Coordination of Groups of Mobile Autonomous Agents Using Nearest Neighbor Rules*

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Abstract

In a recent *Physical Review Letters* article, Vicsek *et al.* propose a simple but compelling discrete-time model of n autonomous agents {i.e., points or particles} all moving in the plane with the same speed but with different headings. Each agent's heading is updated using a local rule based on the average of its own heading plus the headings of its "neighbors." In their paper, Vicsek *et al.* provide simulation results which demonstrate that the nearest neighbor rule they are studying can cause all agents to eventually move in the same direction despite the absence of centralized coordination and despite the fact that each agent's set of nearest neighbors change with time as the system evolves. This paper provides a theoretical explanation for this observed behavior. In addition, convergence results are derived for several other similarly inspired models. The Vicsek model proves to be a graphic example of a switched linear system which is stable, but for which there does not exist a common quadratic Lyapunov function.

1 Introduction

In a recent paper [1], Vicsek *et al.* propose a simple but compelling discrete-time model of n autonomous agents {i.e., points or particles} all moving in the plane with the same speed but with different headings. Each agent's heading is updated using a local rule based on the average of its own heading plus the headings of its "neighbors." Agent i's *neighbors* at time t, are those agents which are either in or on a circle of pre-specified radius r centered at agent i's current position. The Vicsek model turns out to be a special version of a model introduced previously by Reynolds [2] for simulating visually satisfying flocking and schooling behaviors for the animation industry. In their paper, Vicsek *et al.* provide a variety of interesting simulation results which demonstrate that the nearest neighbor rule they are studying can cause all agents to eventually move in the same direction despite the absence of centralized coordination and despite the fact that each agent's set of nearest neighbors change with time as the system evolves. In this paper we provide a theoretical explanation for this observed behavior.

There is a large and growing literature concerned with the coordination of groups of mobile autonomous agents. Included here is the work of Czirok *et al.* [3] who propose one-dimensional models which exhibit the same type of behavior as Vicsek's. In [4, 5], Toner and Tu construct a continuous "hydrodynamic" model of the group of agents, while other authors such as Mikhailov and Zanette [6] consider the behavior of populations of self propelled particles with long range interactions. Schenk *et al.* determined interactions between individual self-propelled spots from underlying reaction-diffusion equation [7]. Meanwhile in modelling biological systems, Grünbaum and Okubo use statistical methods to analyze group behavior in animal aggregations.

In addition to these modelling and simulation studies, research papers focusing on the detailed mathematical analysis of emergent behaviors are beginning to appear. For example, Lui *et al.* [8] use Lyapunov methods and Leonard *et al.* [9] and Olfati and Murray [10] use potential function theory to understand flocking behavior while Fax and Murray [11] and Desai *et al.* [12] employ graph theoretic techniques for the same purpose. The one feature which sharply distinguishes previous analyses from that undertaken here is that the latter explicitly takes into account possible changes in nearest neighbors over time, whereas the former do not. Changing nearest neighbor sets is an inherent property of the Vicsek model and in the other models we consider. To analyze such models, it proves useful to appeal to well-known results [13, 14] characterizing the convergence of infinite products of certain types of non-negative matrices. The study of infinite matrix products is ongoing [15, 16, 17, 18, 19, 20] and is undoubtedly producing results which will find application in the theoretical study of emergent behaviors.

Vicsek's model is set up in Section 2 as a system of n simultaneous, one-dimensional recursion equations, one for each agent. A family of simple graphs on n vertices is then introduced to characterize all possible neighbor relationships. Doing this makes it possible to represent the Vicsek model as an n-dimensional switched linear system whose switching signal takes values in the set of indices which parameterize the family of graphs. The matrices which are switched within the system turn out to be non-negative with special structural properties. By exploiting these properties and making use of a classical convergence result due to Wolfowitz [13], we prove that all n agents' headings converge to a common steady state heading provided the n agents are all "linked together" via their neighbors with sufficient frequency as the system evolves. The model under consideration turns out to provide a graphic example of a switched linear system which is stable, but for which there does not exist a common quadratic Lyapunov function.

In Section 2.2 we define the notion of an average heading vector in terms of graph Laplacians [21] and we show how this idea leads naturally to the Vicsek model as well as to other decentralized control models which might be used for

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the same purposes. We propose one such model which assumes each agent knows an upper bound on the number of agents in the group, and we explain why this model has the convergence properties similar to Vicsek's.

In Section 3 we consider a modified version of Vicsek's discrete-time system consisting of the same group of n agents, plus one additional agent, labelled 0, which acts as the group's leader. Agent 0 moves at the same constant speed as its n followers but with a fixed heading θ_0. The ith follower updates its heading just as in the Vicsek model, using the average of its own heading plus the headings of its neighbors. For this system, each follower's set of neighbors can also include the leader and does so whenever the leader is within the follower's neighborhood defining circle of radius r. We prove that the headings of all n agents must converge to the leader's provided all n agents are " linked to their leader" together via their neighbors frequently enough as the system evolves.

