upsampling/Downsampling:

Upsampling by L (↑L): $X_{up} = X(e^{j\omega L})$

Downsampling by M (↓M): $X_{down} = \frac{1}{M} \sum_{i=0}^{M-1} X(e^{j(\frac{\omega}{M} - \frac{2\pi}{M} i)})$

Interchange Identities:



1. (20 points) Consider the system shown in Figure 1 for discrete-time processing of the continuous-time input signal $g_c(t)$. The input signal $g_c(t)$ is of the form $g_c(t) = f_c(t) + e_c(t)$. where the Fourier transforms of $f_c(t)$ and $e_c(t)$ are shown in Figure 2. Since the input signal is not bandlimited, a continuous-time antialiasing filter $H_{aa}(j\Omega)$ is used. The frequency response for $H_{aa}(j\Omega)$ is shown in Figure 3. Assume the C/D and D/C blocks are ideal as in class. Ideal means that the D/C converter contains an ideal lowpass reconstruction filter with a bandwidth of π/T and a gain of T.

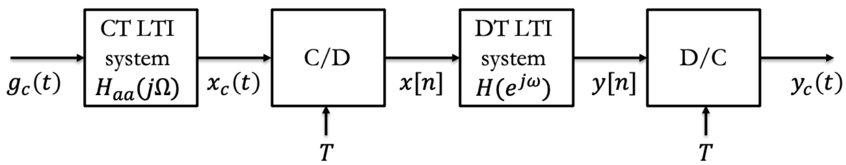


Figure 1

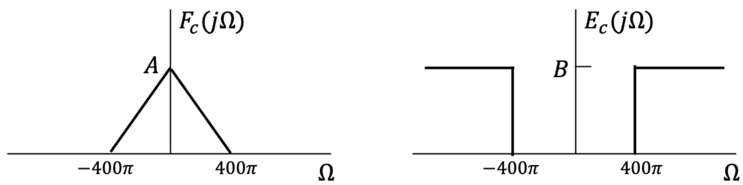


Figure 2

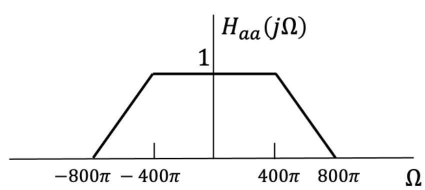
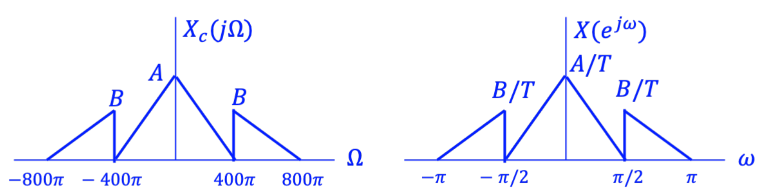


Figure 3

(a) If the sampling rate is $\Omega_S = 1600\pi$, determine $H(e^{j\omega})$, the frequency response of the discrete-time system, so that the output is $y_c(t) = f_c(t)$.

After the anti-aliasing filter and sampling, the $X_c(j\Omega)$ and $X(e^{j\omega})$ are as follows:



Therefore we just need a half-band LPF to keep $f_c(t)$ and filter out everything else before the D/C converter.

$$H(e^{j\omega}) = \begin{cases} 1 & \text{for } |\omega| \leq \frac{\pi}{2} \\ 0 & \text{else} \end{cases} \tag{1}$$

(b) It turns out that since we are only interested in obtaining $f_c(t)$ at the output, we can use a lower sampling rate than $\Omega_S = 1600\pi$ while still using the antialiasing filter in Figure 3. Determine the minimum value of Ω_S such that $y_c(t) = f_c(t)$? Determine $H(e^{j\omega})$ for this new choice of Ω_S.

