Optimal Control of Spatially Distributed Systems[†]

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***Abstract*— In this paper, we study the structural properties of optimal control of spatially distributed systems. Such systems consist of an infinite collection of possibly heterogeneous linear control systems that are spatially interconnected via certain distant dependent coupling functions over arbitrary graphs. The key idea of the paper is the introduction of a special class of operators called spatially decaying (SD) operators. We study the structural properties of infinite-horizon linear quadratic optimal controllers for such systems by analyzing the spatial structure of the solution to the corresponding operator Lyapunov and Riccati equations. We prove that the kernel of the optimal feedback of each subsystem decays in the spatial domain at a rate proportional to the inverse of the corresponding coupling function of the system.**

I. INTRODUCTION

Analysis and synthesis of distributed coordination and control algorithms for networked dynamic systems has become a vibrant part of control theory research. Several authors have studied the problem of optimal control of certain classes of spatially distributed systems with symmetries in their spatial structure. In [1], Bamieh *et al.* used spatial Fourier transforms and operator theory to study optimal control of linear spatially invariant systems with standard $\mathcal{H}_2$ (LQ), and $\mathcal{H}_\infty$ criteria. It was shown that such problems can be tackled by solving a parameterized family of finite-dimensional problems in Fourier domain. Furthermore, the authors show that the resulting optimal controllers have an inherent spatial locality similar to the underlying system.

Another interesting related work in this area is reported in [2] where the authors use operator theoretic tools, motivated by results of [3] to analyze time-varying systems, and design optimal controllers for heterogeneous systems which are not shift invariant with respect to spatial or temporal variables. In [4], the authors introduce the notion of *quadratic invariance* for a constraint set (e.g. sparsity constraints on communication structure of plant and controller). Using this notion, the authors show that the problem of constructing optimal controllers with certain sparsity patterns on the information structure can be cast as a convex optimization problem.

This paper is very close in spirit to [1]. The objective of this paper is to analyze the spatial structure of infinite horizon optimal controllers of spatially distributed systems. Here, we extend the results of [1] to *heterogeneous* systems with *arbitrary* spatial structure and show that quadratically optimal controllers inherit the same spatial structure as the original plant. The key point of departure from [1] is that the systems considered in this work are not spatially invariant and the corresponding operators are not translation invariant either. The spatial structures studied in [1] are Locally Compact Abelian (LCA) groups [5] such as $(\mathbb{Z},+)$ and $(\mathbb{Z}_n,\oplus)$. As a result, the group operation naturally induces a translation operator for functions defined on the group. However, when the dynamics of individual subsystems are not identical and the spatial structure does not necessarily enjoy the symmetries of LCA groups, standard tools such as Fourier analysis cannot be used to analyze the system.

To address this issue, a new class of linear operators, called *spatially decaying* (SD) operators, are introduced that are natural extension of linear translation invariant operators. It is shown that such operators exhibit a localized behavior in spatial domain, i.e., the norm of blocks in the matrix representation of the operator decay in space. It turns out that the coupling between subsystems in many well-known cooperative control and networked control problems can be characterized by an SD operator. A linear control system is called *spatially decaying* if the operators in its state-space representation are SD. It is shown that the unique solution of *Lyapunov* and algebraic *Riccati* equations (ARE) corresponding to SD system are indeed SD themselves. As a result, the corresponding optimal controllers are SD and spatially localized, meaning that in the optimal controller, the gain of subsystems that are "farther away" from a given subsystem decays in space and the resulting controller is inherently localized.

The machinery developed in this paper can be used to analyze the spatial structure of a broader range of optimal control problems such as constrained, finite horizon control or Model Predictive Control of spatially distributed systems. This problem has been analyzed in detail in [6] and [7].

This paper is organized as follows. We introduce the notation and the basic concepts used throughout the paper in Section II. The optimal control problem for spatially distributed linear systems is presented in Section III. The concept of spatially decaying operators and their properties are introduced in Section IV. The structural properties of quadratically optimal controllers are addressed in Section V. Simulation results are included in Section VI. Finally, our concluding remarks are presented in Section VII.

II. PRELIMINARIES

$\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^+$ the set of non-negative real numbers, and $\mathbb{C}$ the set of complex numbers. Consider an undirected connected graph with a nonempty set $\mathbb{G}$ of nodes. We refer to $\mathbb{G}$ as the spatial domain. $|\,.\,|$ and $\|\,.\,\|$ denote the Euclidean vector norm and its corresponding induced matrix norm, respectively. The Banach space $\ell_p(\mathbb{G})$ for $1 \leq p < \infty$ is defined to be the set of all sequences

