A Foundations-of-Computation Approach to Formalizing Musical Analysis

Halley Young
University of Pennsylvania
halleyy@seas.upenn.edu

ABSTRACT

In the last century, two trends have increased the scope of musical analysis: music theorists have provided mathematical insight into specific musical scenarios, while musicologist have examined the nature of musical analysis as a cultural, cognitive, and scholarly endeavor [1] [2] [3] [4] [5]. This paper intends to bring these two strands of research together by providing a constructive mathematical foundation for the process of musical analysis. By establishing a mathematical description of the generation of an analysis of a piece of music, useful mathematical tools for performing operations frequently used in analysis, and possible precise definitions for loaded terms such as "musical similarity" and "musical form", I will extend the analyst and the meta-analyst's ability to create abstractions from musical surfaces, the core of every process of analysis.

1. INTRODUCTION

In this age of globalism and prestige given to scientific thought, it is unsurprising that both the study of musical universals and mathematical abstractions of music are receiving much attention. Authors such as Steven Brown and Joseph Jordania [6], Leonard Meyer [7], Jay Rahn [8], and most recently Samuel Mehr [9] have taken various approaches including the methodologies of cognitive science, semiotics, corpus and field studies, and anthropological theory in order to make statements about "musical universals," or facts that seem to describe the way many peoples perform, compose, and listen to music. At the same time, specific musical practices, from African rhythms to the Classical diatonic scale, have been scrutinized in order to understand the mathematical abstractions that can explain how and why these practices manifest [1] [2]. However, few scholars have studied the mathematical universals of music analysis, or formalizations which could apply to the analysis of any musical phenomenon. Theory of computation and foundations of computation provide a paradigm for making statements about music analysis in general with mathematical rigor.

In this paper, I shall introduce several concepts borrowed from computer science to generalize aspects of music analysis. I will first discuss the basic features of a relevant computation on a piece of music, which I term a "property", and will provide a constructive basis for determining the relevance of such a computation. I shall then invoke type theory to describe the type-theoretic foundations for deriving such properties from the "basic" components of sound (pitch, duration, loudness, and timbre). I will discuss comparisons both between different property types, as well as between different property values and between different musical pieces as mathematical objects from which properties are derived. I will then focus on a particular type of property whose definition is recursive, and will introduce two such properties - tonal function and meter in common practice music.

2. THE HORIZONTAL-VERTICAL-ONTOLOGICAL COMPLEX

The juxtaposition of "horizontal" and "vertical" views of music and the realization that both need to be considered simultaneously is well understood in music pedagogy (for instance, the title to New York University's introduction to music course is "Harmony and Counterpoint"). It is equally well understood that such simultaneous comprehension is cognitively not trivial. Since at least the 1760's, musicologists have complained about the relative importance assigned by composers to one dimension to the neglect of the other (see Rousseau on harmony and melody) [10], and it is widely agreed that Baroque music was "more contrapuntal" than the "harmony-based" music that followed. However, I argue that in every type of music where more than one thing is occurring in a temporal moment, there are properties of the music that emerge out of changes that occur as the piece progresses, and properties that emerge out of the simultaneity of parts being conceived as a unified whole.

However, the horizontal and vertical dimensions of music only describe half of the picture. Consider a single, monophonic melody. Its perception is not only as a gestalt, but also as a complex of different properties that it exhibits - one hears the song "Hot Cross Buns", but also the narrow range, the diatonicity, the downwards motion, the repetition, and so forth. Each of these properties is a part of my experience of the song. Similarly, consider a single chord. Musicologists will probe that chord for its root tone, for its overall consonance, for its interval vector, and for its interval set class, because all of these are supposed to provide additional information that's useful for analysis. Notice that from one perspective, information doesn't increase when you compute the interval vector of a chord - you're merely describing a property that is always derivable from the information you have, that of the chord's pitch classes. Nonetheless, calculating the interval vector tells the composer something important about the chord, as it allows comparison to other chords in order to estimate its potential effect. The potential for calculation is not the same as the actualization of calculation, and our understanding of music is fundamentally shaped by the calculations we (consciously or unconsciously) produce on horizontal, vertical, and horizontal-and-vertical aspects of the music. We will refer to every calculable aspect of music as a "property". This label includes the four properties (pitch, duration, dynamics, timbre) which are often cited in educational literature as the "fundamental elements of music", as well as more complicated properties like harmonic function or a representation of musical contour such as CSEGs.