2 Leaderless Coordination

The system studied by Vicsek *et al.* in [1] consists of n autonomous agents {e.g., points or particles}, labelled 1 through n, all moving in the plane with the same speed but with different headings. Each agent's heading is updated using a simple local rule based on the average of its own heading plus the headings of its "neighbors." Agent i's *neighbors* at time t, are those agents which are either in or on a circle of pre-specified radius r centered at agent i's current position. In the sequel $\mathcal{N}_i(t)$ denotes the set of labels of those agents which are neighbors of agent i at time t. Agent i's heading, written θ_i, evolves in discrete-time in accordance with a model of the form

$$\theta_i(t+1) = <\theta_i(t)>_r \tag{1}$$

where t is a discrete-time index taking values in the non-negative integers $\{0, 1, 2, \ldots\}$, and $<\theta_i(t)>_r$ is the average of the headings of agent i and agent i's neighbors at time t; that is

$$<\theta_i(t)>_r = \frac{1}{1+n_i(t)}\left(\theta_i(t) + \sum_{j\in\mathcal{N}_i(t)} \theta_j(t)\right) \tag{2}$$

where $n_i(t)$ is the number of neighbors of agent i at time t. Here and elsewhere in this paper, headings are represented as real numbers between $-\infty$ and ∞. We could also represent headings as numbers between 0 and 2π without changing any of the results which follow; this is a consequence of the fact that if all headings are between 0 and 2π before an averaging step, they will all be between 0 and 2π after averaging.

The explicit form of the update equations determined by (1) and (2) depends on the relationships between neighbors which exist at time t. These relationships can be conveniently described by a simple, undirected graph [1] with vertex set $\{1, 2, \ldots, n\}$ which is defined so that (i, j) is one of the graph's edges just in case agents i and j are neighbors. Since the relationships between neighbors can change over time, so can the graph which describes them. To account for this we will need to consider all possible such graphs. In the sequel we use the symbol $\mathcal{P}$ to denote a suitably defined set, indexing the class of all simple graphs $\mathbb{G}_p$ defined on n vertices. The set of agent heading update rules defined by (1) and (2), can be written in state form. Toward this end, for each $p \in \mathcal{P}$, define

$$F_p = (I + D_p)^{-1}(A_p + I) \tag{3}$$

where A_p is the adjacency matrix of graph $\mathbb{G}_p$ and D_p the diagonal matrix whose *ith* diagonal element is the valence of vertex i within the graph. Then

$$\theta(t+1) = F_{\sigma(t)}\theta(t), \quad t \in \{0, 1, 2, \ldots\} \tag{4}$$

where θ is the heading vector $\theta = [\,\theta_1 \quad \theta_2 \quad \ldots \quad \theta_n\,]'$ and $\sigma : \{0, 1, \ldots\} \to \mathcal{P}$ is a switching signal whose value at time t, is the index of the graph representing the agents' neighbor relationships at time t. A complete description of this system would have to include a model which explains how σ changes over time as a function of the positions of the n agents in the plane. While such a model is easy to derive and is essential for simulation purposes, it would be difficult to take into account in a convergence analysis. To avoid this difficulty, we shall adopt a more conservative approach which ignores how σ depends on the agent positions in the plane and assumes instead that σ might be any switching signal in some suitably defined set of interest.

Our goal is to show for a large class of switching signals and for any initial set of agent headings that the headings of all n agents will converge to the same steady state value θ_{ss}. Convergence of the θ_i to θ_{ss} is equivalent to the state vector θ converging to a vector of the form $\theta_{ss}\mathbf{1}$ where $\mathbf{1} \stackrel{\Delta}{=} [\,1 \quad 1 \quad \ldots \quad 1\,]'_{n\times 1}$. Naturally there are situations where convergence to a common heading cannot occur. The most obvious of these is when one agent - say the *ith* - starts so far away from the rest that it never acquires any neighbors. Mathematically this would mean not only that $\mathbb{G}_{\sigma(t)}$ is never connected at any time t, but also that vertex i remains an isolated vertex of $\mathbb{G}_{\sigma(t)}$ for all t. This situation is likely to be encountered if r is very small. At the other extreme, which is likely if r is very large, all agents might remain neighbors of all others for all time. In this case, σ would remain fixed along such a trajectory at that value in $p \in \mathcal{P}$ for which $\mathbb{G}_p$ is a complete graph. Convergence of θ to $\theta_{ss}\mathbf{1}$ can easily be established in this special case because with σ so fixed, (4) is a linear, time-invariant, discrete-time system. The situation of perhaps the greatest interest is between these two extremes when $\mathbb{G}_{\sigma(t)}$ is not necessarily complete or even connected for any $t \geq 0$, but when no strictly proper subset of $\mathbb{G}_{\sigma(t)}$'s vertices is isolated from the rest for all time. Establishing convergence in this case is challenging because σ changes with time and (4) is not time-invariant. It is this case which we intend to study. Towards this end, we denote by $\mathcal{Q}$ the subset of $\mathcal{P}$ consisting of the indices of the connected graphs in $\{\mathbb{G}_p : p \in \mathcal{P}\}$. Our first result establishes the convergence of θ for the case when σ takes values only in $\mathcal{Q}$.

Theorem 1 *Let $\theta(0)$ be fixed and let $\sigma : \{0, 1, 2, \ldots\} \to \mathcal{P}$ be a switching signal satisfying $\sigma(t) \in \mathcal{Q}$, $t \in \{0, 1, \ldots\}$. Then*

$$\lim_{t\to\infty} \theta(t) = \theta_{ss}\mathbf{1} \tag{5}$$

where θ_{ss} is a number depending only on $\theta(0)$ and σ.

