Pareto-Optimal Learning Algorithms for Repeated Games

Penn Theory Seminar

Eshwar Ram Arunachaleswaran, Natalie Collina, Jon Schneider

February 20, 2024

University of Pennsylvania, Google Research

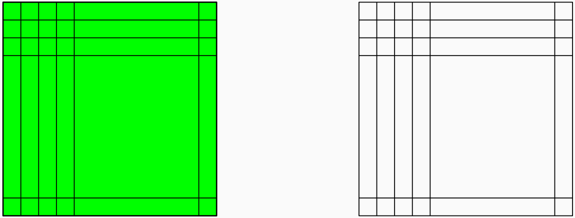
Table of contents

Intro

Some Vibes

What is a good algorithm to commit to in a repeated 2-player game?

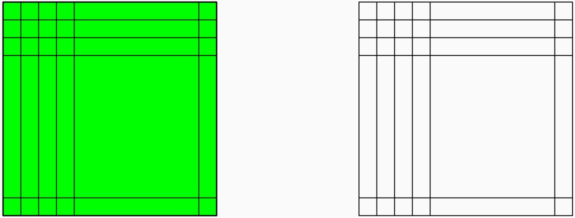
(Bimatrix game, linear payoff functions)



What is a good algorithm to commit to in a repeated 2-player game?

Assumption

The other player, called an optimizer, knows your algorithm and will best-respond (non-myopically).



Some Vibes

What is a good algorithm to commit to in a repeated 2-player game?

Full Information

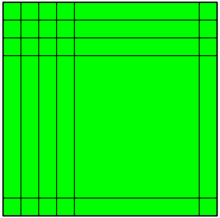
Knowing the optimizer's payoff means we can design optimal algorithms to play with (Stackelberg).

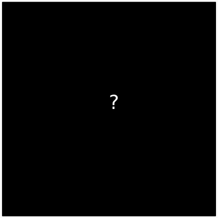
Some Vibes

What is a good algorithm to commit to in a repeated 2-player game?

Assumption

You do not know the optimizer's payoffs.



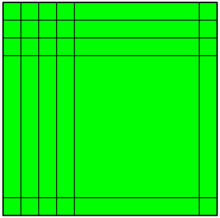


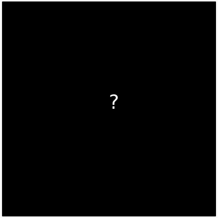
Some Vibes

What is a good algorithm to commit to in a repeated 2-player game?

Our Setting

Starting with no information with the other player, what is a reasonable guarantee to ask for?





Some Vibes

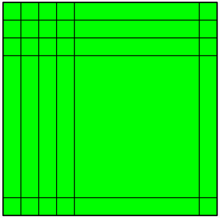
What is a good algorithm to commit to in a repeated 2-player game?

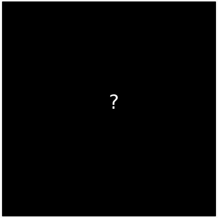
Our Setting

Starting with no information with the other player, what is a reasonable guarantee to ask for?

Optimistic

Pointwise (over all optimizers) optimality





Some Vibes

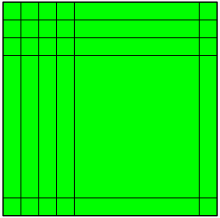
What is a good algorithm to commit to in a repeated 2-player game?

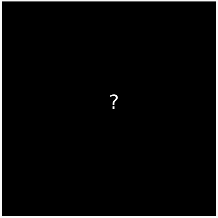
Our Setting

Starting with no information with the other player, what is a reasonable guarantee to ask for?

Pessimistic

The maximin value, on average.





Some Vibes

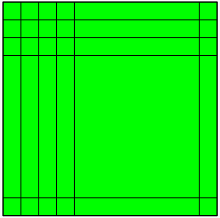
What is a good algorithm to commit to in a repeated 2-player game?

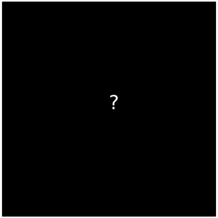
Our Setting

Starting with no information with the other player, what is a reasonable guarantee to ask for?

A Little Less Pessimistic

Low Regret on every transcript.





Some Vibes

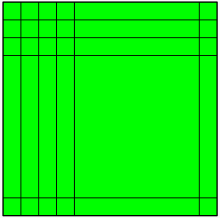
What is a good algorithm to commit to in a repeated 2-player game?

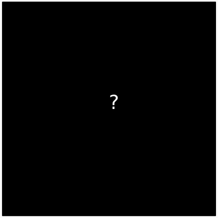
Our Setting

Starting with no information with the other player, what is a reasonable guarantee to ask for?

Our answer

Pareto-Optimality (based on a Partial Ordering over Algorithms) and No-Regret.





Some Vibes

Pareto Optimality

A property of algorithms based upon a partial order over algorithms.Two Algorithms A and B are compared over all possible optimizer payoffs

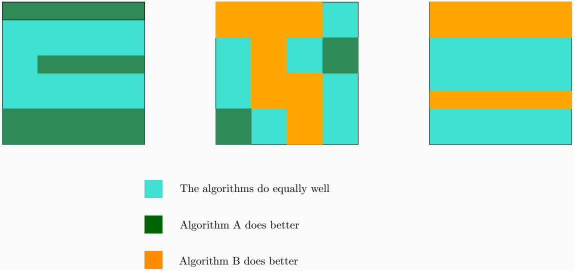


Figure 1: Space of Optimizer Payoffs : Three Scenarios

Overview of Main Results

Main Results

- All No-Swap-Regret Algorithms are Pareto-optimal.

Overview of Main Results

Main Results

- All No-Swap-Regret Algorithms are Pareto-optimal.
- Not all No-Regret algorithms are Pareto-optimal.

Overview of Main Results

Main Results

- All No-Swap-Regret Algorithms are Pareto-optimal.
- Not all No-Regret algorithms are Pareto-optimal. Specifically, Follow-the-Regularized-Leader (FTRL) based algorithms (which includes Multiplicative Weights Update, Online Gradient Descent) are Pareto-dominated.