3. MUSICAL OBJECTS, CONSTRUCTIVISM, AND EXPRESSIVITY

Constructive mathematics, which became the foundation for much programming language theory in the 1920's, is different than classical mathematics in its insistence that if one wants to show that an object exists, one has to construct the object in a well-defined logic, rather than proving that it exists indirectly. In terms of music, we may extend the metaphor of constructive and classical logics to distinguish between "classical" and "constructive" definitions of properties. A property could be defined as any function from a piece of music to some domain, including one which is non-computable (or non-expressible in any interpretable language). However, as an analytical tool, the musical properties which can be described in an appropriately expressive language are much more useful, as this definition can then be applied to any arbitrary piece of music to draw similarities between the two. In contrast, claiming that "there exists a property f such that $f(mozart\_sonata\_in\_C) = 2$ is rather meaningless for the analyst. Furthermore, in computer science, it is understood that using the least complicated language possible for expressing a given property is optimal. One of the goals of this paper is to figure out how to define common musical properties in constructive ways and using as simple a language as possible.

4. WHAT MAKES A PROPERTY USEFUL/RELEVANT?

We have informally claimed that certain properties are relevant to some styles but not others. It is difficult to quantify relevance absolutely, but we shall do so relative to a classification task T and a classification model type M with fixed computing resources. Such a classification task could be determining who wrote a piece, at which point in a composer's career it was written, what style a piece was written in, or whether a piece is enjoyed by a given person. Such a fixed classification model could be a neural network with a fixed number of weights, a decision tree with a fixed number of nodes, or a search algorithm with a fixed max depth and breath. Then relative to this task T and model strength, we can say a property P is significant if a model in class M can do better on T if it knows the values of P for all the music it is classifying as opposed to being given random valuations instead. Furthermore, relevance is inductive: If a property P1 is relevant, any property P2 that needs to be known in order to compute P1 is also relevant.

5. COMMON TYPE CONSTRUCTORS WHICH PRODUCE MANY PROPERTIES

In computer science and category theory, there are several standard type constructors. These are constructs which take types as arguments, and return a more complex type which depends on the input types. Many properties frequently referenced in music analysis can be thought of as the result of passing a primitive type (time, duration, pitch, dynamics) to a type constructor.

5.1 n-Product Type

A product type, as defined by type theory, is a type that results from a combination of two or more other types [11]. For instance, the assignment of specific intervals to specific instruments, as can be heard in Elliot Carter's string quartet [12], can be thought of as a property which consists of a pair of an interval and an instrument. Another product type might be the association of a certain scale degree with a certain type of ornamentation, as happens in Hindustani music [13]. If we limit ourselves to triads harmonically, we can represent chords as products of a pitch-set class and one of the interval-sets $\{[037], [047]\}$. What we typically hear as an instantiated note can be thought of as an n-product of a pitch, a duration, a timbre, and a dynamic (the four primitive properties).

5.2 n-Vector Type

An n-Vector Type consists of a fixed number of values of the same type. Examples include all dyads, triads, or tetrads; a traditional score (which is a fixed number of parts), and a representation of a 12-tone scale as a set of 12 0/1 bits.

5.3 List Type

A list consists of a variable number of values of the same type. Examples include a motif, which can be any number of notes long; a chord, which can be any number of notes long; or a chord progression, which can contain any number of chords.

5.4 Transformation Type

Transformation types are homeomorphic functions from one value to another of the same type. Examples include the function which maps one rhythmic pattern to the same rhythmic pattern in augmentation. Any possible list of durations could be passed into this function, but it is simple to calculate its output regardless of the function. Functions generally have to be more than computable (have a finite representation) to be noticeable, however. It is not even enough for the functions to be computationally simple (as defined by the shortest way of describing the function using English or pseudocode). In addition, they must be perceptually simple according to the principles of cognitive musicology, which may seem somewhat arbitrary from a computational perspective. This is exemplified by the extreme difficulty in recognizing retrograde inversions - the retrograde function in Haskell (a typical programming language) is roughly the same length as the inversion function, yet one is perceptually more salient than the other.[14].