[1]By an undirected *graph* $\mathbb{G}$ on vertex set $\mathcal{V} = \{1, 2, \ldots n\}$ is meant $\mathcal{V}$ together with a set of unordered pairs $\mathcal{E} = \{(i, j) : i, j \in \mathcal{V}\}$ which are called $\mathbb{G}$'s edges. Such a graph is *simple* if it has no self-loops {i.e., $(i, j) \in \mathcal{E}$ only if $i \neq j$} or repeated edges {i.e., $\mathcal{E}$ contains only distinct elements}. By the *valence* of a vertex v of $\mathbb{G}$ is meant the number of edges of $\mathbb{G}$ which are "incident" on v where by an *incident* edge on v is meant an edge (i, j) of $\mathbb{G}$ for which either $i = v$ or $j = v$. The *adjacency matrix* of $\mathbb{G}$ is an $n \times n$ matrix of whose ijth entry is 1 if (i, j) is one of $\mathbb{G}$'s edges and 0 if it is not.

It is possible to establish convergence to a common heading under conditions which are significantly less stringent that those assumed in Theorem 1. To do this we need to introduce several concepts. By the *union* of a collection of simple graphs, $\{\mathbb{G}_{p_1}, \mathbb{G}_{p_2}, \ldots, \mathbb{G}_{p_m}\}$, each with vertex set $\mathcal{V}$, is meant the simple graph $\mathbb{G}$ with vertex set $\mathcal{V}$ and edge set equaling the union of the edge sets of all of the graphs in the collection. We say that such a collection is *jointly connected* if the union of its members is a connected graph. Note that if such a collection contains at least one graph which is connected, then the collection must be jointly connected. On the other hand, a collection can be jointly connected even if none of its members are connected.

It is natural to say that the n agents under consideration are *linked together* across a time interval $[t, \tau]$ if the collection of graph $\{\mathbb{G}_{\sigma(t)}, \mathbb{G}_{\sigma(t+1)}, \ldots, \mathbb{G}_{\sigma(\tau)}\}$ encountered along the interval, is jointly connected. Theorem 1 says, in essence, that convergence of all agent's headings to a common heading is for certain provided all n agents are linked together across each successive interval of length one {i.e., all of the time}. Of course there is no guarantee that along a specific trajectory the n agents will be so linked. Perhaps a more likely situation, at least when r is not too small, is when the agents are linked together across contiguous intervals of arbitrary but finite length. If the lengths of such intervals are uniformly bounded, then in this case too convergence to a common heading proves to be for certain.

Theorem 2 *Let $\theta(0)$ be fixed and let $\sigma : \{0, 1, 2, \ldots\} \to \mathcal{P}$ be a switching signal for which there exists an infinite sequence of contiguous, non-empty, bounded, time-intervals $[t_i, t_{i+1})$, $i \geq 0$, starting at $t_0 = 0$, with the property that across each such interval, the n agents are linked together. Then*

$$\lim_{t \to \infty} \theta(t) = \theta_{ss}\mathbf{1} \tag{6}$$

where θ_{ss} is a number depending only on $\theta(0)$ and σ.

The hypotheses of Theorem 2 require each of the collections $\{\mathbb{G}_{\sigma(t_i)}, \mathbb{G}_{\sigma(t_i+1)}, \ldots, \mathbb{G}_{\sigma(t_{i+1}-1)}\}$, $i \geq 0$, to be jointly connected. Although no constraints are placed on the intervals $[t_i, t_{i+1})$, $i \geq 0$, other than that they be of finite length, the constraint on σ is more restrictive than one might hope for. What one would prefer instead is to show that (6) holds for every switching signal σ for which there is an infinite sequence of bounded, non-overlapping {but not necessarily contiguous} intervals across which the n agents are linked together. Whether or not this is true remains to be seen.

A sufficient but not necessary condition for σ to satisfy the hypotheses of Theorem 2 is that on each successive interval $[t_i, t_{i+1})$, σ take on at least one value in $\mathcal{Q}$. Theorem 1 is thus an obviously a consequence of Theorem 2 for the case when all intervals are of length 1. For this reason we need only develop a proof for Theorem 2. To do this we will make use of certain structural properties of the F_p. As defined, each F_p is square and non-negative, where by a *non-negative* matrix is meant a matrix whose entries are all non-negative. Each F_p also has the property that its row sums all equal 1 {i.e., $F_p\mathbf{1} = \mathbf{1}$}. Matrices with these two properties are called *stochastic* [22]. The F_p have the additional property that their diagonal elements are all non-zero. For the case when $p \in \mathcal{Q}$ {i.e., when $\mathbb{G}_p$ is connected}, it is known that $(I + A_p)^m$ becomes a matrix with all positive entries for m sufficiently large [22]. It is easy to see that if $(I+A_p)^m$ has all positive entries, then so does F_p^m. Such $(I + A_p)$ and F_p are examples of "primitive matrices" where by a *primitive* matrix is meant any square, non-negative matrix M for which M^m is a matrix with all positive entries for m sufficiently large [22]. It is known [22] that among the n eigenvalues of a primitive matrix, there is exactly one with largest magnitude, that this eigenvalue is the only one possessing an eigenvector with all positive entries, and that the remaining $n-1$ eigenvalues are all strictly smaller in magnitude than the largest one. This means that for $p \in \mathcal{Q}$, 1 must be F_p's largest eigenvalue and all remaining eigenvalues must lie within the open unit circle. As a consequence, each such F_p must have the property that $\lim_{i \to \infty} F_p^i = \mathbf{1}c_p$ for some row vector c_p. Any stochastic matrices M for which $\lim_{i \to \infty} M^i$ is a matrix of rank 1 is called *ergodic* [22]. Primitive stochastic matrices are thus ergodic matrices. To summarize, each F_p is a stochastic matrix with positive diagonal elements and if $p \in \mathcal{Q}$ then F_p is also primitive and hence ergodic. The crucial convergence result upon which the proof of Theorem 2 depends is classical [13] and is as follows.

