Fall 2008 CIS 260

Mathematical Foundations of Computer Science

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Homework 9

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Problem 1. Consider the function, $J\colon \mathbb{N}\times\mathbb{N}\to\mathbb{N}$, given by

$$J(m,n) = \frac{1}{2}[(m+n)^2 + 3m + n].$$

(a) Prove that for any $z\in\mathbb{N}$, if $J(m,n)=z$, then

$$8z+1 = (2m+2n+1)^2 + 8m.$$

Deduce from the above that

$$2m+2n+1 \leq \sqrt{8z+1} < 2m+2n+3.$$

(b) If $x\mapsto \lfloor x\rfloor$ is the function from $\mathbb{R}$ to $\mathbb{N}$ (the *floor function*), where $\lfloor x\rfloor$ is the largest integer $\leq x$ (for example, $\lfloor 2.3\rfloor = 2$, $\lfloor\sqrt{2}\rfloor = 1$), then prove that

$$\lfloor\sqrt{8z+1}\rfloor + 1 = 2m+2n+2 \quad\text{or}\quad \lfloor\sqrt{8z+1}\rfloor + 1 = 2m+2n+3,$$

so that

$$\lfloor(\lfloor\sqrt{8z+1}\rfloor + 1)/2\rfloor = m+n+1.$$

(c) Since $J(m,n)=z$ means that

$$2z = (m+n)^2 + 3m + n,$$

prove that m and n are solutions of the system

$$\begin{aligned} m+n &= \lfloor(\lfloor\sqrt{8z+1}\rfloor + 1)/2\rfloor - 1\\ 3m+n &= 2z - (\lfloor(\lfloor\sqrt{8z+1}\rfloor + 1)/2\rfloor - 1)^2. \end{aligned}$$

If we let

$$\begin{aligned} Q_1(z) &= \lfloor(\lfloor\sqrt{8z+1}\rfloor + 1)/2\rfloor - 1\\ Q_2(z) &= 2z - (\lfloor(\lfloor\sqrt{8z+1}\rfloor + 1)/2\rfloor - 1)^2 = 2z - (Q_1(z))^2, \end{aligned}$$

then prove that $Q_2(z) - Q_1(z)$ is even and that

$$\begin{aligned} m &= \frac{1}{2}(Q_2(z) - Q_1(z)) = K(z) \\ n &= Q_1(z) - \frac{1}{2}(Q_2(z) - Q_1(z)) = L(z). \end{aligned}$$

Conclude that J is a bijection between $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$, with

$$\begin{aligned} m &= K(J(m,n)) \\ n &= L(J(m,n)). \end{aligned}$$

Remark: It can also be shown that $J(K(z), L(z)) = z$.

Problem 2. Let $S_{n\,p}$ be the number of surjections from the set $\{1, \ldots, n\}$ onto the set $\{1, \ldots, p\}$, where $1 \leq p \leq n$. Observe that $S_{n\,1} = 1$.

(a) Recall that $n!$ (factorial) is defined for all $n \in \mathbb{N}$ by $0! = 1$ and $(n+1)! = (n+1)n!$. Also recall that $\binom{n}{k}$ (n choose k) is defined for all $n \in \mathbb{N}$ and all $k \in \mathbb{Z}$ as follows:

$$\begin{aligned} \binom{n}{k} &= 0, \quad \text{if} \quad k \notin \{0, \ldots, n\} \\ \binom{0}{0} &= 1 \\ \binom{n}{k} &= \binom{n-1}{k} + \binom{n-1}{k-1}, \quad \text{if} \quad n \geq 1. \end{aligned}$$

Prove by induction on n that

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

(b) Prove that

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n \quad (n \geq 0) \quad \text{and} \quad \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0 \quad (n \geq 1).$$

Hint. Use the *binomial formula* shown in the class notes: For all $a, b \in \mathbb{R}$ and all $n \geq 0$,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

(c) Prove that

$$p^n = S_{n\,p} + \binom{p}{1} S_{n\,p-1} + \binom{p}{2} S_{n\,p-2} + \cdots + \binom{p}{p-1}.$$

(d) For all $p \geq 1$ and all i, k, with $0 \leq i \leq k \leq p$, prove that

$$\binom{p}{i}\binom{p-i}{k-i} = \binom{k}{i}\binom{p}{k}.$$

Use the above to prove that

$$\binom{p}{0}\binom{p}{k} - \binom{p}{1}\binom{p-1}{k-1} + \cdots + (-1)^k \binom{p}{k}\binom{p-k}{0} = 0.$$

(e) Prove that

$$S_{n\,p} = p^n - \binom{p}{1}(p-1)^n + \binom{p}{2}(p-2)^n + \cdots + (-1)^{p-1}\binom{p}{p-1}.$$

Hint. Write all p equations given by (c) for $1, 2, \ldots, p-1, p$, multiply both sides of the equation involving $(p-k)^n$ by $(-1)^k\binom{p}{k}$, add up both sides of theses equations and use (b) to simplify the sum on the righthand side.