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$x = (x_i)_{i\in\mathbb{G}}$ in which $x_i \in \mathbb{R}^{n_i}$ satisfying $\sum_{i\in\mathbb{G}} |x_i|^p < \infty$ endowed with the norm $\|x\|_p := \left(\sum_{i\in\mathbb{G}} |x_i|^p\right)^{\frac{1}{p}}$. The Banach space $\ell_\infty(\mathbb{G})$ denotes the set of all bounded sequences endowed with the norm $\|x\|_\infty := \sup_{i\in\mathbb{G}} |x_i|$. Throughout the paper, we will use the shorthand notation ℓ_p for $\ell_p(\mathbb{G})$. The space ℓ_2 is a Hilbert space with inner product $\langle x, y\rangle := \sum_{i\in\mathbb{G}} \langle x_i, y_i\rangle$ for all $x, y \in \ell_2$. An operator $\mathcal{Q} : \ell_p \to \ell_q$ for $1 \le p, q \le \infty$ is bounded if it has a finite induced norm, i.e., the following quantity

$$\|\mathcal{Q}\|_{p,q} := \sup_{\|x\|_p = 1} \|\mathcal{Q}x\|_q$$

is bounded. The identity operator is denoted by $\mathcal{I}$. The set of all bounded linear operators of ℓ_p into itself is denoted by $\mathscr{B}(\ell_p)$. An operator $\mathcal{Q} \in \mathscr{B}(\ell_p)$ has an *algebraic* inverse if it has an inverse $\mathcal{Q}^{-1}$ in $\mathscr{B}(\ell_p)$ [8]. The adjoint operator of $\mathcal{Q} \in \mathscr{B}(\ell_2)$ is the operator $\mathcal{Q}^*$ in $\mathscr{B}(\ell_2)$ such that $\langle \mathcal{Q}x, y\rangle = \langle x, \mathcal{Q}^* y\rangle$ for all $x, y \in \ell_2$. An operator $\mathcal{Q}$ is self-adjoint if $\mathcal{Q} = \mathcal{Q}^*$. An operator $\mathcal{Q} \in \mathscr{B}(\ell_2)$ is *positive definite*, shown as $\mathcal{Q} \succ 0$, if there exists a number $\alpha > 0$ such that $\langle x, \mathcal{Q}x\rangle > \alpha \|x\|_2^2$ for all nonzero $x \in \ell_2$.

The set of all functions from $A \subseteq \mathbb{R}$ into $\mathbb{R}$ is a vector space $\mathscr{F}$ over $\mathbb{R}$. For $f_1, f_2 \in \mathscr{F}$, the notation $f_1 \prec f_2$ will be used to mean the pointwise inequality $f_1(s) < f_2(s)$ for all $s \in A$. A family of *seminorms* on $\mathscr{F}$ is defined as $\{\|\,.\,\|_T \mid T \in \mathbb{R}^+\}$ in which $\|f\|_T := \sup_{s\le T} |f(s)|$ for all $f \in \mathscr{F}$. The topology generated by all open $\|\,.\,\|_T$-balls is called the topology generated by the family of seminorms and is denoted by $\|\,.\,\|_T$-topology. Continuity of a function in this topology is equivalent to continuity in every seminorm in the family. Although the results of section III is set up in a general framework, in this paper we are interested in linear operators which have matrix representations.

III. Optimal Control of Spatially Distributed Systems

We begin by considering a continuous-time linear model for spatially distributed systems over a discrete spatial domain $\mathbb{G}$ described by

$$\frac{d}{dt}\psi(t) = (\mathcal{A}\psi)(t) + (\mathcal{B}u)(t) \qquad (1)$$

$$y(t) = (\mathcal{C}\psi)(t) + (\mathcal{D}u)(t) \qquad (2)$$

with the initial condition $\psi(0) = \psi_0$. All signals are assumed to be in $L_2([0,\infty); \ell_2)$ space: at each time instant $t \in [0,\infty)$, signals $\psi(t)$, $u(t)$, $y(t)$ are assumed to be in ℓ_2. The state-space operators $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ are assumed to be constant functions of time from ℓ_2 to itself. The semigroup generated by $\mathcal{A}$ is *strongly continuous* on ℓ_2. This assumption guarantees existence and uniqueness of classical solutions of the system given by (1)-(2) (cf. Chapter 3 of [9]).

Example 1: Consider the general one-dimensional heat equation for a bi-infinite bar [10]

$$\frac{\partial}{\partial t}\psi(x,t) = \frac{\partial}{\partial x}\left(c(x)\frac{\partial}{\partial x}\psi(x,t)\right) + b(x)u(x,t)$$

where x is the spatial independent variable, t is the temporal independent variable, $\psi(x,t)$ is the temperature of the bar, and $u(x,t)$ is a distributed heat source. The thermal conductivity c is only a function of x and is differentiable with respect to x. The boundary conditions are assumed to be $\psi(\infty,t) = \psi(-\infty,t) = 0$. By inserting finite difference approximation for the spatial partial derivatives, the following continuous-time, discrete-space model can be obtained:

$$\frac{\partial}{\partial t}\psi(x_k,t) = c'(x_k)\left(\frac{\psi(x_{k-1},t) - \psi(x_k,t)}{\delta}\right) + c(x_k)\left(\frac{\psi(x_{k-1},t) - 2\psi(x_k,t) + \psi(x_{k+1},t)}{\delta^2}\right) + b(x_k)u(x_k,t)$$

where $c'(x) = \frac{d}{dx}c(x)$. The discretization is performed with equal spacing $\delta = x_k - x_{k-1}$ of the points x_k such that there is an integer number of points in space. Hence, after discretization the spatial domain is $\mathbb{G} = \mathbb{Z}$. This model can be represented as linear system (1) in which the infinite-tuples $\psi(t) = (\psi(x_k,t))_{k\in\mathbb{G}}$ and $u(t) = (u(x_k,t))_{k\in\mathbb{G}}$ are the state and control input variables of the infinite-dimensional system and the block elements of the state-space operators $\mathcal{A}$ and $\mathcal{B}$ are defined as follows