Theorem 3 (Wolfowitz) *Let $M_1, M_2, \ldots, M_m$ be a finite set of ergodic matrices with the property that for each sequence $M_{i_1}, M_{i_2}, \ldots, M_{i_j}$ of positive length, the matrix product $M_{i_j} M_{i_{j-1}} \cdots M_{i_1}$ is ergodic. Then for each infinite sequence , $M_{i_1}, M_{i_2}, \ldots$ there exists a row vector c such that*

$$\lim_{j \to \infty} M_{i_j} M_{i_{j-1}} \cdots M_{i_1} = \mathbf{1}c$$

The finiteness of the set $M_1, M_2, \ldots, M_m$ is crucial to Wolfowitz's proof. This finiteness requirement is also the reason why we need to assume *contiguous*, bounded intervals in the statement of Theorem 2.

In order to make use of Theorem 3, we need a few facts concerning products of the types of matrices we are considering. First we point out that the class of $n \times n$ stochastic matrices with positive diagonal elements is closed under matrix multiplication. This is because the product of two non-negative matrices with positive diagonals is a matrix with the same properties and because the product of two stochastic matrices is stochastic. Proof of Theorem 2depends on Theorem 3 and the following key lemma [23]:

Lemma 1 *Let $\{p_1, p_2, \ldots, p_m\}$ be a set of indices in $\mathcal{P}$ for which $\{\mathbb{G}_{p_1}, \mathbb{G}_{p_2}, \ldots, \mathbb{G}_{p_m}\}$ is a jointly connected collection of graphs. Then the matrix product $F_{p_1} F_{p_2} \cdots F_{p_m}$ is ergodic.*

2.1 Quadratic Lyapunov Functions

As we have already noted, $F_p\mathbf{1} = \mathbf{1}$, $p \in \mathcal{P}$. Thus span $\{\mathbf{1}\}$ is an F_p-invariant subspace. From this and standard existence conditions for solutions to linear algebraic equations, it follows that for any $(n-1) \times n$ matrix P with kernel spanned by $\mathbf{1}$, the equations

$$PF_p = \tilde{F}_p P, \; p \in \mathcal{P} \tag{7}$$

have unique solutions $\tilde{F}_p$, $p \in \mathcal{P}$, and moreover that

$$\text{spectrum } F_p = \{1\} \cup \text{spectrum } \tilde{F}_p, \quad p \in \mathcal{P} \tag{8}$$

As a consequence of (7) it can easily be seen that for any sequence of indices $p_0, p_1, \ldots p_i$ in $\mathcal{P}$,

$$\tilde{F}_{p_i} \tilde{F}_{p_{i-1}} \cdots \tilde{F}_{p_0} P = P F_{p_i} F_{p_{i-1}} \cdots F_{p_0} \tag{9}$$

Since P has full row rank and $P\mathbf{1} = 0$, the convergence of a product of the form $F_{p_i} F_{p_{i-1}} \cdots F_{p_0}$ to $\mathbf{1}c$ for some row vector c, is equivalent to convergence of the corresponding product $\tilde{F}_{p_i} \tilde{F}_{p_{i-1}} \cdots \tilde{F}_{p_0}$ to the zero matrix. Thus, for example, if

$p_0, p_1, \ldots$ is an infinite sequence of indices in $\mathcal{Q}$, then, in view of Theorem 3,

$$\lim_{i\to\infty} \tilde{F}_{p_i}\tilde{F}_{p_{i-1}}\cdots\tilde{F}_{p_0} = 0 \quad (10)$$

Some readers might be tempted to think, as we first did, that the validity of (10) could be established directly by showing that the $\tilde{F}_p$ in the product share a common quadratic Lyapunov function. More precisely, (10) would be true if there were a single positive definite matrix M such that all of the matrices $\tilde{F}_p' M \tilde{F}_p - M, \quad p \in \mathcal{Q}$ were negative definite. Although each $\tilde{F}_p, \ p \in \mathcal{Q}$ can easily be shown to be discrete-time stable, there are classes of F_p for which that no such common Lyapunov matrix M exists. While we've not been able to construct a simple analytical example which demonstrates this, we have been able to determine, for example, that no common quadratic Lyapunov function exists for the class of all F_p whose associated graphs have 10 vertices and are connected. One can verify that this is so by using semidefinite programming and restricting the check to just those connected graphs on 10 vertices with either 9 or 10 edges.

