Graph Signal Processing: An Introductory Overview

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Goals

- Some background and basic concepts
- A bit of history and what's going on now (at the workshop!)
- FAQs and challenges

- Disclaimers

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  - Partial overview!

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- Disclaimers
  - Partial overview!
  - More questions than answers

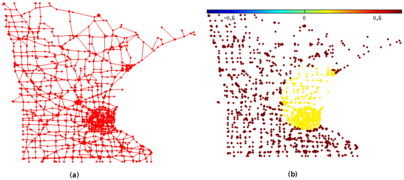
Outline

This is the 2nd GSP workshop

- First workshop was held with IEEE GlobalSIP, Dec 2013.
- Much more activity in this field!
- More than double the number of presentations, 1 day vs 3 days.

Graph signal processing: why now?

1796 Philadelphia roadmap, Library of Congress



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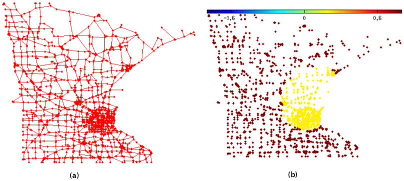

1796 Philadelphia roadmap, Library of Congress

Standard questions: What is the shortest path? What is safest path?

Graph signal processing: why now?

- Going from physical graphs to information graphs:
  - From:
    - Roads and rail
    - Telephone Networks
  - To:
    - Web
    - Online social network

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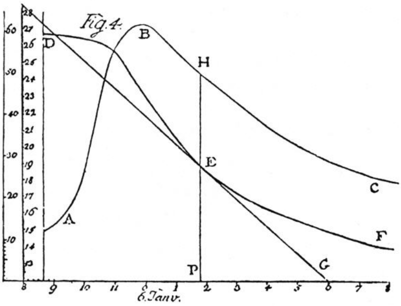
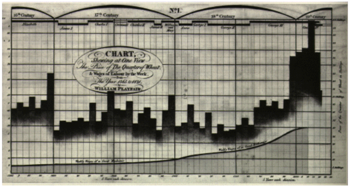
- Sensing technology allows us to measure "on a graph"
- Where is GSP being used?
  - Physical networks
  - Information networks
  - Regular signals

Graph signal processing: why now?





Lambert, 1765, Playfair ca. 1820

See (E. Tufte, The Visual Display of Quantitative Information, '83)

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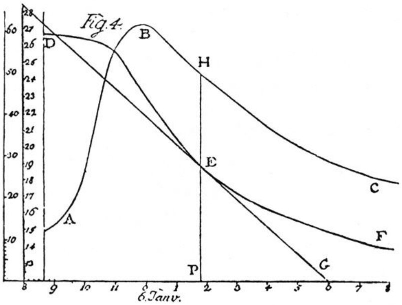

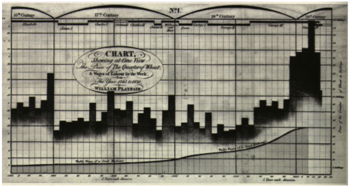

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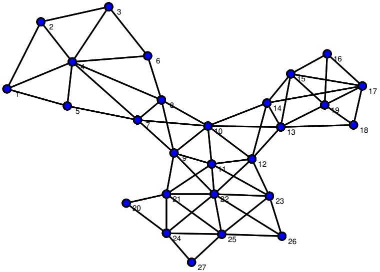
See (E. Tufte, The Visual Display of Quantitative Information, '83)
Do we know how to think about and visualize graph signals?

Outline

Multiple algebraic representations



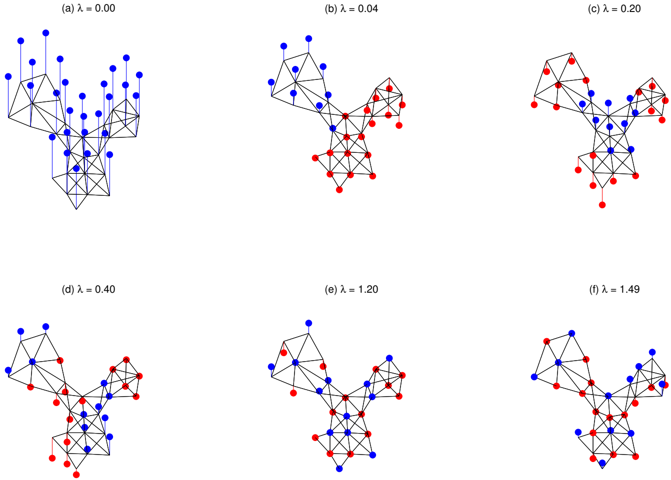
- Graph $G = (\mathcal{V}, E, w)$.
- Adjacency $\mathbf{A}$, a_{ij}, a_{ji} = weights of links between i and j (could be different if graph is directed.)
- Degree $\mathbf{D} = diag\{d_i\}$, in case of undirected graph.
- Various algebraic representations
  - normalized adjacency $\frac{1}{|\lambda_{max}|}\mathbf{A}$
  - Laplacian matrix $\mathbf{L} = \mathbf{D} - \mathbf{A}$.
  - Symmetric normalized Laplacian $\mathcal{L} = \mathbf{D}^{-1/2}\mathbf{L}\mathbf{D}^{-1/2}$
- Graph Signal $\mathbf{f} = \{f(1), f(2), ..., f(N)\}$