Here we recognize that we can tolerate aliasing as long as we preserve $f_c(t)$. This occurs for $\Omega_S = 1200\pi$, such that for $\omega \leq \frac{2}{3}\pi$ we have the spectra of $f_c(t)$ uncorrupted. This leads to:

$$H(e^{j\omega}) = \begin{cases} 1 & \text{for } |\omega| \leq \frac{2}{3}\pi \\ 0 & \text{else} \end{cases} \tag{2}$$

2. (30 points) The discrete-time filtering system shown in Figure 4 comprises a C/D converter sampling at rate $f_1 = 2000Hz$, a filter with frequency response $H(e^{j\omega})$, a resampler that resamples at a rate of $D : U$ (downsample by D, and upsample by U) and an ideal D/C converter at rate f_2. Ideal means that the D/C converter contains an ideal lowpass reconstruction filter with a bandwidth of πf_2 and a gain of $1/f_2$. The spectrum of the input, $X(j\Omega)$, and frequency response, $H(e^{j\omega})$, are shown in Figure 5.

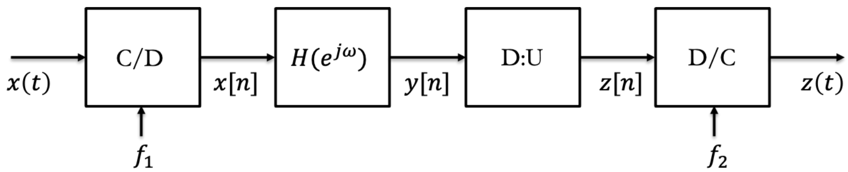


Figure 4

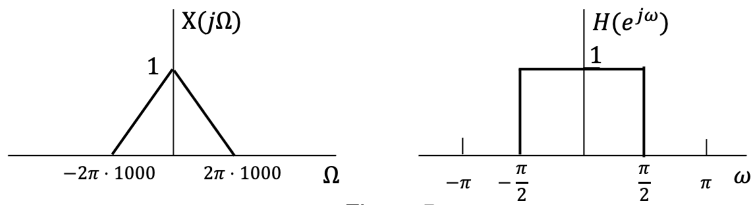


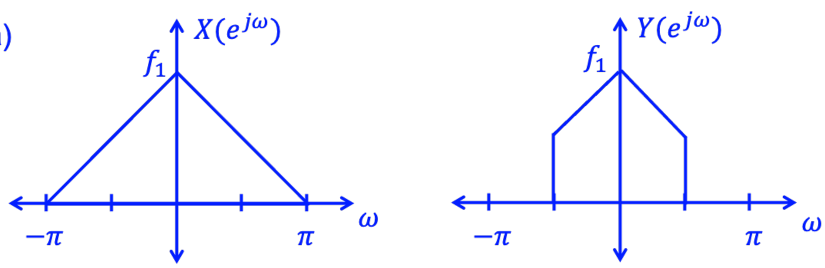
Figure 5

(a) Plot the spectra $X(e^{j\omega})$ and $Y(e^{j\omega})$. Label all axes and relevant features.

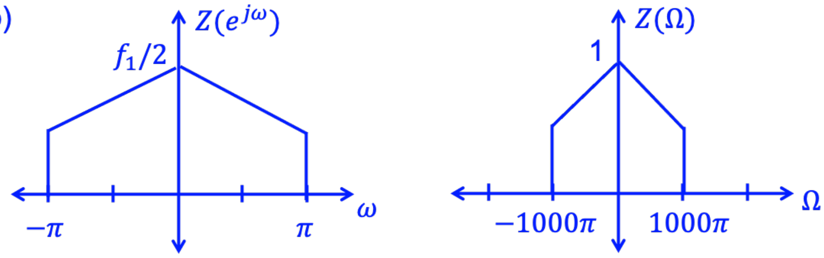
(b) Given $f_2 = 1000Hz$, $U = 1$, and $D = 2$, plot the spectra $Z(e^{j\omega})$ and $Z(j\Omega)$. Label all axes and relevant features.

(c) Given $f_2 = 4000Hz$, $U = 2$, and $D = 1$, plot the spectra $Z(e^{j\omega})$ and $Z(j\Omega)$. Label all axes and relevant features.

a)

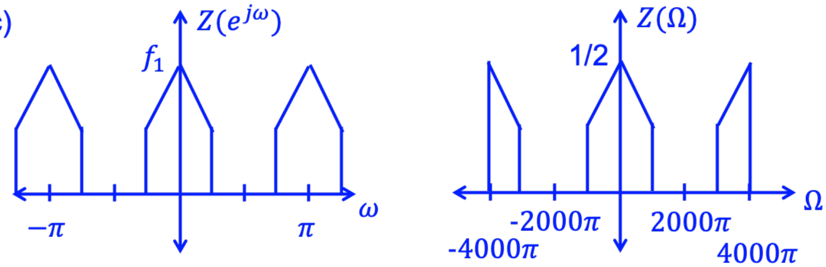


b)