It is worth noting that existence of a common quadratic Lyapunov function for all discrete time stable $m \times m$ matrices $M_1, M_2, \ldots$ in some given finite set $\mathcal{M}$, is a much stronger condition than is typically needed to guarantee that all infinite products of the M_i converge to zero. It is known [24] that convergence to zero of all such infinite products is in fact equivalent to the "joint spectral radius" of $\mathcal{M}$ being strictly less than 1 where *joint spectral radius of* $\mathcal{M}$ is defined as

$$\rho_{\mathcal{M}} := \limsup_{k\to\infty}\{\max_{N\in\mathcal{M}^k} ||N||\}^{\frac{1}{k}}$$

where $\mathcal{M}^k$ is the set of all matrix products of elements of $\mathcal{M}$ of length k, and $||\cdot||$ is any norm on the space of real $m \times m$ matrices. It turns out that $\rho_{\mathcal{M}}$ does not depend on the choice of norm because all norms on a finite-dimensional space are equivalent. On the other hand, a "tight" sufficient condition for the existence of a common quadratic Lyapunov function for the matrices in $\mathcal{M}$, is $\rho_{\mathcal{M}} \in [0, \frac{1}{\sqrt{m}})$ [25]. This condition is *tight* in the sense that one can find a finite set of $m \times m$ matrices with joint spectral radius $\rho = \frac{1}{\sqrt{m}}$, whose infinite products converge to zero despite the fact that there does not exist common quadratic Lyapunov function for the set. From this one can draw the conclusion that sets of matrices with "large" m are not likely to possess a common quadratic, even though all infinite products of such matrices converge to zero. This can in turn help explain why it has proved to be necessary to go as high as $n = 10$ to find a case where a common quadratic Lyapunov function for a family of F_p does not exist.

2.2 Generalization

It is possible to interpret the Vicsek model analyzed in the last section as the closed-loop system which results when a suitably defined decentralized feedback law is applied to the n-agent heading model

$$\theta(t+1) = \theta(t) + u(t) \quad (11)$$

with open-loop control u. To end up with the Vicsek model, u would have to be defined as

$$u(t) = -(I + D_{\sigma(t)})^{-1} e(t) \quad (12)$$

where e is the *average heading error* vector

$$e(t) \triangleq L_{\sigma(t)}\theta(t) \quad (13)$$

and, for each $p \in \mathcal{P}$, L_p is the symmetric matrix

$$L_p = D_p - A_p \quad (14)$$

known in graph theory as the *Laplacian* of $\mathbb{G}_p$ [26]. It is easily verified that equations (11) to (14) do indeed define the Vicsek model. We've elected to call e the average heading error because if $e(t) = 0$ at some time t, then the heading of each agent with neighbors at that time will equal the average of the headings of its neighbors. In the present context, Vicsek's control (12) can be viewed as a special case of a more general decentralized feedback control of the form

$$u(t) = -G_{\sigma(t)}^{-1} L_{\sigma(t)}\theta(t) \quad (15)$$

where for each $p \in \mathcal{P}$, G_p is a suitably defined, nonsingular diagonal matrix with *ith* diagonal element g_p^i. This, in turn, is an abbreviated description of a system of n individual agent control laws of the form

$$u_i(t) = -\frac{1}{g_i(t)}\left(n_i(t)\theta_i(t) + \sum_{j\in\mathcal{N}_i(t)} \theta_j(t)\right), \quad i \in \{1, 2, \ldots, n\} \quad (16)$$

where for $i \in \{1, 2, \ldots, n\}$, $u_i(t)$ is the ith entry of $u(t)$ and $g_i(t) \triangleq g_{\sigma(t)}^i$. Application of this control to (11) would result in the closed-loop system

$$\theta(t+1) = \theta(t) - G_{\sigma(t)}^{-1} L_{\sigma(t)}\theta(t) \quad (17)$$

Note that the form of (17) implies that if θ and σ were to converge to a constant values $\bar{\theta}$, and $\bar{\sigma}$ respectively, then $\bar{\theta}$ would automatically satisfy $L_{\bar{\sigma}}\bar{\theta} = 0$. This means that control (15) automatically forces each agent's heading to converge to the average of its neighbors, if agent headings were to converge at all. In other words, the choice of the G_p does not effect the requirement that each agent's heading equal the average of the headings of its neighbors, if there is convergence at all.

The preceding suggests that there might be useful choices for the G_p alternative to those considered by Vicsek, which also lead to convergence. One such choice turns out to be

$$G_p = gI, \ p \in \mathcal{P} \quad (18)$$

where g is any number greater than n. Our aim is to show that with the G_p so defined, Theorem 2 continues to be valid. In sharp contrast with the proof technique used in the last section, convergence will be established here using a common quadratic Lyapunov function.