Other Results/ Questions :

Other Results

- A Geometric View of Algorithms
- A characterization of best-responses to a no-regret algorithm
- A characterization of Pareto-optimal No-Regret Algorithms

Model

Model

Two players - Learner and Optimizer

Model

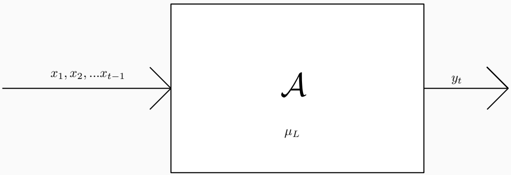
In Each round

- The Learner has an action set Δ_n
- The Optimizer has an action set Δ_m
- They play actions x_t, y_t in the t-th round
- Linear utliity functions u_L, u_O

Model : Learning Algorithms

The Learner Perspective

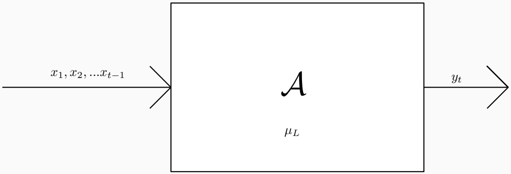
Without seeing u_O, the Learner commits to an algorithm $\mathcal{A}$ mapping (deterministic) from histories of play of length $t-1$ to distributions over actions y_t in round t.



Model: Learning Algorithms

The Learner Perspective

Without seeing u_O, the Learner commits to an algorithm $\mathcal{A}$ mapping (deterministic) from histories of play of length $t-1$ to distributions over actions y_t in round t.



The resulting transcript of play is $(x_1, y_1), (x_2, y_2) \cdots (x_t, y_t)$.

Model : No-Regret

No-Regret

Without seeing u_O, the Learner commits to an algorithm $\mathcal{A}$ mapping (deterministic) from histories of play of length $t-1$ to distributions over actions y_t in round t.

$$\sum_{t=1}^{T} u_L(x_t, y_t) \geq \left(\max_{y^* \in [n]} \sum_{t=1}^{T} u_L(x_t, y^*) \right) - o(T).$$

Model : No-Swap-Regret

No-Regret

A learning algorithm $\mathcal{A}$ is a *no-swap-regret algorithm* if it is the case that, regardless of the sequence of actions $(x_1, x_2, \ldots, x_T)$ taken by the optimizer, the learner's utility satisfies

$$\sum_{t=1}^{T} u_L(x_t, y_t) \geq \max_{\pi:[n]\to[n]} \sum_{t=1}^{T} u_L(x_t, \pi(y_t)) - o(T).$$

Model : No-Regret and No-Swap-Regret

No-Regret and No-Swap-Regret algorithms are known to exist.

Model : Mean-Based Algorithms

Only moves within $o(T)$ being the historical best-response action get non-trivial, i.e., $\Omega_T(1)$ mass.

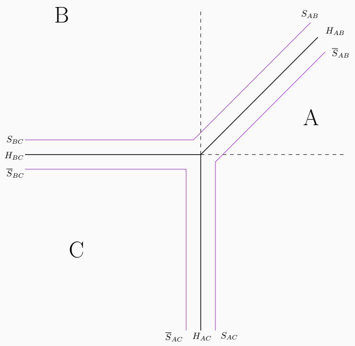


Figure 2: Space of Cumulative Payoff Vectors

Model : Mean-Based Algorithms

Only moves within $o(T)$ of being the historical best-response action get non-trivial, i.e., $\Omega_T(1)$ mass.

Examples of Mean-Based Algorithms

MWU, FTPL, OGD are all mean-based.

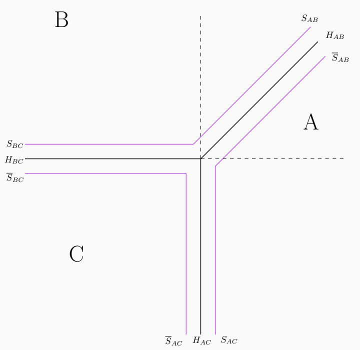


Figure 3: Space of Cumulative Payoff Vectors

Model : Follow-the-Regularized-Leader (FTRL)

Given that R is continuous and strongly-convex, and $\eta_T = \frac{1}{o(T)}$:

$$y_t = \arg\max_{y \in \Delta^n} \left(\sum_{s=1}^{t-1} u_L(x_s, y) - \frac{R(y)}{\eta_T} \right)$$

Model : Follow-the-Regularized-Leader (FTRL)

Given that R is continuous and strongly-convex, and $\eta_T = \frac{1}{o(T)}$:

$$y_t = \arg\max_{y \in \Delta^n} \left(\sum_{s=1}^{t-1} u_L(x_s, y) - \frac{R(y)}{\eta_T} \right)$$

Examples of FTRL Algorithms

MWU, FTPL, OGD.

Model

The Optimizer Perspective

With full information (payoffs, learner algorithm), the optimizer plays a best-response sequence

Model : Optimizer Behaviour

The Optimizer Perspective

With full information (payoffs, learner algorithm), the optimizer plays a best-response sequence [1].

$$x_1, x_2 \cdots x_T \in \underset{(x_1, x_2 \cdots x_T) \in \Delta_m^T}{\arg\max} \frac{1}{T} \sum_{t=1}^{T} u_O(x_t, y_t)$$

where $y_t = \mathcal{A}(x_1, x_2 \cdots x_{t-1})$

[1] Tie-breaking in favor of the learner.

Model : Optimizer Behaviour

The Optimizer Perspective

With full information (payoffs, learner algorithm), the optimizer plays a best-response sequence of actions [2], [3].

$$x_1, x_2 \cdots x_T \in \underset{(x_1, x_2 \cdots x_T) \in \Delta_m^T}{\arg\max} \frac{1}{T} \sum_{t=1}^{T} u_O(x_t, y_t)$$

where $y_t = \mathcal{A}(x_1, x_2 \cdots x_{t-1})$

[2]Tie-breaking in favor of the learner.
[3]Cheating slightly here!