- Discussion:
  1. Undirected graphs easier to work with
  2. Some applications require directed graphs
  3. Graphs with self loops are useful

Graph spectrum, GFT

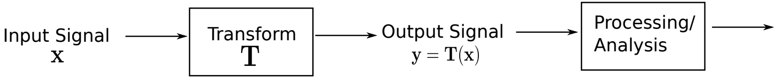
- Different results/insights for different choices of operator
- Laplacian $\mathbf{L} = \mathbf{D} - \mathbf{A} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}'$
- Eigenvectors of $\mathbf{L}$: $\mathbf{U} = \{\mathbf{u}_k\}_{k=1:N}$
- Eigenvalues of $\mathbf{L}$: $diag\{\boldsymbol{\Lambda}\} = \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_N$
- **Eigen-pair system $\{(\lambda_k, \mathbf{u}_k)\}$ provides Fourier-like interpretation — Graph Fourier Transform (GFT)**

Eigenvectors of graph Laplacian



- Basic idea: increased variation **on the graph**, e.g., $\mathbf{f}^t\mathbf{L}\mathbf{f}$, as frequency increases

Graph Transforms and Filters

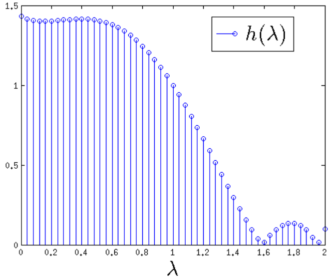


- Properties
  - Invertible
  - Critically sampled/overcomplete
  - Orthogonal/near orthogonal/frames
- **What makes these "graph transforms"?**
  - **Frequency interpretation**
    - Operation is diagonalized by $\mathbf{U}$
  - **Vertex localization**: polynomial of the operator ($\mathbf{A}$, $\mathbf{L}$, etc)
    - Polynomial degree is small (vs a polynomial of degree $N - 1$).

Frequency interpretation

- Spectral Wavelet transforms (Hammond et al, CHA'09):

  Design spectral kernels: $h(\lambda) : \sigma(G) \to \mathbb{R}$.



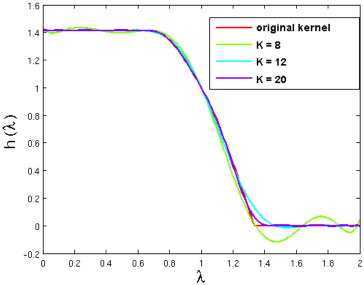
$$\mathbf{T}_h = h(\mathcal{L}) = \mathbf{U} h(\mathbf{\Lambda}) \mathbf{U}^t$$

$$h(\mathbf{\Lambda}) = diag\{h(\lambda_i)\}$$

- Analogy: FFT implementation of filters

Vertex Localization: SGWT

- Polynomial kernel approximation:



$$h(\lambda) \approx \sum_{k=0}^{K} a_k \lambda^k$$

$$\mathbf{T}_h \approx \sum_{k=0}^{K} a_k \mathcal{L}^k$$

- Note that **A** and **L** are both 1-hop operations *K*-hop localized: no spectral decomposition required.

Summary

- Different types of graphs (directed, undirected, with/without self loops)
- Multiple algebraic representations of graphs ($\mathbf{A}, \mathbf{L}, \ldots$)
- Their eigenvectors/eigenvalues induce a notion of variation
- The corresponding operators can be viewed as "shifts" or "elementary operators"
- Polynomials of these operators represent "local" processing on the graph
- Graph filtering: polynomial/diagonal in vertex/frequency

Outline

Graphs Signal Processing: a bit of history

Key point: many threads lead to graph signal processing

- spectral graph theory
- image processing
- semi-supervised learning
- block transforms
- vertex domain transforms
- frequency domain transforms

Spectral graph theory

- Results emerging in '50s and '60s linking algebraic graph structure to graph properties, initial work in Math, later interest in CS
- Classic works
  - (Cvetkovic, Doobs and Sachs, '80)
  - (Chung, '96)
  - (Spielman, '01)
- Primary concerns are linking spectrum and graph properties, no signals