$$[\mathcal{A}]_{ki} = \begin{cases} \frac{c'(x_k)\delta + c(x_k)}{\delta^2} & , \quad i = k-1 \\ -\frac{c'(x_k)\delta + 2c(x_k)}{\delta^2} & , \quad i = k \\ \frac{c(x_k)}{\delta^2} & , \quad i = k+1 \\ 0 & , \quad \text{otherwise} \end{cases}$$

and

$$[\mathcal{B}]_{ki} = \begin{cases} b(x_k) & , \quad i = k \\ 0 & , \quad \text{otherwise} \end{cases}$$

for all $k, i \in \mathbb{G}$. One can show that $\mathcal{A}$ is an unbounded operator on ℓ_2. However, the semigroup generated by $\mathcal{A}$ is strongly continuous on ℓ_2.

A. Exponential Stability

Consider the following autonomous system over $\mathbb{G}$

$$\frac{d}{dt}\psi(t) = (\mathcal{A}\psi)(t) \qquad (3)$$

with initial condition $\psi(0) = \psi_0$. Suppose that $\mathcal{A}$ generates a strongly continuous C_0-semingroup on ℓ_2, denoted by $\mathcal{T}(t)$. The system (3) is *exponentially stable* if

$$\|\mathcal{T}(t)\|_{2,2} \le Me^{-\alpha t} \quad \text{for} \quad t \ge 0$$

for some $M, \alpha > 0$.

Theorem 1 [9]: Let $\mathcal{A}$ be the infinitesimal generator of the C_0-semigroup $\mathcal{T}(t)$ on ℓ_2 and $\mathcal{Q}$ a positive definite operator. Then $\mathcal{T}(t)$ is exponentially stable if and only if the *Lyapunov* equation

$$\langle \mathcal{A}\phi, \mathcal{P}\phi\rangle + \langle \mathcal{P}\phi, \mathcal{A}\phi\rangle + \langle \phi, \mathcal{Q}\phi\rangle = 0 \qquad (4)$$

for all $\phi \in \mathscr{D}(\mathcal{A})$, has a positive definite solution $\mathcal{P} \in \mathscr{B}(\ell_2)$.

B. LQR *control of infinite dimensional systems*

While the main results of this paper are proven for LQ optimal controllers, similar results can be proven for $\mathcal{H}_\infty$ and $\mathcal{H}_2$ problems. In general, the solutions to these problems can be formulated in terms of two operator AREs. Such problems have been addressed in the literature for general classes of distributed parameter systems [9], [11]. An elegant analysis for the spatially invariant case can be found in [1]. Similar to the finite-dimensional case, optimal solutions to infinite-dimensional LQR can be formulated in terms of an operator Riccati equation. Consider the quadratic cost functional given by

$$\mathfrak{J} = \int_0^\infty \langle \psi(t), \mathcal{Q}\psi(t)\rangle + \langle u(t), \mathcal{R}u(t)\rangle \ dt. \quad (5)$$

The system (1)-(2) with cost (5) is said to be *optimizable* if for every initial condition $\psi(0) = \psi_0 \in \ell_2$, there exists an input function $u \in L_2([0,\infty);\ell_2)$ such that the value of (5) is finite [9]. Note that if $(\mathcal{A},\mathcal{B})$ is *exponentially stabilizable*, then the system (1)-(2) is optimizable.

Theorem 2 [9]: Let operators $\mathcal{Q} \succeq 0$ and $\mathcal{R} \succ 0$ be in $\mathscr{B}(\ell_2)$. If the system (1)-(2) with cost functional (5) is optimizable and $(\mathcal{A}, \mathcal{Q}^{1/2})$ is exponentially detectable, then there exists a unique nonnegative, self-adjoint operator $\mathcal{P} \in \mathscr{B}(\ell_2)$ satisfying the ARE

$$\begin{aligned}\langle \varphi, \mathcal{P}\mathcal{A}\phi\rangle + \langle \mathcal{P}\mathcal{A}\varphi, \phi\rangle \quad &+ \quad \langle \varphi, \mathcal{Q}\phi\rangle \\ &- \quad \langle \mathcal{B}^*\mathcal{P}\varphi, \mathcal{R}^{-1}\mathcal{B}^*\mathcal{P}\phi\rangle = 0 \quad (6)\end{aligned}$$

for all $\varphi, \phi \in \mathscr{D}(\mathcal{A})$ such that $\mathcal{A} - \mathcal{B}\mathcal{R}^{-1}\mathcal{B}^*\mathcal{P}$ generates an exponentially stable C_0-semigroup. Moreover, the optimal control $\tilde{u} \in L_2([0,\infty);\ell_2)$ is given by the feedback law

$$\tilde{u}(t) = -\mathcal{R}^{-1}\mathcal{B}^*\mathcal{P} \ \tilde{\psi}(t)$$

where $\tilde{\psi}$ is the solution of

$$\frac{d}{dt}\tilde{\psi}(t) = (\mathcal{A} - \mathcal{B}\mathcal{R}^{-1}\mathcal{B}^*\mathcal{P}) \ \tilde{\psi}(t) \quad (7)$$

with initial condition ψ_0.