As before, we will use the model

$$\theta(t+1) = F_{\sigma(t)}\theta(t) \quad (19)$$

where, in view of the definition of the G_p in (18), the F_p are now symmetric matrices of the form

$$F_p = I - \frac{1}{g}L_p, \qquad p \in \mathcal{P} \quad (20)$$

To proceed we need to review a number of well known and easily verified properties of graph Laplacians relevant to the problem at hand. For this, let $\mathbb{G}$ be any given simple graph with n vertices. Let D be a diagonal matrix whose diagonal elements are the valences of $\mathbb{G}$'s vertices and write A for $\mathbb{G}$'s adjacency matrix. Then, as noted before, the Laplacian of $\mathbb{G}$ is the symmetric matrix $L = D - A$. The definition of L clearly implies that $L\mathbf{1} = 0$. Thus L must have an eigenvalue at zero and $\mathbf{1}$ must be an eigenvector for this eigenvalue. It turns out that L is a symmetric positive semidefinite matrix, and the number of connected components of $\mathbb{G}$ is exactly the same as the multiplicity of L's eigenvalue at 0. Thus $\mathbb{G}$ is a connected graph if and only if L has exactly one eigenvalue at 0. Also, the largest eigenvalue of L is less than or equal to n [26]. This means that the eigenvalues of $\frac{1}{g}L$ must be smaller than 1 since $g > n$. From these properties it clearly follows that the eigenvalues of $(I - \frac{1}{g}L)$ must all be between 0 and 1, and that if $\mathbb{G}$ is connected, then all will be strictly less than 1 except for one eigenvalue at 1 with eigenvector $\mathbf{1}$. Since each F_p is of the form $(I - \frac{1}{g}L)$, each F_p possesses all of these properties.

Let σ be a fixed switching signal with value $p_t \in \mathcal{Q}$ at time $t \geq 0$. What we would like to do is to prove that as $i \to \infty$, the matrix product $F_{p_i}F_{p_{i-1}} \cdots F_{p_0}$ converges to $\mathbf{1}c$ for some row vector c. As noted in the section 2.1, this matrix product will so converge if and only if

$$\lim_{i\to\infty} \tilde{F}_{p_i}\tilde{F}_{p_{i-1}} \cdots \tilde{F}_{p_0} = 0 \tag{21}$$

where as in section 2.1, $\tilde{F}_p$ is the unique solution to $PF_p = \tilde{F}_pP,\ p \in \mathcal{P}$ and P is any full rank $(n-1) \times n$ matrix satisfying $P\mathbf{1} = 0$. For simplicity and without loss of generality we shall henceforth assume that the rows of P form a basis for the orthogonal complement of the span of $\mathbf{e}$. This means that PP' equals the $(n-1) \times (n-1)$ identity $\tilde{I}$, that $\tilde{F}_p = PF_pP',\quad p \in \mathcal{P}$, and thus that each $\tilde{F}_p$ is symmetric. Moreover, in view of (8) and the spectral properties of the $F_p,\ p \in \mathcal{Q}$, it is clear that each $\tilde{F}_p,\ p \in \mathcal{Q}$ must have a real spectrum lying strictly inside of the unit circle. This plus symmetry means that for each $p \in \mathcal{Q}$, $\tilde{F}_p - \tilde{I}$ is negative definite, that $\tilde{F}_p'\tilde{F}_p - \tilde{I}$ is negative definite and thus that $\tilde{I}$ is a common discrete-time Lyapunov matrix for all such $\tilde{F}_p$. Using this fact it is straight forward to prove that Theorem 1 holds for system (17) provided the G_p are defined as in (18) with $g > n$. Proving Theorem 2 in this case is more involved and can be found in [23].

3 Leader Following

In this section we consider a modified version of Vicsek's discrete-time system consisting of the same group of n agents as before, plus one additional agent, labelled 0, which acts as the group's leader. Agent 0 moves at the same constant speed as its n followers but with a fixed heading θ_0. The *ith* follower updates its heading just as before, using the average of its own heading plus the headings of its neighbors. The difference now is that each follower's set of neighbors can include the leader and does so whenever the leader is within the follower's neighborhood defining circle of radius r. Agent i's update rule thus is of the form

$$\theta_i(t+1) = \frac{1}{1 + n_i(t) + b_i(t)}\left(\theta_i(t) + \sum_{j\in\mathcal{N}_i(t)} \theta_j(t) + b_i(t)\theta_0\right) \tag{22}$$

where as before, $\mathcal{N}_i(t)$ is the set of labels of agent i's neighbors from the original group of n followers, and $n_i(t)$ is the number of labels within $\mathcal{N}_i(t)$. Agent 0's heading is accounted for in the ith average by defining $b_i(t)$ to be 1 whenever agent 0 is a neighbor of agent i and 0 otherwise.

The explicit form of the n update equations exemplified by (22), depends on the relationships between neighbors which exist at time t. Like before, each of these relationships can be conveniently described by a simple undirected graph. In this case, each such graph has vertex set $\{0, 1, 2, \ldots, n\}$ and is defined so that (i, j) is one of the graph's edges just in case agents i and j are neighbors. For this purpose we consider an agent - say i - to be a neighbor of agent 0 whenever agent 0 is a neighbor of agent i. We will need to consider all possible such graphs. In the sequel we use the symbol $\bar{\mathcal{P}}$ to denote a set indexing the class of all simple graphs $\bar{\mathbb{G}}_p$ defined on vertices $0, 1, 2, \ldots, n$. We will also continue to make reference to the set of all simple graphs on vertices $1, 2, \ldots, n$. Such graphs are now viewed as subgraphs of the $\bar{\mathbb{G}}_p$. Thus, for $p \in \bar{\mathcal{P}}$, $\mathbb{G}_p$ now denotes the subgraph obtained from $\bar{\mathbb{G}}_p$ by deleting vertex 0 and all edges incident on vertex 0.