- Link to GSP is weaker than it should be: because we are interested in working on arbitrary graphs (more on this later)

Linking graph spectrum to graph structure

DCT and KLT

- (Ahmed, Natarajan, Rao, T. on Computers, '74) Optimality of DCT for high correlation random vectors (close to 1)
- (Strang, SIAM'99) Graph interpretation (eigenvectors of line graphs with weight one), connection to DST
- (Püschel & Moura, SIAM'03) Generalization,
- (Püschel & Moura, TSP'08) General Algebraic signal processing perspective:
  - DCT as basis for a signal space of finite signals under different boundary conditions (Sandryhaila & Moura, '14) (see afternoon talk!)
- (Shen et al, PCS'10) regular graphs with irregular weights (use GFT of the graph)
- (Zhang, Florencio & Chou, SPL'13), General case of graphs obtained from precision matrices corresponding to Gauss Markov Random Fields.

Eigenvectors of graphs with regular connectivity and unequal weights/self-loops

Image processing

- (Wu & Leahy, PAMI, '93), (Shi & Malik, PAMI, '00): graph cuts for image segmentation, smaller edge weights across image boundaries
- (Tomasi & Manduchi, '98) Bilateral filtering, filter weights function of pixel and photometric distances
- (Elmoataz, Lezoray, Bougleux, TIP'08), (Osher et al, SIAM'07) Graph Laplacians for regularization
- (Milanfar, SPM'13) Various signal dependent image filters from a graph perspective

Weighted graphs with edges a function of pixel distance and intensity differences

Semi-supervised learning

- Learning from labeled and unlabeled training data
  - Estimate labels for unlabeled data
  - Decide what data to label
  - Consider kNN data graph
- (Belkin, Niyogi, '03 NIPS), (Zhou et al, NIPS'04), (Smola & Kondor, COLT'03), (Zhu et al, ML'03) Regularization on graphs, semi-supervised learning, label propagation
  - Generally use $\mathbf{L}$ to favor smooth signals on the graph
- (Anis et al, KDD'14) Graph signal sampling interpretation
- Why should a "label" signal be smooth?

Graphs connecting datapoints in feature space (e.g., kNN), labels should be slow varying

Graph Transforms: vertex domain approaches

- (Schroeder & Sweldens, '95), (DeRose & Salesin, '95) Transforms for attributes defined on meshes, often use lifting based techniques
- Network graphs, (Crovella& Kolaczyk, INFOCOM'03), graphs with arbitrary connectivity, analysis tool, overcomplete
- Sensor networks, (Baraniuk et al, IPSN'06), (Wang & Ramchandran, '06), (Ciancio, et al, IPSN'06) (Shen & Ortega, IPSN'08, TSP'10)
  - Emphasis on distributed operation, approaches sometimes use structure (e.g., trees, tesselations)
- Graph lifting (Narang & Ortega, APSIPA'09), (Janson et al, Royal Statistical Society'09)
  - Even/odd assignment in regular signals correspond to bipartite approximation of a graph

Vertex domain approaches require graph partitions

Graph Transforms: frequency domain approaches

- Diffusion wavelets (Coiffman & Maggioni'06):
  - Use successive applications of a diffusion to create subspaces of lower graph frequency content (and less localization in vertex domain)
  - Eigendecomposition of powers of $\mathbf{L}$
  - No exact localization in vertex domain
- Spectral Graph Wavelets (Hammond et al, CHA '09)
  - Spectral domain design (kernel having desirable properties and its scalings)
  - Polynomial approximation for localization, no need to explicit frequency decomposition
  - Nice vertex/frequency interpretation, overcomplete
- Filterbanks (Narang & Ortega, TSP'12, TSP'13)
  - Critically sampled, orthogonal/bi-orthogonal solutions
  - Exact solutions, only bi-partite graphs

Different trade-offs possible depending on whether critical sampling is required

Sampling

- Irregular sampling in regular domains, e.g., properties that guarantee reconstructions (Gröchenig, 92), (Aldroubi & Gröchenig, '01)
  - Focus is on reconstruction based on regular domain properties (frequency)
- Optimality conditions for combinatorial graphs (Pesenson'08)
- Various approaches for sample set selection and reconstruction (Anis et al, '14), (Shomorony & Avestimehr, 14), (Chen et al, '14)

Selection of nodes that are most informative

Conference topics

- Distributed processing
- Graph learning
- Filtering
- Fundamentals
- Sampling
- Statistical Graph Signal Processing
- Applications