Solving equations (4) and (6) can be a tedious task in general. However, the complexity of the problem will reduce significantly if the underlying system is spatially invariant with respect to $\mathbb{G}$ (cf. Section III.B of [1]). The main objective of this paper is to analyze the spatial structure of the solutions of operator equations (4) and (6) rather than solving them explicitly.

IV. Spatially Decaying Operators

The key difficulty in extending the results of [1] is that the notion of spatial invariance was critical in being able to use Fourier methods which greatly simplified the analysis. Simply put, if we replace "space" with "time", we get a more familiar analogue of this problem: Fourier methods can not be used directly for analysis of linear time-varying systems. In the following, we will generalize the notion of regions of analyticity of transforms to a larger class of linear operators. Without loss of generality, in the following definitions it is assumed that all operators are self-adjoint.

Definition 1: A distance function on a discrete topology with a set of nodes $\mathbb{G}$ is defined as a single-valued, nonnegative, real function $\mathrm{dis}(k,i)$ defined for all $k,i,j \in \mathbb{G}$ which has the following properties:

(i) $\mathrm{dis}(k,i) = 0$ iff $k = i$.
(ii) $\mathrm{dis}(k,i) = \mathrm{dis}(i,k)$.
(iii) $\mathrm{dis}(k,i) \le \mathrm{dis}(k,j) + \mathrm{dis}(j,i)$.

Definition 2: A nondecreasing continuous function $\chi : \mathbb{R}^+ \to [1,\infty)$ is called a coupling characteristic function if $\chi(0) = 1$ and $\chi(s+t) \le \chi(s)\ \chi(t)$ for all $s,t \in \mathbb{R}^+$. The constant coupling characteristic function with unit value everywhere is denoted by $\mathbf{1}$.

In order to be able to characterize rates of decay we define a one-parameter family of coupling characteristic functions as follows.

Definition 3: A one-parameter family of coupling characteristic functions $\mathscr{C}$ is defined to be the set of all characteristic functions χ_α, χ_β for $\alpha,\beta \in \mathbb{R}^+$ such that

(i) $\chi_0 = \mathbf{1}$.
(ii) $\chi_\alpha\chi_\beta = \chi_{\alpha+\beta}$.
(iii) For $\alpha < \beta$, relation $\chi_\alpha \prec \chi_\beta$ holds .
(iv) χ_α is a continuous function of α in $\|\,.\,\|_T$-topology.

Using this definition, we can now formally define a spatially decaying (SD) operator.

Definition 4: Suppose that a distance function $\mathrm{dis}(.,.)$ and a one-parameter family of parameterized coupling characteristic functions $\mathscr{C}$ are given. A linear operator $\mathcal{Q}$ is SD with respect to $\mathscr{C}$ if there exists $\tau > 0$ such that the auxiliary operator $\tilde{\mathcal{Q}}$, defined block-wise as

$$[\tilde{\mathcal{Q}}]_{ki} = [\mathcal{Q}]_{ki}\ \chi_\alpha(\mathrm{dis}(k,i))$$

is bounded on ℓ_p for all $0 \le \alpha < \tau$. The number τ is referred to as the *decay margin*.

In general, determining the boundedness of the auxiliary operator depends on the choice of p. The result of lemma 1 gives us a simple sufficient condition for an operator to be SD in terms of all ℓ_p-norms. Under the assumptions of definition 4, we also assume that the following condition holds

$$\sup_{k\in\mathbb{G}} \sum_{i\in\mathbb{G}} \chi_\alpha(\mathrm{dis}(k,i))^{-1} < \infty$$

for all $0 < \alpha < \tau$.

Lemma 1: A linear operator $\mathcal{Q}$ is SD with respect to the one-parameter family of coupling characteristic functions $\mathscr{C}$ on all ℓ_p if there exists $\tau > 0$ such that the following holds

$$\sup_{k\in\mathbb{G}} \sum_{i\in\mathbb{G}} \|[\mathcal{Q}]_{ki}\|\ \chi_\alpha(\mathrm{dis}(k,i)) < \infty \quad (8)$$

for all $0 \le \alpha < \tau$.

Proof: See [12] for a proof. ■

Examples of SD operators appear naturally in many applications. Intuitively, we may interpret the norm of each block element $[\mathcal{Q}]_{ki}$ as the coupling strength between subsystems k and i. Given the one-parameter family of coupling characteristic functions $\mathscr{C}$, fix a value for $\alpha \in [0,\tau)$. For an infinite graph, if we fix a node k and move on the graph away from node k, the coupling strength decays proportional to the inverse of the coupling characteristic function $\chi_{\hat{\alpha}}$ with $\alpha < \hat{\alpha} < \tau$ so that relation (8) holds. The notion of an SD operator will be key in proving spatial locality of optimal controllers. Throughout the rest of the paper, we say an operator is SD if it satisfies condition (8).

A. Examples of Spatially Decaying Operators

The following class of operators which are used extensively in cooperative and distributed control are interesting special classes of SD operators.