Model : Optimizer Behaviour

The Optimizer Perspective

With full information (payoffs, learner algorithm), the optimizer plays a best-response sequence of actions

$$x_1, x_2 \cdots x_T \in \underset{(x_1, x_2 \cdots x_T) \in \Delta_m^T}{\arg\max} \frac{1}{T} \sum_{t=1}^{T} u_O(x_t, y_t)$$

The learner gets payoff $V_L(\mathcal{A}, u_O, T) = \frac{1}{T} \sum_{t=1}^{T} u_O(x_t, y_t)$

Model : Asymptotics

Limit Payoffs

- The learner's limit payoff is $V_L(\mathcal{A}, u_O) = \lim_{T \to \infty} V_L(\mathcal{A}, u_O, T)$.

Model : Asymptotics

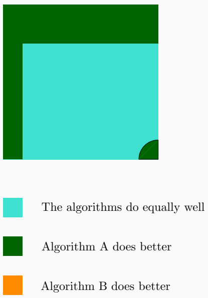
Limit Payoffs

- The learner's limit payoff is $V_L(\mathcal{A}, u_O) = \lim_{T \to \infty} V_L(\mathcal{A}, u_O, T)$.
- Motivation : Do not care about $o_T(1)$ differences in average payoff.

Model : Pareto-Domination

Algorithm $\mathcal{A}$ dominates algorithm $\mathcal{B}$ for some payoff u_L if:

- For all μ_O : $V_L(\mathcal{A}, u_O) \geq V_L(\mathcal{B}, u_O)$.
- $\exists \mu_O$ s.t. $V_L(\mathcal{A}, u_O) > V_L(\mathcal{B}, u_O)$ [4].



[4] In fact equivalent to a positive measure set

Model : Pareto-Domination

Algorithm $\mathcal{A}$ dominates algorithm $\mathcal{B}$ for some payoff u_L if:

- For all μ_O : $V_L(\mathcal{A}, u_O) \geq V_L(\mathcal{B}, u_O)$.
- $\exists \mu_O$ s.t. $V_L(\mathcal{A}, u_O) > V_L(\mathcal{B}, u_O)$ [5].

All our Pareto-domination results are for a positive-measure set of learner payoffs.

[5] In fact equivalent to a positive measure set

Model : Pareto-Optimality

Pareto-optimality of Algorithms

Algorithm $\mathcal{A}$ is Pareto-optimal if it is not Pareto-dominated by any other algorithm $\mathcal{B}$.

Related Work

- Learning in Games - [BSV24], [DSS19], [MMSS22]
- Stackelberg Equilibria in Repeated Games - [CAK23], [HLNW22]

William Brown, Jon Schneider, and Kiran Vodrahalli.
Is learning in games good for the learners?
Advances in Neural Information Processing Systems, 36, 2024.

Natalie Collina, Eshwar Ram Arunachaleswaran, and Michael Kearns.
Efficient stackelberg strategies for finitely repeated games.
In *Proceedings of the 2023 International Conference on Autonomous Agents and Multiagent Systems*, pages 643–651, 2023.

Yuan Deng, Jon Schneider, and Balasubramanian Sivan.
Strategizing against no-regret learners.
Advances in neural information processing systems, 32, 2019.

- Nika Haghtalab, Thodoris Lykouris, Sloan Nietert, and Alexander Wei.
  Learning in stackelberg games with non-myopic agents.
  In *Proceedings of the 23rd ACM Conference on Economics and Computation*, pages 917–918, 2022.
- Yishay Mansour, Mehryar Mohri, Jon Schneider, and Balasubramanian Sivan.
  Strategizing against learners in bayesian games.
  In *Conference on Learning Theory*, pages 5221–5252. PMLR, 2022.

Menus

Menus : The Optimizer Best-Response, An Alternate View

Correlated Strategy Pairs (CSPs)

Consider all possible distribution of action pairs generated over sequences over optimizers.

Menus : The Optimizer Best-Response, An Alternate View

Correlated Strategy Pairs (CSPs)

Consider all possible distribution of action pairs generated over sequences over optimizers.

$$\left\{\varphi \in \Delta_{mn} : \exists x_1, x_2 \cdots x_T \text{ s.t. } \varphi = \frac{1}{T}\sum_{t=1}^{T} x_t \otimes y_t\right\}$$

Menus : The Optimizer Best-Response, An Alternate View

Correlated Strategy Pairs (CSPs)

Consider all possible distribution of action pairs generated over sequences over optimizers.

$$\left\{\varphi \in \Delta_{mn} : \exists x_1, x_2 \cdots x_T \text{ s.t. } \varphi = \frac{1}{T}\sum_{t=1}^{T} x_t \otimes y_t\right\}$$

Take their convex hull and call this set the menu $\mathcal{M}(\mathcal{A}_T)$.

Menus : The Optimizer Best-Response, An Alternate View

Correlated Strategy Pairs (CSPs)

Consider all possible distribution of action pairs generated over sequences over optimizers.

Take their convex hull and call this set the menu $\mathcal{M}(\mathcal{A}_T)$.

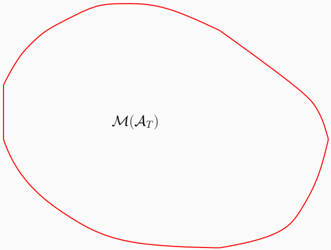


Figure 4: A Simple Menu

Menus : The Optimizer Best-Response, An Alternate View

Correlated Strategy Pairs (CSPs)

- Consider all possible distribution of action pairs generated over sequences over optimizers.
- Take their convex hull and call this set the menu $\mathcal{M}(\mathcal{A}_T)$.

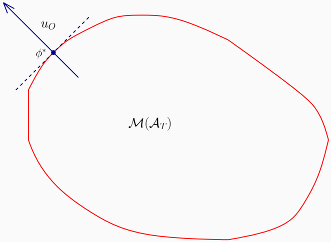


Figure 5: An Optimizer's Choice on a Simple Menu

Menus : The Optimizer Best-Response, An Alternate View

Recall that the learner's limit payoff is $V_L(\mathcal{A}, u_O) = \lim_{T\to\infty} V_L(\mathcal{A}, u_O, T)$.