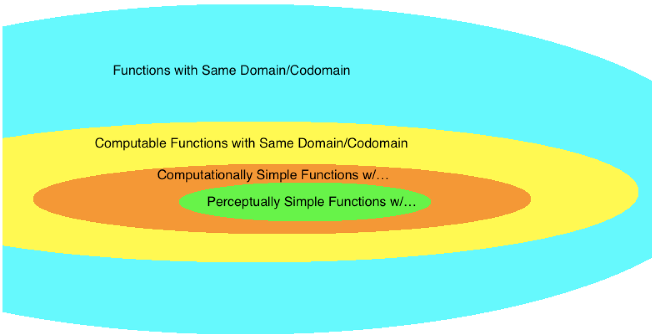

Figure 1. Venn Diagram of Musically Relevant Function Hierarchy

There are two ways to view transformation types, and the one used affects relevance and tractability in analysis. One way to view transformation types is as a (potentially) infinite class of functions over the same domain and codomain, in which case each transformation can be represented by a unique natural number, and the "value" of the transformation is that natural number. This works well over small domains - for instance, neo-Riemannians have essentially done this with the finite and relatively small set of transformations between major and minor chords (of which there are only 24). However, in other instances, such as with all transformations over arbitrary-sized rhythms (a list value), there are infinitely many such transformations. Furthermore, not only are there hypothetically an infinite number of natural numbers that these transformations map to, but in practice, the minimum number of transformations needed to describe a pair of rhythms (r_1, r_2) from a set $R = r_0 \dots r_m$ of length $rhythmLen$ can grow exponentially in $rhythmLen$. Therefore, it may be useful to invoke dependent type theory, and say that a "transformation of type T(t)" on x_1 and x_2 is the boolean value 0/1 of whether $t(x_1, x_2)$ holds.

6. ONTOLOGICAL, HORIZONTAL, AND VERTICAL COMPRESSIONS AND THE OPERATORS THAT PRODUCE THEM

Given that music is a temporal art form, many properties are of a list type, and therefore if there are m possible values and n elements in the list, the number of possible values grows exponentially $(O(m^n))$ with the number of elements. In some computational tasks, having a compressed representation is therefore necessary in order to meaningfully reason about a collection of such properties. Here I will present an example of compression which is both elegant and used in practice, indicating its cognitive relevance.

Consider the case of melodic contour. A common way to describe contour is to abstract the individual pitches into a line into relative sizes, so the sequence $69, 70, 69, 68, 71$ would be described as $1, 2, 1, 0, 3$, suggesting that the first and third pitch are the second lowest, the second pitch is the second highest, the fourth is the lowest, and the fifth is the highest. This is indeed a compression, as it maps any transposition of the line or proportional increase of each interval in the line to the same contoural value. It is also perceptually relevant, as the human tendency to abstract absolute to relative sizes is well documented.

Figure 2. Two differing rhythms

We can reduce further. The contoural property above still has a list type (i.e., there can be infinite contours of different sizes). If we want to reduce this list type to a finite type, we can apply Morris's CSEG reduction, which only eliminates all notes except the first, highest other than the first or last, lowest other than the first or last, and last note. This again could be formulated as a perceptual hypothesis that such notes are the most salient in actually perceiving contours. Any list type can be reduced to an n-Vector type by producing a saliency metric and eliminating all but the most salient.

Now, consider Tigran Hamasyan's frequent usage of contrasting sections which can be said to have the same "rhythmic contour" but different meter (such as a groove of 3 beats, 3 beats, 4 beats, 2 beats in the first section, and a groove of 4 beats, 4 beats, 5 beats, 2 beats in the second section) [15]. Here the same ontological reduction can be applied on duration values instead of on pitch values. We could logically infer therefore what a "rhythmic CSEG of Hamasyan's grooves would look like", although it's usefulness may be in question.

7. DISTANCE(S)

As Belkin points out in his composition textbook, most music, regardless of genre, depends on the notion of repetition and contrast [16]. The very notion of repetition and contrast rely on some way of measuring the extent to which two musical objects are similar or different. For instance, Belkin uses a 1-5 metric to measure differences between motifs, where a score of 1 indicates two motifs which are perceived as extremely similar (such as transposition a whole tone up), while a score of 5 indicates two motifs which are perceived as extremely different (such as a motif and its retrograde transpose in a different tempo). There is no mathematical argument that suggests that retrograde would be perceived of as less similar than inversion, and yet it clearly is [**?**]. However, formalisms can still help in defining distances. For instance, any two lists of objects can be compared by minimal edit distance, where the edits allowed constitute the transformations possible on the object. As an example, consider the two rhythms in figure 2. If we assume that a retrograde operation has a distance of 2 units, a single division operation (splitting a note into two equal parts) has a distance of 1 unit, and an operation combining the duration of two notes has a distance of 1 unit, it can be shown that they have a minimum edit distance of 3. On the other hand, the rhythms in figure 3 have an edit distance of 1, and hence are more similar according to the assumed metric.