1) Spatially Truncated Operators: These are operators with finite range couplings. Examples of such operators arise in motion coordination of autonomous agents such as the

Laplacian operator. Given the coupling range $T > 0$, the following class of linear operators are SD with respect to every coupling characteristic functions

$$[\mathcal{Q}]_{ki} = \begin{cases} Q_{ki} & \text{if } \operatorname{dis}(k,i) \leq T \\ 0 & \text{if } \operatorname{dis}(k,i) > T \end{cases} \quad (9)$$

where $Q_{ki} \in \mathbb{R}^{n\times n}$. For such operators and every given node $k \in \mathbb{G}$, we have that

$$\sum_{i\in\mathbb{G}} \|[\mathcal{Q}]_{ki}\| \; \chi_\alpha(\operatorname{dis}(k,i)) \leq \sum_{i\sim k} \|[\mathcal{Q}]_{ki}\| \; \chi_\alpha(T) < \infty. \quad (10)$$

The relation $\sim$ is the neighborhood relation defined as $i \sim k$ if and only if $\operatorname{dis}(k,i) \leq T$. Inequality (10) shows that $\mathcal{Q}$ is SD with respect to every $\mathscr{C}$ and the decay margin is $\tau = \infty$.

2) Exponentially Decaying Operators: Consider the one-parameter family of coupling characteristic functions $\mathscr{C}_E$ defined by

$$\chi_\zeta(s) = (1+\zeta)^s \quad (11)$$

where $\zeta \in \mathbb{R}^+$. Operator $\mathcal{Q}$ is said to be *exponentially* SD if condition (8) holds with respect to $\mathscr{C}_E$ defined by (11) for all $\zeta \in [0,\tau)$ where $\tau > 0$ is the decay margin. An important example of exponentially SD operators is the class of translation invariant operators with $\mathbb{G} = \mathbb{Z}$. It can be shown that, under some mild assumptions, a translation invariant operator in $\mathscr{B}(\ell_2)$ is exponentially SD with $\operatorname{dis}(k,i) = |k-i|$ as a natural notion of distance [12]. The decay margin of $\mathcal{Q}$ is equal to r, the distance of the nearest pole of the Fourier transform of $\mathcal{Q}$ to the unit circle in $\mathbb{C}$.

3) Algebraically Decaying Operators: Consider the parameterized family of characteristic functions $\mathscr{C}_A$ defined as

$$\chi_\nu(s) = (1+\lambda s)^\nu \quad (12)$$

in which $\lambda > 0$ and $\nu \in \mathbb{R}^+$. Operator $\mathcal{Q}$ is said to be *algebraically* SD if condition (8) holds with respect to $\mathscr{C}_A$ defined by (12) for all $\nu \in [0,\tau)$ where $\tau > 0$ is the decay margin. Such functions are often used as pair-wise potentials among agents in flocking and cooperative control problems [13]. Another example of such coupling functions arises in wireless networks. The coupling between nodes, which is considered as the power of the communication signal between agents, decays with the inverse fourth power law, i.e., $\frac{1}{\operatorname{dis}(k,i)^4}$.

B. Properties of SD Operators

We define the operator norm

$$\|\mathcal{Q}\|_\tau^\star = \sup_{\alpha\in[0,\tau)} \sup_{k\in\mathbb{G}} \sum_{i\in\mathbb{G}} \|[\mathcal{Q}]_{ki}\| \; \chi_\alpha(\operatorname{dis}(k,i))$$

and the normed vector space

$$\mathcal{S}_\tau(\mathscr{C}) = \{\mathcal{Q} \;:\; \|\mathcal{Q}\|_\tau^\star < \infty\}.$$

It can be shown that the operator norm satisfies the following properties [12], for all $\mathcal{Q},\mathcal{P} \in \mathcal{S}_\tau(\mathscr{C})$ and $c \in \mathbb{C}$,

(i) $\|\mathcal{Q}\|_\tau^\star \geq 0$ and $\|\mathcal{Q}\|_\tau^\star = 0$ iff $\mathcal{Q} = 0$.
(ii) $\|c\;\mathcal{Q}\|_\tau^\star = |c| \; \|\mathcal{Q}\|_\tau^\star$.
(iii) $\|\mathcal{Q}+\mathcal{P}\|_\tau^\star \leq \|\mathcal{Q}\|_\tau^\star + \|\mathcal{P}\|_\tau^\star$.
(iv) $\|\mathcal{Q}\mathcal{P}\|_\tau^\star \leq \|\mathcal{Q}\|_\tau^\star \; \|\mathcal{P}\|_\tau^\star$.

Property (iv) is called the submultiplicative property.

Theorem 3: Given a one-parameter family of coupling characteristic functions $\mathscr{C}$ and $\tau > 0$, the operator space $\mathcal{S}_\tau(\mathscr{C})$ forms a *Banach Algebra* with respect to the operator norm $\|.\|_\tau^\star$ under the operator composition operation.

Proof: See [12] for a proof. ■

The above theorem is a key ingredient in proving that optimal controllers of SD systems are SD. We have shown that operator space $\mathcal{S}_\tau(\mathscr{C})$ is closed under addition, multiplication, and limit properties (cf. [12], Theorem 5). Furthermore, if an SD operator has an algebraic inverse on $\mathscr{B}(\ell_2)$, the inverse operator $\mathcal{Q}^{-1}$ is also SD [14]. It is straightforward to check that the serial, parallel, and well-posed feedback interconnection of two SD systems are also SD. In the next section, using the closure properties of SD operators, it is shown that the solution of differential Lyapunov and Riccati equations converge to an SD operator.

V. Structure of Quadratically Optimal Controllers

As discussed in section III, our aim is not to solve the Lyapunov equation (4) and ARE (6) explicitly but to study the spatial structure of the solution of these algebraic equations by means of tools developed in the previous sections. In the following, it is shown that the solution of equations (4) and (6) have an inherent spatial locality and the characteristics of the coupling function will determine the degree of localization.