Menus : The Optimizer Best-Response, An Alternate View

- Recall that the learner's limit payoff is $V_L(\mathcal{A}, u_O) = \lim_{T\to\infty} V_L(\mathcal{A}, u_O, T)$.
- So, we would have to optimize over an infinite sequence of menus and take the limit.

Menus : The Optimizer Best-Response, An Alternate View

- Recall that the learner's limit payoff is $V_L(\mathcal{A}, u_O) = \lim_{T\to\infty} V_L(\mathcal{A}, u_O, T)$.
- So, we would have to optimize over an infinite sequence of menus and take the limit.
- Instead, take the limit menu and optimize over it!

Menus : The limit Menu Suffices

- So, we would have to optimize over an infinite sequence of menus and take the limit.
- Instead, take the limit menu and optimize over it!
- The limit menu is defined as $\mathcal{M}(\mathcal{A}) = \lim_{T\to\infty} \mathcal{M}(\mathcal{A}_T)$.

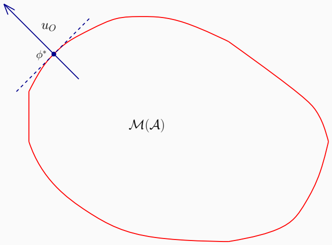


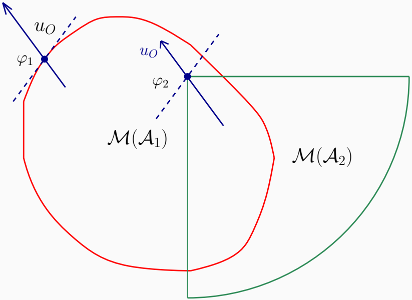
Figure 6: An Optimizer's Choice on the limit Menu

Menus are all you need

Comparing two algorithms $\mathcal{A}_1$ and $\mathcal{A}_2$ for a given u_O:

Key Idea

The learner (and optimizer) payoffs can be entirely inferred from the limit menus.

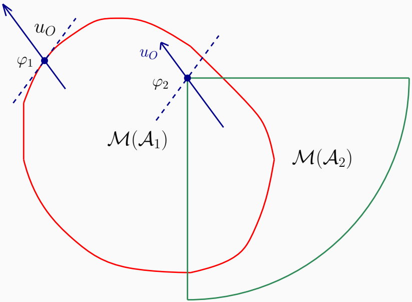


Menus are all you need

Comparing two algorithms $\mathcal{A}_1$ and $\mathcal{A}_2$ for a given u_O:

Key Idea

Algorithms can be replaced by their limit menus while discussing Pareto-domination (and optimality).



Menus : Examples

$$\begin{array}{c} \\ P \\ Q \end{array} \begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{bmatrix} X & X \\ X & X \end{bmatrix} \end{array}$$

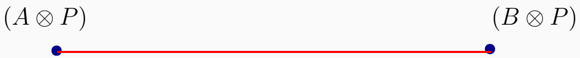
Menus: Examples

$$\begin{array}{c} \\ P \\ Q \end{array}\begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{bmatrix} X & X \\ X & X \end{bmatrix} \end{array}$$

Learning Algorithm $\mathcal{A}_1$: Always play P

Menus: Examples

Learning Algorithm $\mathcal{A}_1$: Always play P



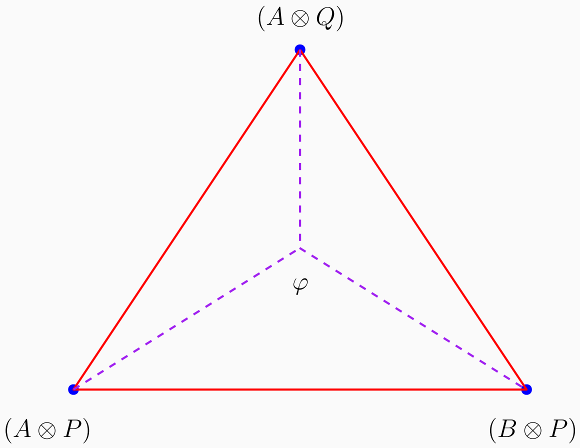
Menus: Examples

$$\begin{array}{c} \\ P \\ Q \end{array}\begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{bmatrix} X & X \\ X & X \end{bmatrix} \end{array}$$

Learning Algorithm $\mathcal{A}_2$: Play Q as long as the Optimizer has always played A. Otherwise, play P.

Menus: Examples

Learning Algorithm $\mathcal{A}_2$: Play Q as long as the Optimizer has always played A. Otherwise, play P.



What do menus look like in general?

What do menus look like in general?

Approachable Sets

A set S is approachable if, for every $x \in \Delta_m$, there exists a $y \in \Delta_n$ such that $x \otimes y \in S$.

What do menus look like in general?

Approachable Sets

A set S is approachable if, for every $x \in \Delta_m$, there exists a $y \in \Delta_n$ such that $x \otimes y \in S$.

Theorem

A closed, convex subset $\mathcal{M} \subseteq \Delta_{mn}$ is an limit menu iff it is approachable.

Menu Properties

Approachable Sets

A set S is approachable if, for every $x \in \Delta_m$, there exists a $y \in \Delta_n$ such that $x \otimes y \in S$.

- For every convex approachable set S, there is some $\mathcal{M} \subseteq S$ which is a valid menu.

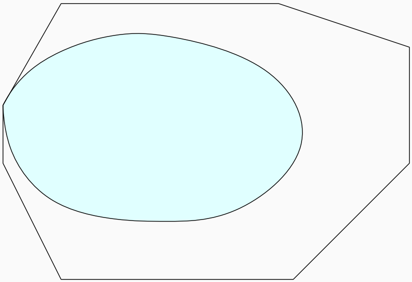
Menu Properties

Approachable Sets

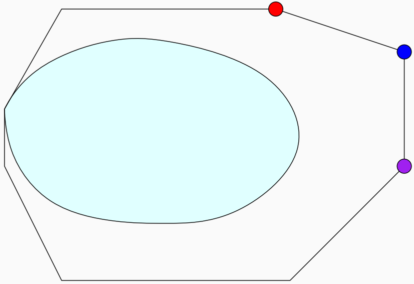
A set S is approachable if, for every $x \in \Delta_m$, there exists a $y \in \Delta_n$ such that $x \otimes y \in S$.