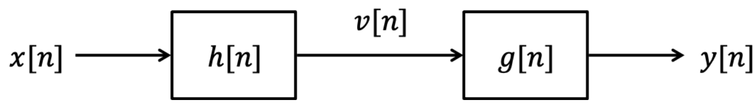
For part b, we are only downsampling, so we see the spectrum stretched in frequency before the D/C block.

c)



For part c, we are only upsampling, so we see the spectrum shrunk in frequency before the D/C block. Note the replica around 2π now lands at π.

3. (30 points) A stable system is shown below, comprising a cascade of two LTI discrete-time filters such that $y[n] = x[n]$.



The first filter has an unknown impulse response, $h[n]$. The second filter is defined by the difference equation:

$$y[n] = v[n] - \frac{3}{4}v[n-1] + \frac{1}{8}v[n-2] \tag{3}$$

For each filter $G(z)$ and $H(z)$ in the combined system:

(a) Find the transfer function and impulse response.

(b) Plot the filter pole-zero plot and indicate the ROC.

(c) Indicate if the filter is all-pass, min-phase, or neither. Explain your reasoning.

(a) From the difference equation, we can take the z-transform of both sides and solve:

$$Y(z) = V(z) - \frac{3}{4}Y(z)z^{-1} + \frac{1}{8}Y(z)z^{-2}$$
$$G(z) = \frac{Y(z)}{V(z)} = 1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}$$

Taking the inverse z-transform, we get $g[n] = \delta[n] - \frac{3}{4}\delta[n-1] + \frac{1}{8}\delta[n-2]$.

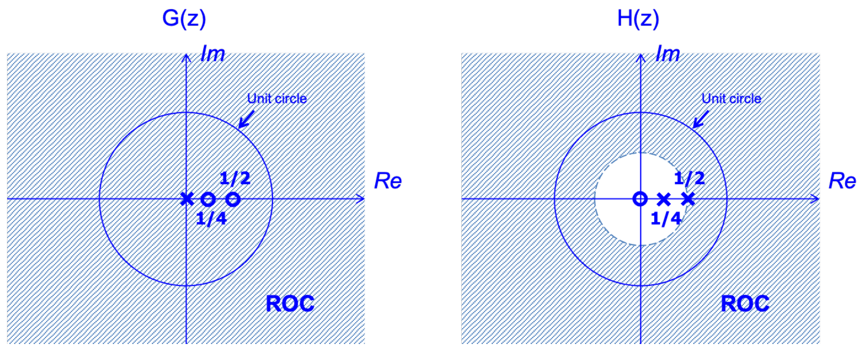
Since $y[n] = x[n]$, we know $1 = G(z) \times H(z)$ and can solve and use partial fractions to derive $H(z)$:

$$H(z) = \frac{1}{G(z)} = \frac{1}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}$$
$$H(z) = \frac{1}{(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{4}z^{-1})}$$
$$H(z) = \frac{2}{1 - \frac{1}{2}z^{-1}} + \frac{-1}{1 - \frac{1}{4}z^{-1}}$$

Taking the inverse z-transform and because we know the system is stable:

$$h[n] = 2\left(\frac{1}{2}\right)^n u[n] - \left(\frac{1}{4}\right)^n u[n]$$

(b) Pole-zero plot of each filter:



(c) Both systems are min-phase because all their poles and zeros are inside the unit circle.

4. (20 points) For each $H(z)$ in the table below, select the matching magnitude response from the magnitude plots M1-M6 also shown below. It is possible that some pictures may apply to more than one system, and some pictures will not apply to any system. If none of the pictures is possible, write none.

The easiest way to do this was to plug in $z = \pm 1$ and match accordingly.

$H(z)$	**Magnitude Plot**
$z - 2$	M1
$\frac{3}{2}\frac{1+jz}{z-j}$	M3
$2(z + \frac{1}{2})$	M4
$\frac{3}{2}\frac{z+1}{2z-1}$	M6
$\frac{3}{2}(z-1)$	M5
$z + 2$	M4