Theorem 5: Assume that operators $\mathcal{A},\mathcal{Q} \in \mathcal{S}_\tau(\mathscr{C})$ and $\mathcal{Q}$ is positive definite. If $\mathcal{A}$ is the infinitesimal generator of an exponentially stable C_0-semigroup $\mathcal{T}(t)$ on ℓ_2, then the unique positive definite solution of operator Lyapunov equation (4) satisfies $\mathcal{P} \in \mathcal{S}_\tau(\mathscr{C})$.

Proof: See [12] for a proof. ■

In the next theorem without loss of generality, we will assume that $\mathcal{R} = \mathcal{I}$. Otherwise, by only assuming that $\mathcal{R}$ has an algebraic inverse on $\mathscr{B}(\ell_2)$, it can be shown that $\mathcal{R}^{-1}$ is SD [14]. According to the closure under multiplication property of SD operators, if $\mathcal{P}$ and $\mathcal{B}$ are SD, then the optimal feedback operator $\mathcal{K} = -\mathcal{R}^{-1}\mathcal{B}^*\mathcal{P}$ will be SD.

Theorem 6: Let $\mathcal{A},\ \mathcal{B},\ \mathcal{Q} \in \mathcal{S}_\tau(\mathscr{C})$ and $\mathcal{Q} \succeq 0$. Moreover, assume that conditions of Theorem 2 hold. Then the unique positive definite solution of operator ARE (6) satisfies $\mathcal{P} \in \mathcal{S}_\tau(\mathscr{C})$.

Proof: Consider the Differential Riccati Equation

$$\begin{aligned}\frac{d}{dt}\langle\varphi,\mathcal{P}(t)\phi\rangle &= \langle\varphi,\mathcal{P}(t)\mathcal{A}\phi\rangle + \langle\mathcal{P}(t)\mathcal{A}\varphi,\phi\rangle + \langle\varphi,\mathcal{Q}\phi\rangle \\ &\quad - \langle\mathcal{B}^*\mathcal{P}(t)\varphi,\mathcal{B}^*\mathcal{P}(t)\phi\rangle\end{aligned}$$

with $\mathcal{P}(0) = 0$. We denote the unique solution of this differential Riccati equation in the class of strongly continuous, self-adjoint operators in $\mathscr{B}(\ell_2)$ by the one-parameter family of operator-valued function $\mathcal{P}(t)$ for $t \geq 0$. The nonnegative operator $\mathcal{P}$, the unique solution of ARE, is the strong limit of $\mathcal{P}(t)$ on ℓ_2 as $t \to \infty$ (see theorem 6.2.4 of [9]). Therefore, we have that

$$\lim_{t\to\infty} \|\mathcal{P}(t) - \mathcal{P}\|_{2,2} = 0. \quad (13)$$

From the differential Riccati equation, it follows that

$$\frac{d}{dt}[\mathcal{P}(t)]_{ki} = [\mathcal{A}^*\mathcal{P}(t) + \mathcal{P}(t)\mathcal{A} - \mathcal{P}(t)\mathcal{B}\mathcal{B}^*\mathcal{P}(t) + \mathcal{Q}]_{ki}$$

for all $k, i \in \mathbb{G}$. For a differentiable matrix $X(t) \in \mathbb{C}^{n\times n}$ for $t \geq 0$, we have the following inequality

$$\begin{aligned}\frac{d}{dt}\|X(t)\| &= \lim_{\delta\to 0}\frac{\|X(t+\delta)\| - \|X(t)\|}{\tau} \\ &\leq \left\|\lim_{\delta\to 0}\frac{X(t+\delta) - X(t)}{\delta}\right\| \leq \left\|\frac{d}{dt}X(t)\right\|. \quad (14)\end{aligned}$$

Using inequality (14), we have

$$\begin{aligned}\frac{d}{dt}\|\mathcal{P}(t)\|_\tau^\star &\leq \sup_{k\in\mathbb{G}}\sum_{i\in\mathbb{G}} \|\frac{d}{dt}[\mathcal{P}(t)]_{ki}\| \ \chi_\alpha(\mathrm{dis}(k,i)) \\ &\leq \|\mathcal{A}^*\mathcal{P}(t) + \mathcal{P}(t)\mathcal{A} - \mathcal{P}(t)\mathcal{B}\mathcal{B}^*\mathcal{P}(t) + \mathcal{Q}\|_\tau^\star.\end{aligned}$$