- For every convex approachable set S, there is some $\mathcal{M} \subseteq S$ which is a valid menu
- Menus are Upwards-Closed

Upwards Closedness



Upwards Closedness



Menu Properties

Approachable Sets

A set S is approachable if, for every $x \in \Delta_m$, there exists a $y \in \Delta_n$ such that $x \otimes y \in S$.

- For every convex approachable set S, there is some $\mathcal{M} \subseteq S$ which is a valid menu
- Menus are Upwards-Closed

Putting these together:
Every approachable set S is a valid menu

Menu Characterization

Approachable Sets

A set S is approachable if, for every $x \in \Delta_m$, there exists a $y \in \Delta_n$ such that $x \otimes y \in S$.

Theorem

A closed, convex subset $\mathcal{M} \subseteq \Delta_{mn}$ is an limit menu iff it is approachable.

No(-Swap)-Regret Redux

No(-Swap)-Regret is a property of just the CSPs.

No(-Swap)-Regret Redux

No(-Swap)-Regret is a property of just the CSPs.

$$\sum_{t=1}^{T} u_L(x_t, y_t) \geq \max_{\pi:[n]\to[n]} \sum_{t=1}^{T} u_L(x_t, \pi(y_t)).$$

No(-Swap)-Regret Redux

No(-Swap)-Regret is a property of just the CSPs.

A CSP φ is *no-swap-regret* if, for each $j \in [n]$, it satisfies

$$\sum_{i \in [m]} \varphi_{ij} u_L(i, j) \geq \max_{j^* \in [n]} \sum_{i \in [m]} \varphi_{ij} u_L(i, j^*).$$

where $\varphi = \frac{1}{T} \sum_{t=1}^{T} x_t \otimes y_t$.

No(-Swap)-Regret Redux

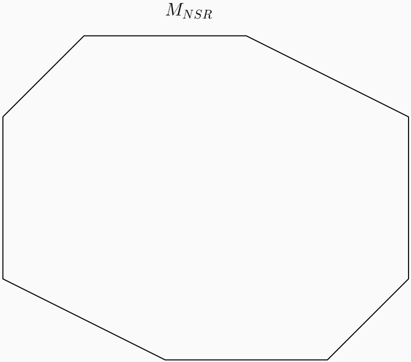
A natural set of CSPs vis-a-vis no-regret:

$\mathcal{M}_{NSR}$ is the set of all CSPs that are no-swap-regret.

A natural set of CSPs vis-a-vis no-regret:

$\mathcal{M}_{NSR}$ is the set of all CSPs that are no-swap-regret.

Observation
M_{NSR} is a polytope.

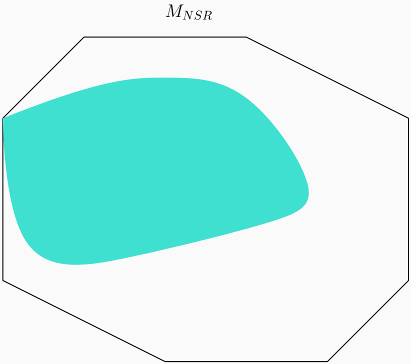


No(-Swap)-Regret Redux

A natural set of CSPs vis-a-vis no-regret:

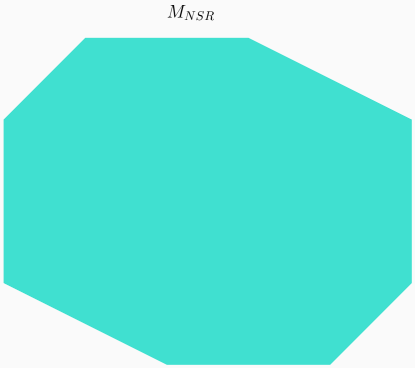
$\mathcal{M}_{NR}$ is the set of all CSPs that are no-regret.

Observation
The limit menu $\mathcal{M}$ of any no-swap-regret algorithm is contained in $\mathcal{M}_{NSR}$.



Third Main Result

Theorem
All no-swap-regret algorithms $\mathcal{A}$ have the same limit menu, which is $\mathcal{M}_{NSR}$.



Third Main Result

Theorem
All no-swap-regret algorithms $\mathcal{A}$ have the same limit menu, which is $\mathcal{M}_{NSR}$.

Particularly interesting in the context of multiple, seemingly different, approaches to NSR algorithms.

No-Swap-Regret Algorithms are Pareto-Optimal

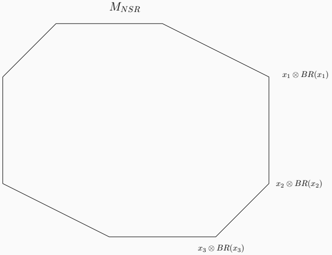
No Swap Regret Algorithms are Pareto-Optimal

Theorem
All no-swap-regret algorithms $\mathcal{A}$ have the same limit menu, which is $\mathcal{M}_{NSR}$.

No Swap Regret Algorithms are Pareto-Optimal

Theorem
All no-swap-regret algorithms $\mathcal{A}$ have the same limit menu, which is $\mathcal{M}_{NSR}$.

Theorem: $\mathcal{M}_{NSR}$ Characterization
$\mathcal{M}_{NSR}$ is the convex hull of all CSPs of the form $x \otimes y$, with $x \in \Delta_m$ and $y \in \mathrm{BR}_L(x)$.



No Swap Regret Algorithms are Pareto-Optimal

Theorem
All no-swap-regret algorithms $\mathcal{A}$ have the same limit menu, which is $\mathcal{M}_{NSR}$.

Theorem: $\mathcal{M}_{NSR}$ Characterization
$\mathcal{M}_{NSR}$ is the convex hull of all CSPs of the form $x \otimes y$, with $x \in \Delta_m$ and $y \in \mathrm{BR}_L(x)$.