Thus, like relevance of property, distance is not absolute, but relative to a transformation metric T.

Figure 3. Two differing rhythms

8. THE SPECTRUM OF MUSICAL FORM

Form is one of the most abstract musical parameters, and one which seems quite removed from the basic elements of pitch, duration, timbre, and dynamics. We will here constrain our analysis to forms consisting of multiple sections. This includes common Western forms such as Sonata form as well as Theme and Variations form, while common Hindustani form includes the Khyal and the Dhrupad. These forms vary in what they entail regarding the specific properties associated within the sections of the form, the relationships between the sections, and the distance metric between sections. On one end of the spectrum, Matteo Magarotto convincingly argues that, in addition to the thematic connections between the material in the exposition and development of the sonata, specific Galant schemata and rhetorical devices are associated with each section of the sonata, such that the primary theme section (P) is more likely to be associated with a Cudworth sequence than the following transition section (TR) [17]. On the other end of the spectrum, theme and variations are defined only by the relative similarity between the successive presentations of the theme. Khyal involves a specific relationship between the first and second section - the second section is in the same raga and follows much the same melodic path as the first section, but is faster. We can conclude that form can consist of a product of the following:

- General distance constraints which should hold between sections
- Properties which should be associated with each section
- Transformations of properties which should be applied between sections

9. COMPARING DIFFERENT STYLES

Given that properties of music that are considered relevant are not culturally universal, one can distinguish between styles according to the differences in what constitutes a relevant property, or choose to consider only those properties which are comparable in both styles.

9.1 Syntacto-Semantic Universes

One can define the "syntacto-semantic universe" of a style as those properties which are meaningful to that style. For instance, the property of "raga" is as irrelevant to Classical music as "subdominant" is to serialist music (see section 8 on property relevance). One can enumerate for any style the set of relevant properties, and see how it relates (using set-theoretic operations such as symmetric difference and intersection) to that of another style.

9.2 Semantic Fields

In the special case where two styles both contain the same type of property, it is possible to compare the field of values that are considered valid for each style. For instance, consider the property of "chord transformation". Jazz can be distinguished from common practice for the validity of the tritone substitution as a type of chord transformation, while the mapping of an equal-tempered to just-tempered triad is found very rarely in contemporary music but quite often in the music of Jacob Collier [18]. Of course, common practice and jazz (including Jacob Collier's work) share several types of chord transformation, such as the addition of a seventh (although adding a seventh is conditional on chord function in common practice).

9.3 Homologous Properties

Properties are homologous when they describe values of the same type as defined in type theory, but are derived from the musical surface in a different way. For instance, consider a scale. Under one definition, a scale is defined as a single pitch class (the root) together with a set of pitch classes (the notes in the scale). However, the criteria used to define a piece of music as "being in" a given scale vary largely depending on the genre. For instance, in Renaissance music a scale was determined by what notes were used (excluding moments of musica ficta) and the final cadence, in Classical music the scale was expected to define the syntax of harmonic movement, and in certain types of jazz the scale and root define whether the soloist is playing a "dominant chord" or another category of chord relative to the rhythm section [19]. These three examples constitute not only different uses of the $(List\{PC\}, PC)$ type, but would presumably have different optimal extraction functions in order to calculate what the values should be from a given surface - applying jazz theory to parsing Renaissance music would lead to a less relevant representation of the Renaissance music's "scale" than if it was parsed using the assumption that scales determine what notes to use.

10. COMPARING PROPERTIES FOR SUBSET/SUPERSET RELATIONSHIPS

Instantiations of properties can be related to each other in three general ways.

1. Temporal dominance - a property instantiation x is a temporal subset of the property instantiation y if y is ontologically a non-strict superset of x, and y describes a moment in the music that includes the moment described by x. Thus, an antecedent is temporally dominated by the period which contains it.

2. Vertical dominance - a property instantiation x is a vertical subset of the property instantiation y if y is ontologically a non-strict superset of x, and y describes a set of simultaneously occurring events that includes the events described by x. Thus, the first violin's part is vertically dominated by the string section parts as a set.

3. Ontological dominance - a property instantiation x is an ontological subset of the property instantiation

y if x can be computed from y. Thus, the contour is strictly ontologically dominated by the list of intervals, because it is possible to compute the contour from the list of intervals (and not the reverse).