For simplicity in notations, denote $\pi(t) = \|\mathcal{P}(t)\|_\tau^\star$. Using the triangle inequality and the fact that norm $\|\,.\,\|_\tau^\star$ is sub-multiplicative, we have the following differential inequality

$$\dot{\pi}(t) \ \leq \ 2\ \|\mathcal{A}\|_\tau^\star\ \pi(t) + (\|\mathcal{B}\|_\tau^\star)^2\ \ \pi(t)^2 + \|\mathcal{Q}\|_\tau^\star \qquad (15)$$

with initial condition $\pi(0) = 0$ and constraint $\pi(t) \geq 0$ for all $t \geq 0$. All coefficients $\|\mathcal{A}\|_\tau^\star, \|\mathcal{B}\|_\tau^\star, \|\mathcal{Q}\|_\tau^\star$ in the right hand side of the inequality (15) are finite numbers. If $\pi(t)$ for $t \geq 0$ is a solution of the differential inequality (15), then it is also a solution of the following differential inequality

$$\dot{\pi}(t) \leq \ \lambda\ (\pi(t) + 1)^2 \qquad (16)$$

with $\pi(0) = 0$ and $\lambda = \max(\|\mathcal{A}\|_\tau^\star, (\|\mathcal{B}\|_\tau^\star)^2, \|\mathcal{Q}\|_\tau^\star)$. In other words, the set of feasible solutions of (15) is a subset of solutions of (16). From (16), we have

$$-\frac{d}{dt}\left(\frac{1}{\pi(t)+1}\right) \leq \lambda$$

which has the set of solutions

$$\frac{1}{\pi(t)+1} \geq \frac{e^{-\lambda t}}{\pi(0)+1}.$$

Using the fact that $\pi(t) \geq 0$ for all $t \geq 0$ and $\pi(0) = 0$, it follows that

$$\pi(t) \leq e^{\lambda t} - 1.$$

The above inequality is feasible, i.e., there exists at least one sequence of solutions satisfying $\pi(t) \geq 0$ for all $t \geq 0$. The above inequality also proves that $\pi(t) < \infty$ for all $t \geq 0$. Thus, we have that $\mathcal{P}(t) \in \mathcal{S}_\tau(\mathscr{C})$ for all $t \geq 0$. According to Theorem 5 in [12], we can use this result and (13) to conclude that $\mathcal{P} \in \mathcal{S}_\tau(\mathscr{C})$. This completes the proof. ■

VI. Simulation

We consider a large network of N linear subsystems coupled on an arbitrary graph which can be described by

$$\frac{d}{dt}\psi(t) = (\mathcal{A}\psi)(t) \ + \ (\mathcal{B}u)(t).$$

The coupling characteristic function is χ and the system operators are given by $[\mathcal{A}]_{ki} = \frac{10}{\chi(\mathrm{dis}(k,i))}$ and $\mathcal{B} = \mathcal{I}$. The distance function is Euclidean. We will study the LQR problem discussed in Section III with weighting operators $\mathcal{R} = \mathcal{I}$ and $\mathcal{Q}$ being the corresponding unweighted graph Laplacian. The corresponding ARE is given by

$$\mathcal{A}^*\mathcal{P} + \mathcal{P}\mathcal{A} - \mathcal{P}^2 + \mathcal{Q} = 0. \qquad (17)$$

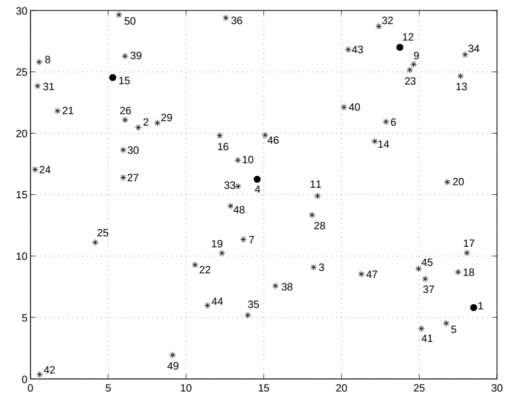

Fig. 1. N=50 nodes are randomly and uniformly distributed in a region of area 30×30 (units)^2. Each node is a linear subsystem which is coupled to other subsystems through their dynamic and a central cost function by a given coupling characteristic function.

Then the LQR optimal feedback is given by $\mathcal{K} = -\mathcal{P}$. In the following simulations, it is assumed that $N = 50$ nodes are randomly and uniformly distributed in a region of area 30×30 (units)^2. Each node is assumed to be a linear system which is coupled through its dynamic and the LQR cost functional to other subsystems. In the sequel, three different scenarios are considered for the coupling characteristic function: algebraical decay, exponential decay, and nearest neighbor coupling. The results are shown, respectively, in Figures 2, 3, and 4 where the norm of the LQR feedback gains $[\mathcal{K}]_{ki}$ corresponding to agents $k = 1, 4, 12, 15$ (their locations are marked by bold stars in Figure 1) is depicted versus the distance of other subsystems to subsystem k. As seen from these simulations, for every subsystem k the norm of the optimal feedback kernel $[\mathcal{K}]_{ki}$ is enveloped by the inverse of the coupling characteristic function. Therefore, the spatial decay rate of the optimal controller can be determined priory only using the information of the coupling characteristic function.

A. Spatial Truncation

Let $\mathcal{K}_T$ be the spatially truncated operator defined by

$$[\mathcal{K}_T]_{ki} = \begin{cases} [\mathcal{K}]_{ki} & \text{if} \quad \mathrm{dis}(k,i) \leq T \\ 0 & \text{if} \quad \mathrm{dis}(k,i) > T. \end{cases}$$

By applying the small-gain stability argument, one can obtain a truncation length T_s for which $\mathcal{K}_T$ is stabilizing for all $T \geq T_s$ (cf. Section V.B in [1]). Figure 5 illustrates the performance loss percentage defined as $\left|\frac{\mathrm{Trace}(\mathcal{P}_T) - \mathrm{Trace}(\mathcal{P})}{\mathrm{Trace}(\mathcal{P})}\right| \times 100$ versus different values of $T \geq T_s$ for different coupling characteristic functions where $\mathcal{P}_T$ satisfies

$$(\mathcal{A} + \mathcal{B}\mathcal{K}_T)^*\mathcal{P}_T + \mathcal{P}_T(\mathcal{A} + \mathcal{B}\mathcal{K}_T) + \mathcal{Q} + \mathcal{K}_T^*\mathcal{R}\mathcal{K}_T = 0.$$

As seen from Figure 5, the larger values of truncation length T ensue better closed-loop performance.