Theorem: $\mathcal{M}_{NSR}$ Minimality
$\mathcal{M}_{NSR}$ is inclusion-minimal and includes φ^+.

No Swap Regret Algorithms are Pareto-Optimal

Theorem
All no-swap-regret algorithms $\mathcal{A}$ have the same limit menu, which is $\mathcal{M}_{NSR}$.

Theorem: $\mathcal{M}_{NSR}$ Characterization
$\mathcal{M}_{NSR}$ is the convex hull of all CSPs of the form $x \otimes y$, with $x \in \Delta_m$ and $y \in \mathrm{BR}_L(x)$.

Theorem: $\mathcal{M}_{NSR}$ Minimality
$\mathcal{M}_{NSR}$ is inclusion-minimal and includes φ^+.

Theorem: φ^+-minimality implies optimality
Every inclusion-minimal menu that contains u_L^+ is pareto-optimal.

No Swap Regret Algorithms are Pareto-Optimal

Theorem
All no-swap-regret algorithms $\mathcal{A}$ have the same limit menu, which is $\mathcal{M}_{NSR}$.

Theorem: $\mathcal{M}_{NSR}$ Characterization
$\mathcal{M}_{NSR}$ is the convex hull of all CSPs of the form $x \otimes y$, with $x \in \Delta_m$ and $y \in \mathrm{BR}_L(x)$.

Theorem: $\mathcal{M}_{NSR}$ Minimality
$\mathcal{M}_{NSR}$ is inclusion-minimal and includes φ^+.

Theorem: φ^+-minimality implies optimality
Every inclusion-minimal menu that contains u_L^+ is pareto-optimal.

Hey what the heck are these new definitions

Definition: Inclusion-Minimality

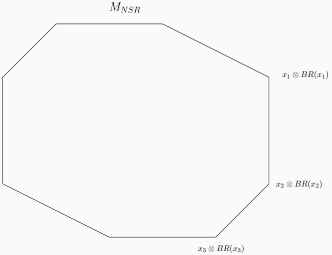
A menu $\mathcal{M}_1$ is *inclusion-minimal* if there is no menu $\mathcal{M}_2$ such that $\mathcal{M}_2 \subsetneq \mathcal{M}_1$.

Definition: φ^+

$u_L^+ = x^* \otimes y^*$, where $(x^*, y^*) = \arg\max_{(x,y)} u_L(x, y)$.

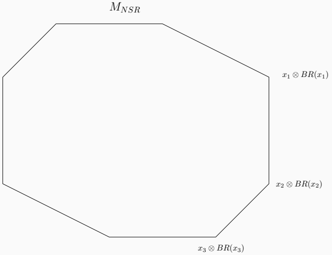
NSR is minimal

Recall: $\mathcal{M}_{NSR}$ is the convex hull of all CSPs of the form $x \otimes y$, with $x \in \Delta_m$ and $y \in \text{BR}_L(x)$.



NSR includes φ^+

Recall: $\mathcal{M}_{NSR}$ is the convex hull of all CSPs of the form $x \otimes y$, with $x \in \Delta_m$ and $y \in \text{BR}_L(x)$.



No Swap Regret Algorithms are Pareto-Optimal

Theorem
All no-swap-regret algorithms $\mathcal{A}$ have the same limit menu, which is $\mathcal{M}_{NSR}$.

Theorem: $\mathcal{M}_{NSR}$ Characterization
$\mathcal{M}_{NSR}$ is the convex hull of all CSPs of the form $x \otimes y$, with $x \in \Delta_m$ and $y \in \mathrm{BR}_L(x)$.

Theorem: $\mathcal{M}_{NSR}$ Minimality
$\mathcal{M}_{NSR}$ is inclusion-minimal and includes φ^+.

Theorem: φ^+-minimality implies optimality
Every inclusion-minimal menu that contains u_L^+ is pareto-optimal.

No Swap Regret Algorithms are Pareto-Optimal

Theorem
All no-swap-regret algorithms $\mathcal{A}$ have the same limit menu, which is $\mathcal{M}_{NSR}$.

Theorem: $\mathcal{M}_{NSR}$ Characterization
$\mathcal{M}_{NSR}$ is the convex hull of all CSPs of the form $x \otimes y$, with $x \in \Delta_m$ and $y \in \mathrm{BR}_L(x)$.

Theorem: $\mathcal{M}_{NSR}$ Minimality
$\mathcal{M}_{NSR}$ is inclusion-minimal and includes φ^+.

Theorem: φ^+-minimality implies optimality
Every inclusion-minimal menu that contains u_L^+ is pareto-optimal.

φ^+-minimality implies pareto-optimality

Sufficient to prove:

Lemma
If $\mathcal{M}_1$ contains φ^+ and $\mathcal{M}_2 \backslash \mathcal{M}_1 \neq \emptyset$, then there is an Optimizer payoff u_O such that

$$V_L(\mathcal{M}_1, u_O) > V_L(\mathcal{M}_2, u_O)$$

φ^+-minimality implies pareto-optimality

Lemma
If $\mathcal{M}_1$ contains φ^+ and $\mathcal{M}_2 \backslash \mathcal{M}_1 \neq \emptyset$, then there is an Optimizer payoff u_O such that

$$V_L(\mathcal{M}_1, u_O) > V_L(\mathcal{M}_2, u_O)$$

Proof:

Two cases:

- $\mathcal{M}_2$ does not contain φ^+

φ^+-minimality implies pareto-optimality

Lemma
If $\mathcal{M}_1$ contains φ^+ and $\mathcal{M}_2 \backslash \mathcal{M}_1 \neq \emptyset$, then there is an Optimizer payoff u_O such that

$$V_L(\mathcal{M}_1, u_O) > V_L(\mathcal{M}_2, u_O)$$

Proof:

Two cases:

- $\mathcal{M}_2$ does not contain φ^+ (easy)

φ^+-minimality implies pareto-optimality

Lemma
If $\mathcal{M}_1$ contains φ^+ and $\mathcal{M}_2 \backslash \mathcal{M}_1 \neq \emptyset$, then there is an Optimizer payoff u_O such that

$$V_L(\mathcal{M}_1, u_O) > V_L(\mathcal{M}_2, u_O)$$

Proof:

Two cases:

- $\mathcal{M}_2$ does not contain φ^+ (easy)
- $\mathcal{M}_2$ does contain φ^+

φ^+-minimality implies pareto-optimality

Lemma
If $\mathcal{M}_1$ contains φ^+ and $\mathcal{M}_2 \backslash \mathcal{M}_1 \neq \emptyset$, then there is an Optimizer payoff u_O such that

$$V_L(\mathcal{M}_1, u_O) > V_L(\mathcal{M}_2, u_O)$$

Proof:
Two cases:

- $\mathcal{M}_2$ does not contain φ^+ (easy)
- $\mathcal{M}_2$ does contain φ^+ (a little trickier)

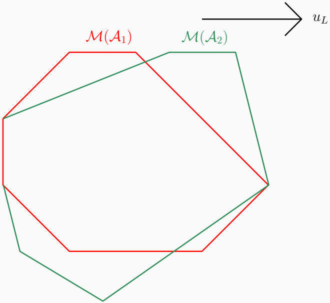
φ^+-minimality implies pareto-optimality

Lemma
If $\mathcal{M}_1$ contains φ^+ and $\mathcal{M}_2 \backslash \mathcal{M}_1 \neq \emptyset$, then there is an Optimizer payoff u_O such that

$$V_L(\mathcal{M}_1, u_O) > V_L(\mathcal{M}_2, u_O)$$

Special Case

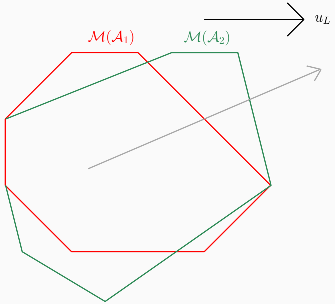
Both Menus are Polytopes.



φ^+-minimality implies pareto-optimality

Lemma
If $\mathcal{M}_1$ contains φ^+ and $\mathcal{M}_2 \setminus \mathcal{M}_1 \neq \emptyset$, then there is an Optimizer payoff u_O such that

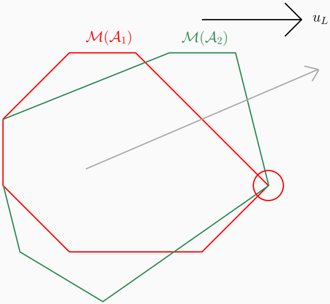
$$V_L(\mathcal{M}_1, u_O) > V_L(\mathcal{M}_2, u_O)$$



φ^+-minimality implies pareto-optimality

Lemma
If $\mathcal{M}_1$ contains φ^+ and $\mathcal{M}_2 \backslash \mathcal{M}_1 \neq \emptyset$, then there is an Optimizer payoff u_O such that

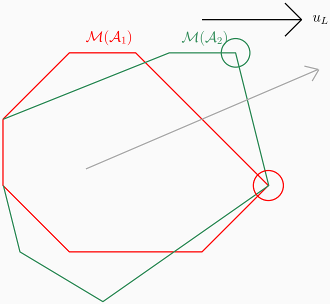
$$V_L(\mathcal{M}_1, u_O) > V_L(\mathcal{M}_2, u_O)$$



φ^+-minimality implies pareto-optimality

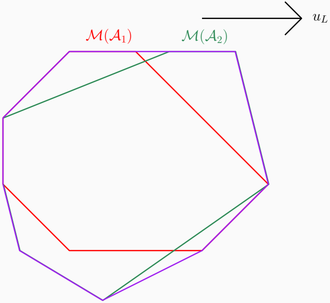
Lemma
If $\mathcal{M}_1$ contains φ^+ and $\mathcal{M}_2 \backslash \mathcal{M}_1 \neq \emptyset$, then there is an Optimizer payoff u_O such that

$$V_L(\mathcal{M}_1, u_O) > V_L(\mathcal{M}_2, u_O)$$

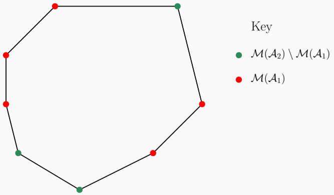


φ^+-minimality implies pareto-optimality

Take the convex hull of the union.

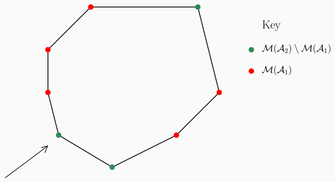


Take the convex hull of the union.



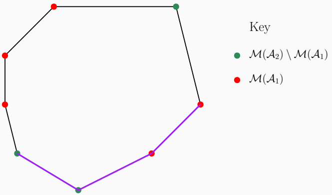
φ^+-minimality implies pareto-optimality

- Start with an “extra” vertex in $\mathcal{M}(\mathcal{A}_2)$.



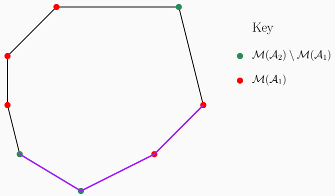
φ^+-minimality implies pareto-optimality

- Start with an "extra" vertex in $\mathcal{M}(\mathcal{A}_2)$.
- Construct a path of strictly increasing u_L value.



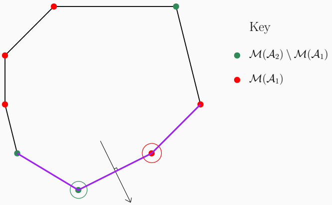
φ^+-minimality implies pareto-optimality

- Start with an “extra” vertex in $\mathcal{M}(\mathcal{A}_2)$.
- Construct a path of strictly increasing u_L value.
- Find a crossover edge.



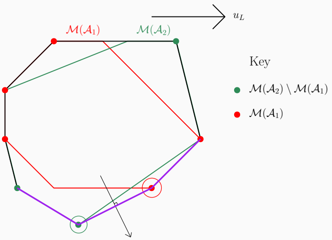
φ^+-minimality implies pareto-optimality

- Start with an "extra" vertex in $\mathcal{M}(\mathcal{A}_2)$.
- Construct a path of strictly increasing u_L value.
- Find a crossover edge.