Graph Filter Design

- Classic problem in DSP
- Goal: Design filters with different properties in terms of localization, orthogonality, etc.
- Different types of filters:
  - Moving average, graph-temporal
  - Graph diffusion
  - graph-temporal
  - lifting approaches
  - representations using dictionaries

Graph Learning

- Goal: learn a graph from data
- Multiple cases: covariance,propagating graph signals, etc.
- Examples: estimate a sparse inverse covariance (precision) matrix, estimate a Laplacian that makes a observed data smooth on average
- Question: what is the advantage of using a graph vs PCA?
  - Interpretation, approximate KLT with polynomial graph operations

Sampling

- Goal: decide which graph signal samples (values associated to vertices) should be observed so that we can reconstruct the others
- Assumptions about smoothness of signal
- (almost) any random sampling works when signals are exactly bandlimited and noise free
- Noise and non-bandlimited behavior make things complicated
  - Robust sampling methods (randomize, iterative, with/without knowledge of the GFT)
  - New criteria for signals to be sampled (piecewise smooth)
  - Distributed sampling

Statistical GSP

- Random signals on graphs
- Definition of Stationary Graph Signals
- Time varying signals over graphs
- PCA on graphs

Applications

- Image regularization
- Compression
- Computer vision applications (e.g., motion analysis)
- Origin-Destination traffic matrices
- Tracing of outbreaks
- Wireless network optimization
- Brain connectivity

Outline

Scale

- Can we really apply our tools to large scale datasets?
  - Facebook 1G+ nodes
- How to interpret results in a large scale graph
- How to interpret locality:
  - Graph diameter vs average?
  - We often do not consider what the real footprint is for a polynomial of degree K.
- Large scale implementation
  - Parallelization
  - GraphLab
    - Basic primitive: each node communicating with its neighbors
    - Algorithms to distribute nodes across processors to preserve locality
    - Example: node requests information from neighbors and computes an output ($\mathbf{Lx}$) or apply this recursively ($\mathbf{L}(\mathbf{Lx})$), i.e. every node stores $\mathbf{Lx}$ and then $\mathbf{L}^2\mathbf{x}$, etc.
  - Should our community contribute?

How to choose a graph

- In some applications graph is given (e.g., social networks)
- In some it is a function of some known information (e.g., distance in a sensor network)
  - How to select weights?
  - e.g., bilateral filter, etc
  - are there optimality results?
- Designing graphs from data
  - Sparse inverse covariance: why is a graph representation of a dataset better?
  - Advantages vs other methods

Shifts and localization

- Most current filtering schemes use an operator (Laplacian or Adjacency)
- Should it be considered a "shift" (note that sometimes signals vanish after being shifted).
- The effect of a shift depends on the eigenvalue associated with it: graphs with same eigenvectors, but different eigenvalues? $\mathbf{L} = \sum_i \lambda_i \mathbf{u}_i \mathbf{u}_i^t$, different graph connectivity, but same frequency interpretations (Gavili & Zhang, Arxiv'15)
- Classes of equivalent graphs?
- Bounds on frequency-vertex domain localization
- Specialize these bounds to specific graph types
- Complicated because of properties of shift
- Several contributions in this workshop

Need to consider special characteristics of graphs

- Example: how to deal with high multiplicity eigenvalues (high dimensional subspace with the same graph frequency) (Zeng et al, ICASSP'16)
- How to assess the impact of "removing" edges?
- What is the best way to approximate a graph?
- Graph reductions/simplifications
- More generally there could be interesting results that apply only to certain classes of graphs?
  - bipartite (Narang & Ortega, '11), circulant (Ekambaran et al, '13), M-block cyclic (Teke & Vaidyanathan, '16)

Datasets and community

- Should we have a set of standard datasets?
- Matlab code: GSP Toolbox EPFL (Perraudin & Paratte)
- Anything else we should do?

What is the killer app?

- Understand in what cases a graph-based approach is better than directly working with signals in $\mathbb{R}^N$.
- Many cases
  - Graph is given (web, social network)
  - Irregular measurements (sensor networks)
  - Graph approaches are an alternative (e.g., images)
  - Data driven methods (e.g., machine learning).
  - Large scale systems (e.g., finite state machines)
- **GSP methods are closely linked to existing approaches**
- New perspectives on existing topics
- perhaps an emerging new way to understand problems

Outline

Conclusions

- GSP has deep roots in the Signal Processing community
- A lot of progress, interesting results
- Many open questions!
- Outcomes
  - Work with massive graph-datasets: potential benefits of localized "frequency" analysis
  - Novel insights about traditional applications (image/video processing)
  - Promising results in machine learning, image processing, among other areas
- To get started (Shuman et al, SPM'13), (Sandryhaila & Moura, SPM'14)
- Enjoy the workshop!