The set of agent heading update rules defined by (22) can be written in state form. Toward this end, for each $p \in \bar{\mathcal{P}}$, let A_p denote the $n\times n$ adjacency matrix of the n-agent graph $\mathbb{G}_p$ and let D_p be the corresponding diagonal matrix of valences of $\mathbb{G}_p$. Then in matrix terms, (22) becomes

$$\theta(t+1) = (I + D_{\sigma(t)} + B_{\sigma(t)})^{-1}((I + A_{\sigma(t)})\theta(t) + B_{\sigma(t)}\mathbf{1}\theta_0) \tag{23}$$

where $\sigma : \{0, 1, \ldots\} \to \bar{\mathcal{P}}$ is now a switching signal whose value at time t, is the index of the graph $\bar{\mathbb{G}}_p$ representing the agent system's neighbor relationships at time t and for $p \in \bar{\mathcal{P}}$, B_p is the $n \times n$ diagonal matrix whose ith diagonal element is 1 if $(i, 0)$ is one of $\bar{\mathbb{G}}_p$'s edges and 0 otherwise.

Our goal here is to show for a large class of switching signals and for any initial set of follower agent headings, that the headings of all n followers converge to the heading of the leader. For convergence in the leaderless case we required all n-agents to be linked together across each interval within an infinite sequence of contiguous, bounded intervals. We will need a similar requirement in the leader following case under consideration. Let us agree to say that the n agents are *linked to the leader* across an interval $[t, \tau]$ if the collection of graphs $\{\bar{\mathbb{G}}_{\sigma(t)}, \bar{\mathbb{G}}_{\sigma(t+1)}, \ldots, \bar{\mathbb{G}}_{\sigma(\tau)}\}$ encountered along the interval is jointly connected. In other words, the n agents are linked to their leader across an interval $\mathcal{I}$ just when the $n+1$-member group consisting of the n agents and their leader is linked together across $\mathcal{I}$. Note that for the n-agent group to be linked to its leader across $\mathcal{I}$ does not mean that the n-agent group must be linked together across $\mathcal{I}$. Nor is the n-agent group necessarily linked its leader across $\mathcal{I}$ when it is linked together across $\mathcal{I}$. Our main result on discrete-time leader following is stated as the following theorem:

Theorem 4 *Let $\theta(0)$ and θ_0 be fixed and let $\sigma : \{0, 1, 2, \ldots\} \to \bar{\mathcal{P}}$ be a switching signal for which there exists an infinite sequence of contiguous, non-empty, bounded, time-intervals $[t_i, t_{i+1})$, $i \geq 0$, starting at $t_0 = 0$, with the property that across each such interval, the n-agent group of followers is linked to its leader. Then*

$$\lim_{t\to\infty} \theta(t) = \theta_0\mathbf{1} \tag{24}$$

The theorem says that the members of the n-agent group all eventually follow their leader provided there is a positive integer T which is large enough so that the n-agent group is linked to its leader across each contiguous, non-empty time-interval of length at most T. In the sequel we outline several preliminary ideas upon which the proof of Theorem 4 depends.

First, we note that to prove that (24) holds is equivalent to proving that $\lim_{t\to\infty} \epsilon(t) \to 0$ where ϵ is the heading error vector $\epsilon(t) \stackrel{\Delta}{=} \theta(t) - \theta_0\mathbf{1}$. From (23) it is easy to deduce that ϵ satisfies the equation

$$\epsilon(t+1) = F_{\sigma(t)}\epsilon(t) \tag{25}$$

where for $p \in \bar{\mathcal{P}}$, F_p is

$$F_p = (I + D_p + B_p)^{-1}(I + A_p) \qquad (26)$$

Note that the partitioned matrices

$$\bar{F}_p \triangleq \begin{bmatrix} F_p & H_p\mathbf{1} \\ 0 & 1 \end{bmatrix}, \quad p \in \bar{\mathcal{P}} \qquad (27)$$

are stochastic where, for $p \in \bar{\mathcal{P}}$,

$$H_p \triangleq (I + D_p + B_p)^{-1} B_p \qquad (28)$$

Due to lack of space, we skip the proof of Theorem 4, and just note that it is a direct consequence of the following proposition [23]:

Proposition 1 *Let T be a finite positive integer. There exists a positive number $\lambda < 1$, depending only on T, for which*

$$||F_{p_{\bar{t}}} F_{p_{\bar{t}-1}} \cdots F_{p_1}|| < \lambda \qquad (29)$$

for every sequence $p_1, p_2, \ldots p_{\bar{t}} \in \bar{\mathcal{P}}$ of length at most T possessing values $q_1, q_2, \ldots, q_m$ which each occur in the sequence at least $n+1$ times and for which $\{\bar{\mathbb{G}}_{q_1}, \bar{\mathbb{G}}_{q_2}, \ldots, \bar{\mathbb{G}}_{q_m}\}$ is a jointly connected collection of graphs.