11. SEPARABLE/SEMI-SEPARABLE/NON-SEPARABLE PROPERTIES

In addition to classifying pairs of properties in terms of dominance relations, properties can be classified in terms of their separability. Some properties are completely separable - for instance, contour (measured as a CSEG) and rhythm. This separability means that one can design a CSEG and a rhythm, and superimpose them on the same snippet of music with no logical impossibilities. There are other properties which are semi-separable - for instance, intervals and contour. Neither completely determines the other, but it is also not possible to assign them independently with reference to a single piece of music. Finally, there are sets of properties where one subset completely determines another subset, such as the set {rhythmic pattern, rhythmic density}. Note that there are cliques of properties which are non-separable but for which any subset is semi-separable. For instance, it is very difficult if not impossible to make significant decisions about which melodic intervals to use if chord progressions and a total description of contour are already determined, but it is possible to make decisions about which intervals to use if only one of the two is predetermined.

12. FUNCTIONAL AND DIRECT PARAMETERS

Meyer distinguishes between ”syntactic” and ”expressive” features. According to him, ”syntactic” features include melody, rhythm, and harmony, and are able to organize qualitative states, while ”expressive” features either express only quantities (e.g., tempo or dynamics) or are not rich enough to be used in most music in a grammatical fashion, such as timbre (although Meyer stresses that in the case of Hindustani tabla music, timbre is converted into a syntactic feature, because there it is used grammatically) [7]. Another way to view these differences is between parameters with combinatorially many possible values, and those with a fixed, relatively limited range of values. These differences are important. However, type theory provides another distinction between parameters, what I will call ”functional” vs ”direct” parameters.

Functional parameters describe a given piece of music, but their values can not be determined only by the piece of music they describe. A piece of music being the secondary theme of a sonata is not a property only of that theme, but of the surrounding material. A dominant chord can be a local phenomenon, but it is only dominant by virtue of its position in a larger tonal complex. On the other hand, parameters such as pitch set class are not dependent on values other than the exact thing they describe.

12.1 The importance of recursivity

One property of many functional parameters is recursivity. The description of a chord as a dominant depends on the description of another chord as the tonic, a meter is in many genres more likely if the same meter existed in the prior measure, a consequent is only a consequent if it is followed by something which exists within the same description-universe as an “antecedent”. Thus, in order to determine a functional parameter’s value, we must presuppose the existence of its type, and that other elements in the piece have values belonging to this type. Such dependence on recursivity can be compared to Chomsky’s recursive formulation of human grammar and all the subsequent linguistic work which depends on human competence in recursion. It also implies that functional properties are extremely dependent on style for their existence - at least one element of the piece has to be somewhat obviously metered/tonal to give any clue what the other’s value on that axis would be.

12.2 Meter

Even assuming that meter’s type theoretic definition as the product of a number of beats per measure (int) and a duration which carries the beat (int) is correct, the correct meter of a piece is notoriously difficult to establish. Furthermore, composers often have trouble deciding in which meter to notate a piece of music of their own creation, suggesting that there is more than one principle by which meter is imposed. Temperley, Hauptmann, and Lerdahl and Jackendoff provide a few optimal properties of meter [20] [21] [22].

- Strong beats in the music (defined by onset, dynamics, tonality, timbre, register, etc.) tend to fall on the first beat of each measure.
- Durations in the music tend to be multiples of the duration which carries the beat.
- There tends to be fewer rather than more different meters imposed at the same time.
- There tends to be more rather than fewer meters at a time.
- The meter tends not to change within a part over time.

Note that, as mentioned before, unmetered music can certainly exist in a way that non-melodic music cannot, and so finding music that can’t be described according to such optimality principles does not invalidate the concept of meter. Note also that the weight on each of these tendencies is different for different styles of music - Stravinsky would write much music where meter varied each bar, as well as music with multiple meters. Finally, note that, in addition to genres where this definition of meter excludes all music, there are certain genres where this formulation is not enough - for instance, Jacob Collier speaks of needing an additional (third) integer for representing how to divide the beat [23]. Like all parameters, in some genres a particular definition of meter is meaningful to analysis, while in others it is not or less meaningful.