φ^+-minimality implies pareto-optimality

- Start with an “extra” vertex in $\mathcal{M}(\mathcal{A}_2)$.
- Construct a path of strictly increasing u_L value.
- Find a crossover edge.



Multiplicative Weights (and friends) are Pareto-Dominated

Theorem
All FTRL algorithms are Pareto-dominated.

Theorem
All FTRL algorithms are Pareto-dominated.

- What's the smallest-size game in which we can hope to prove this?

Theorem
All FTRL algorithms are Pareto-dominated.

- What's the smallest-size game in which we can hope to prove this?
- The optimizer must have more than one action.

Theorem
All FTRL algorithms are Pareto-dominated.

- What's the smallest-size game in which we can hope to prove this?
- The optimizer must have more than one action.
- The Learner must have more than 2 actions.

Theorem
All FTRL algorithms are Pareto-dominated.

- What's the smallest-size game in which we can hope to prove this?
- The optimizer must have more than one action.
- The Learner must have more than 2 actions. Since No-Regret with two actions implies no-swap-regret.

Theorem
All FTRL algorithms are Pareto-dominated.

- What's the smallest-size game in which we can hope to prove this?
- The optimizer must have more than one action.
- The Learner must have more than 2 actions. Since No-Regret with two actions implies no-swap-regret.

We prove this for a non-degenerate set of 3×2 games.

Theorem
All FTRL algorithms are Pareto-dominated.

Proof Sketch

- All FTRL algorithms induce the same menu.
- And the menu is a polytope (with a succinct description) [6]

[6] implicitly gives the optimizer their exact best response information

All FTRL algorithms induce the same menu

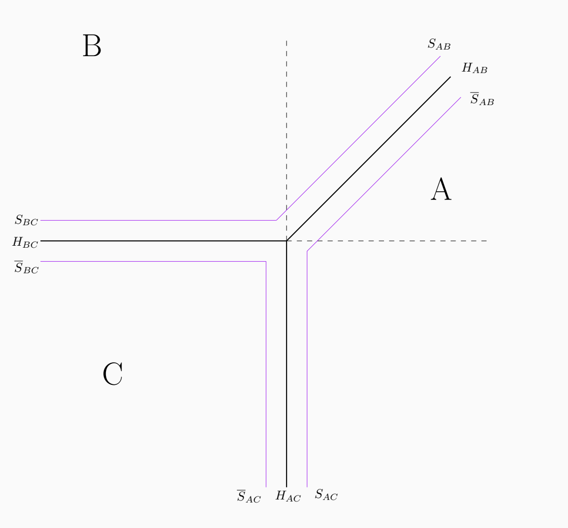


Figure 7: Space of Cumulative Payoffs

All FTRL algorithms induce the same menu

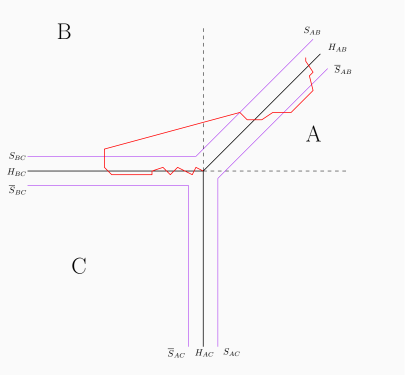
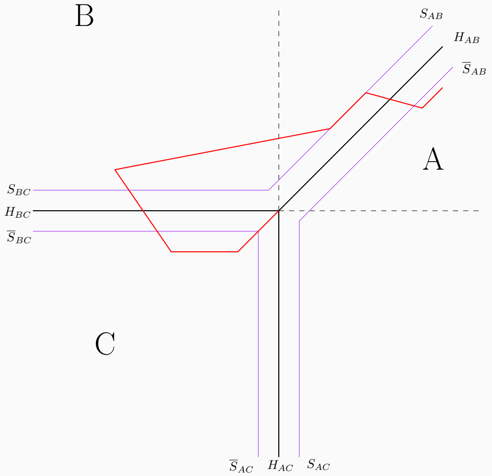


Figure 8: Space of Cumulative Payoffs

All FTRL algorithms induce the same menu

"Mean-Based" Trajectory

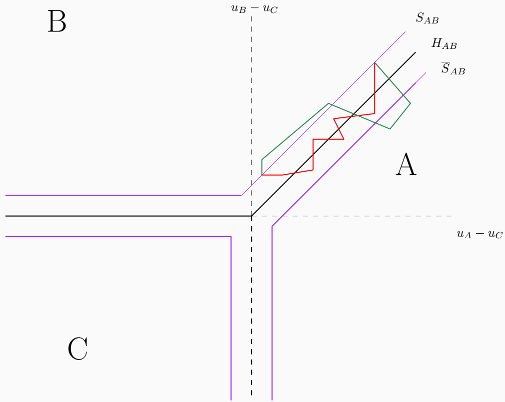
Trajectory has a "clear" leader for all but $o(T)$ time steps.



All FTRL algorithms induce the same menu

"Mean-Based" Trajectory

Convert arbitrary trajectories to mean-based trajectories.



Oh No I Stopped Listening!!!

- hi

Oh No I Stopped Listening!!!

- hi
- it’s not too late

Oh No I Stopped Listening!!!

- hi
- it’s not too late
- here’s what we want you to know

Takeaways

- Pareto-Optimality
- Menus

Takeaways

- Pareto-Optimality
    - Incomparable with No-Regret
- Menus

Takeaways

- Pareto-Optimality
  - Incomparable with No-Regret
  - No-Swap-Regret Algorithms are Pareto-Optimal
- Menus

Takeaways

- Pareto-Optimality
  - Incomparable with No-Regret
  - No-Swap-Regret Algorithms are Pareto-Optimal
- Menus
  - Progress towards understanding FTRL

Takeaways

- Pareto-Optimality
    - Incomparable with No-Regret
    - No-Swap-Regret Algorithms are Pareto-Optimal
- Menus
    - Progress towards understanding FTRL
    - A new paradigm for algorithm design

Thank you!

Figure 11: Us, being happy you listened to our talk