4 Concluding Remarks

The models we have analyzed are of course very simple and as a consequence, they are probably not really descriptive of actual bird-flocking, fish schooling, or even the coordinated movements of envisioned groups of mobile robots. Nonetheless, these models do seem to exhibit some of the rudimentary behaviors of large groups of mobile autonomous agents and for this reason they serve as a natural starting point for the analytical study of more realistic models. It is clear from the developments in this paper, that ideas from graph theory and dynamical system theory will play a central role in both the analysis of such biologically inspired models and in the synthesis of provably correct distributed control laws which produce such emergent behaviors in man-made systems.

References

[1] T. Vicsek, A. Czirok, E. Ben Jacob, I. Cohen, and O. Schochet. Novel type of phase transitions in a system of self-driven particles. *Physical Review Letters*, 75:1226–1229, 1995.

[2] C. Reynolds. Flocks, birds, and schools: a distributed behavioral model. *Computer Graphics*, 21:25–34, 1987.

[3] A. Czirok, A. L. Barabasi, and T. Vicsek. Collective motion of self-propelled particles:kinetic phase transition in one dimension. *Physical Review Letters*, 82:209–212, 1999.

[4] J. Toner and Y. Tu. Flocks, herds, and schools: A quantitative theory of flocking. *Physical Review E.*, 58:4828–4858, 1998.

[5] J. Toner and Y. Tu. Long range order in a two dimensional xy model: How birds fly together. *Physical Review Letters*, 75:4326–4329, 1995.

[6] A. S. Mikhailov and D. Zannette. Noise induced breakdown of collective coherent motion in swarms. *Physical Review E*, 60:4571–4575, 1999.

[7] C. P. Schenk, P Schutz, M. Bode, and H.G. Purwins. Interaction of self organized quasi particles in a two dimensional reaction-diffusion system: The formation of molecules. *Physical Review E*, 58:6480–6486, 1998.

[8] Y. Liu, K.M. Passino, and M. Polycarpou. Stability analysis of one-dimensional asynchronous swarms. In *American Control Conference, Arlington, VA*, pages 716–721, June, 2001.

[9] N. Leonard and E. Friorelli. Virtual leaders, artificial potentials and coordinated control of groups. *IEEE Conference on Decision and Control, Orlando, FL*, 2001.

[10] R. Olfati-Saber and R. M. Murray. Distributed cooperative control of multiple vehicle formations using structural potential functions. In *IFAC World Congress, Barcelona, Spain, To appear*, 2002.

[11] J. A. Fax and R. M. Murray. Graph laplacians and stabilization of vehicle formations. *To appear, 15th IFAC Congress, Barcelona, Spain*, 2002.

[12] J. P. Desai, J.P. Ostrowski, and V. Kumar. Modeling and control of formations of nonholonomic mobile robots. *IEEE transactions on Robotics and Automation*, 17(6):905–908, 2001.

[13] J. Wolfowitz. Products of indecomposable, aperiodic, stochastic matrices. *Proceedings of the American Mathematical Society*, 15:733–737, 1963.

[14] E. Seneta. *Non-negative matrices and Markov chains.* Springer Verlag, New York, 1981.

[15] I. Daubechies and J. C. Lagarias. Sets of matrices all infinite products of which converge. *Linear Algebra and Applications*, 161:227–263, 1992.

[16] I. Daubechies and J. C. Lagarias. Corrigendum/addendum to“sets of matrices all infinite products of which converge”. *Linear Algebra and Applications*, 327:69–83, 2001.

[17] R. Bru, L. Elsner, and M. Neumann. Convergence of infinite products of matrices and inner-outer iteration schemes. *Electronic Transactions on Numerical Analysis*, 2:183–193, 1994.

[18] W. J. Beyn and L. Elsner. Infinite products and paracontracting matrices. *Electronic Journal of Linear Algebra*, 2:1–8, 1997.

[19] A. Rhodius. On the maximum of ergodicity coefficients, the Dobrushin ergodicity coefficient, and products of stochastic matrices. *Linear Algebra and Applications*, 253:141–154, 1997.

[20] A. Vladimirov, E. Elsner, and W.J. Beyn. Stability and paracontractivity of discrete linear inclusions. *Linear Algebra and Applications*, 312:125–134, 2000.

[21] B. Mohar. The Laplacian spectrum of graphs. *in Graph theory, combinatorics and applications (Ed. Y. Alavi G. Chartrand, O. R. Ollerman, and A. J. Schwenk)*, 2:871–898, 1991.

[22] R. Horn and C. R. Johnson. *Matrix Analysis.* Cambridge University Press, New York, 1985.

[23] A. Jadbabaie, J. Lin, and A. S. Morse. Coordination of groups of mobile autonomous agents using nearest neighbor rules. *Accepted for publication in IEEE Transactions on Automatic Control*, July 2002.

[24] M. A. Berger and Y. Wang. Bounded semigroups of matrices. *Linear Algebra and Applications*, 166:21–27, 1992.

[25] M. Ando and M. H. Shih. Simultaneous contractibility. *SIAM Journal on Matrix Analysis and Applications*, 19:487–498, 1998.

[26] C. Godsil and G. Royle. *Algebraic Graph Theory.* Springer Graduate Texts in Mathematics # 207, New York, 2001.