12.3 Tonal Function

According to Miller, harmonic function includes four properties: kinship, identity, quality, and behavior. For instance, subdominant function is defined by "kinship" (sharing at least 2 degrees with the subdominant triad), "identity" (being built on the subdominant degree), "quality" (being a major added-sixth chord), and "behavior" (moving from tonic to dominant) [24]. The prototypical subdominant chord has all of these properties; however, other chords are subdominant to the extent that they have these properties, and to the extent that they don't embody the properties of dominants and tonics. One could imagine establishing, for a given genre, a weighting of each of these properties, and a certain weight at which a chord is termed "substantially tonic".

12.4 Function as monotonically increasing set sequence, or a fixpoint

The recursivity of meter and tonality lead to a mathematical description which is both elegant and constructive (i.e., leads naturally to an algorithm for determining these properties). As mentioned before, for tonality to exist (which we will use as the prototypical functional parameter in the following section), in at least one case there must be reason to believe that a chord has a certain function that is not dependent on another chord having a certain function (or else there would be infinite regress). Once we have established, for instance, that one chord is the tonic, if the following chord can be described as a V chord of the previous chord's root (even if it's not a prototypical dominant-seventh chord), there may be sufficient evidence to view that chord as a dominant chord. This procedure can be performed until no more chords can be identified as having tonal properties, and at that point one can take the set of identified chords as the maximal set of functional chords (an algorithm known as the "fixpoint algorithm"). One can choose to define a piece's tonality only by the maximally inclusive set of functions, or by the sequence of ever-expanding sets of chords that have sufficient weighting as a tonic, subdominant, or dominant to be determined to be functional.

12.5 Modulation

Modulation is a property inherent in many kinds of music, including Bach's key modulation, Elliot Carter's metric modulation, modulation of Makamat in some Turkish music, and modulation of Nusach or Steiger in traditional Ashkenazi music. In all of these systems, there are functional parameters with complex definitions that result in a qualitative (or quantitative but finite) range of values, and a fixpoint algorithm can be used to determine whether a parameter has enough weight to have a given functional value. Thus, there is nothing inherently preventing the existence of a parameter which has properties which suggest two different functional values, or two weights which are significant enough to suggest different values. Modulation is a special case of this in which there is a sequence where the first element has only one functional value A, the second element has both functional values A and B, and the third element has only the functional value B.

12.6 Prolongation and Schenkerism

Tonal function can be described as the set {tonic, subdominant, dominant}, but it can also be described as the product of that set and a pitch-class-set (e.g., the tonic of C major, or of 0). This property is important because it helps establish tonal function as a potentially foldable property. In functional programming, a foldable function takes a list of values, a starting value, and a way of aggregating values, and compresses the list into a value of the same type as the starting value according to this aggregation method. This would be possible without considering roots by compressing every $[tonic, subdominant, dominant, tonic]$ sequence into the value "tonic"; however, according to the assumption inherent in Classical-era music that it is based on a background tonic, such a compression would produce a degenerate distribution (i.e., every piece would produce the compressed value of *tonic*), and would thus be meaningless. Quick, however, claims that the property that a chord that exists as the tonic of the dominant key can prolong a tonic-function area is a) key to any theory of tonic prolongation, b) relates (admittedly rather loosely) to Schenkerism's elevation of the basic line, and c) can be used to generate tonal music [25].

13. CONCLUSION

In conclusion, this paper seeks to formalize the practice of musical analysis through the lens of foundations of computation. It introduces the notion of a "relevant property" as the basic component of analysis, and discusses how these properties can be thought of as computational objects. Individual properties are compared in terms of dominance, type-similarity and separation, while values of properties are compared relative to existing transformation metrics. Special emphasis is placed on understanding transformation properties (properties which themselves are parameterized by a function in a dependent-type-theoretic fashion) and understanding "functional" properties, my term for properties whose evaluation requires a certain type of recursivity.

One of the benefits of such a principled approach is the promise of automation in analysis, and there is much future work to be done in using the principles described here to automate aspects of music formalization and generation. It is quite possible that new relevant properties could be discovered which are necessary to better understand a style of music even as well-theorized as Galant music, while automatic discovery of relevant properties could provide much-needed insight into non-Eurocentric music. New musics could also be generated by asserting as a constraint the relevance of a particular property that is not usually relevant, including functional parameters besides tonality and meter. Finally, the basis of this work in constructive logic allows for "generation by definition", as research in computer science has generation based on specifications as rich as dependent type theory [26][27]. Therefore, there is room to believe that, in addition to the mathematical elegance of a formal approach to analysis, there will be artifacts partially derived from this line of research.

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