$2x_1 + 5x_2 = 60$  \[ x_1 + x_2 = 18 \]

$3x_1 + x_2 = 44$

$(13, 5)$ optimal solution

$(x_1, x_2) = (13, 5)$ is optimal with $z = 31$
9.1-9. First note that \((2,3)\) always satisfies the 3 constraints. (i.e. \((2,3)\) is always feasible.)

In fact, since \(kx_1 + x_2 = 2k + 3\) at \((2,3)\) for any \(k\), the third constraint is always binding. We only need to check if \((2,3)\) is optimal.

Since the line \(kx_1 + x_2 = 2k + 3\) always passes through the point \((2,3)\), changing \(k\) simply rotates the line. Rewriting: \(x_2 = -kx_1 + (2k + 3)\), we see that the slope of the line is \(-k\), and therefore, the slope ranges from \(0\) to \(-\infty\).

As we can see, \((2,3)\) is optimal as long as the slope of the 3rd constraint line is less than \(-\frac{1}{2}\) (the slope of the objective line). If \(k < \frac{1}{2}\), then we can increase the objective by traveling along the 3rd constraint to point \((2 + \frac{2}{k}, 0)\) which has an objective value of \(2 + \frac{2}{k} > 8\) if \(k < \frac{1}{2}\). Therefore, \((2,3)\) is optimal for \(k \geq \frac{1}{2}\).
Let \( t_{ijk} \) = # of units of paper type \( k \) shipped from paper mill \( i \) to customer \( j \).

\( y_{ijk} \) = # of units of paper type \( k \) produced on machine type \( k \) at mill \( i \).

We want to:

\[
\text{minimize } \sum_{i} \sum_{j} \sum_{k} t_{ijk} t_{ijk} + \sum_{i} \sum_{k} \sum_{l} p_{ik} y_{ikl}
\]

Subject to:

\[
\sum_{i} t_{ijk} \geq d_{jk} \quad \forall j, k \quad \text{(demand met)}
\]

\[
\sum_{j} y_{ijk} = \sum_{i} t_{ijk} \quad \forall i, k \quad \text{(amt produced of paper type \( k \), at mill \( i \) = amt shipped)}
\]

\[
\sum_{i} \sum_{k} y_{ik} r_{ikm} \leq r_{im} \quad \forall i, m \quad \text{(raw material available)}
\]

\[
\sum_{k} c_{ik} y_{ik} \leq C_{il} \quad \forall i, l \quad \text{(machine capacity)}
\]

\[
t_{ijk} \geq 0 \quad \forall i, j, k
\]

\[
y_{ijk} \geq 0 \quad \forall i, k, l
\]

---

### Table: Coefficient of

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Optimal Soln: \( x_1^* = 6 \frac{2}{3}, x_2^* = 0, x_3^* = 3 \frac{5}{6} \) and \( Z^* = 66 \frac{2}{3} \).
4.5.3. a) The constraints of any LP problem can be expressed in matrix notation as:
\[ Ax = b, \quad x \geq 0 \]
If \( x^1, x^2, \ldots, x^N \) are feasible solutions and
\[ x = \sum_{k=1}^{N} w_k x^k \quad \text{with} \quad \sum_{k=1}^{N} w_k = 1, \quad w_k \geq 0 \quad (k=1, \ldots, N) \]
then
\[ Ax = A\left(\sum_{k=1}^{N} w_k x^k\right) = \sum_{k=1}^{N} w_k Ax^k = \sum_{k=1}^{N} w_k b = b \]
So \( x \) is also a feasible solution.

b) Since basic feasible solutions are feasible solutions, the argument in part (a) shows any weighted average of them is also feasible.

4.5.4. a) If \( z^* \) is the value of the objective function for an optimal solution, and \( x^1, x^2, \ldots, x^N \) is the set of optimal basic feasible solutions, then for \( x = \sum_{k=1}^{N} t_k x^k \) with \( \sum_{k=1}^{N} t_k = 1, \quad t_k \geq 0 \) for \( k=1, 2, \ldots, N \), problem 4.13 shows \( x \) is feasible.

The objective function is of the form \( c^T x \). So the feasible solution \( x \) we have
\[ c^T x = c^T \left( \sum_{k=1}^{N} t_k x^k \right) = \sum_{k=1}^{N} t_k c^T x^k = \sum_{k=1}^{N} w_k r^k = z^* \]
so \( x \) is also an optimal solution.

b) Let \( x \) be a feasible solution which is not a weighted average of the set of optimal basic feasible solutions, \( x^1, x^2, \ldots, x^N \). \( x \) must be a weighted average of basic feasible solutions, not all of which are optimal. If \( x^1, x^2, \ldots, x^L \) are the basic feasible solutions which are not optimal, we can express \( x \) as:
\[ x = \sum_{k=1}^{L} \alpha_k x_k + \sum_{k=L+1}^{N} \alpha_k x^k \quad \text{with} \]
\[ \sum_{k=1}^{L} \alpha_k + \sum_{k=L+1}^{N} \alpha_k = 1, \quad \alpha_k \geq 0 \quad (k=1, 2, \ldots, N) \]
\[ \alpha_k > 0 \quad \text{for} \quad c^T x_k \quad \text{and} \quad \text{some} \quad \alpha_k < 0. \]

We can conclude:
\[ c^T x = c^T \left( \sum_{k=1}^{L} \alpha_k x_k + \sum_{k=L+1}^{N} \alpha_k x^k \right) = \sum_{k=1}^{L} \alpha_k c^T x_k + \sum_{k=L+1}^{N} \alpha_k c^T x^k \]
\[ < z^* \left( \sum_{k=1}^{L} \alpha_k + \sum_{k=L+1}^{N} \alpha_k \right) = z^* \]
so \( x \) is not an optimal solution.