In the above simulations, the extension of this surprising locality result to finite-dimensional systems is due to the fact that the matrices in system's state-space representation are

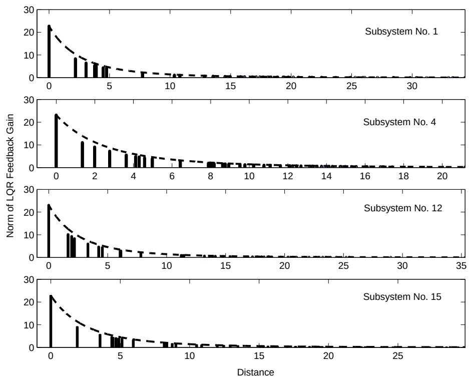

Fig. 2. Norm of LQR feedback gain $\|[\mathcal{K}]_{ki}\|$ (bar) and function $\frac{\|[\mathcal{K}]_{kk}\|}{\chi_\nu(\mathrm{dis}(k,i))}$ (which is algebraically decaying) when $\lambda = 0.1$ and $\nu = 4$ (dashed) for subsystems $k = 1, 4, 12, 15$, respectively, from top to bottom.

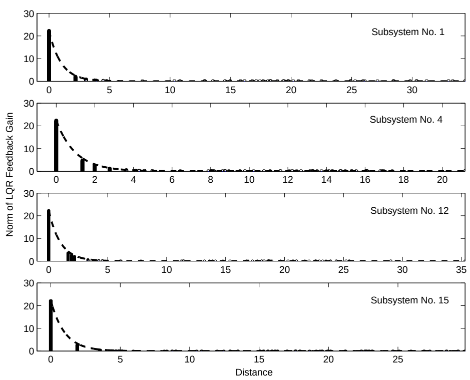

Fig. 3. Norm of LQR feedback gain $\|[\mathcal{K}]_{ki}\|$ (bar) and function $\frac{\|[\mathcal{K}]_{kk}\|}{\chi_\zeta(\mathrm{dis}(k,i))}$ (which is exponentially decaying) when $\zeta = e - 1$ (dashed) for subsystems $k = 1, 4, 12, 15$, respectively, from top to bottom.

defined such that the norm of blocks in the matrix decay as a function of distance between subsystems.

VII. Conclusions

In this paper we studied the spatial structure of infinite horizon optimal controllers for spatially distributed systems. By introducing the notion of SD operators we extended the notion of analytic continuity to operators that are not spatially invariant. Furthermore, we proved that SD operators form a Banach algebra. We used this to prove that solutions of Lyapunov and Riccati equations for SD systems are themselves SD. This result was utilized to show that the kernel of optimal LQ feedback is also SD. Although these results were proven for LQ problems, they can be easily extended to general $\mathcal{H}_2$, and $\mathcal{H}_\infty$ optimal control problems as the key enabling property is the spatial decay of solution of the corresponding Riccati equations. One major implication of these results is that the optimal control problem for spatially decaying systems lends itself to distributed solutions without too much loss in performance as even the centralized solutions are inherently localized.

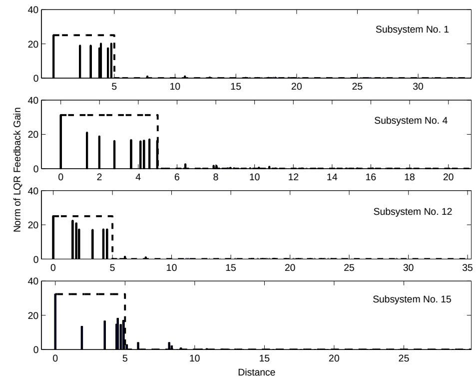

Fig. 4. Norm of LQR feedback gain $\|[\mathcal{K}]_{ki}\|$ (bar) and $\|[\mathcal{K}]_{ki}\| \times$ pulse function (which represents the nearest neighbor coupling) with length $T = 5$ (dashed) for subsystems $k = 1, 4, 12, 15$, respectively, from top to bottom.

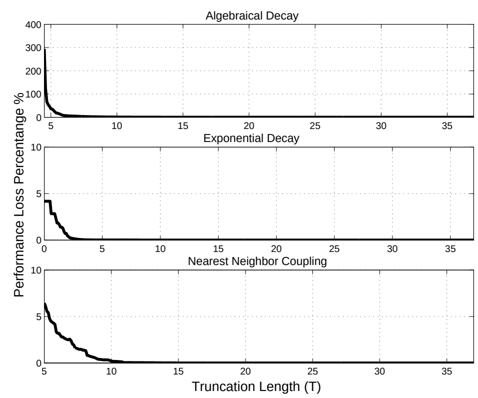

Fig. 5. Performance Loss percentage of LQR controller after spatial truncation for different types of couplings: (i) algebraical decay (ii) exponential decay (iii) nearest neighbor coupling.

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