Fall, 2017 CIS 262

Automata, Computability and Complexity
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Practice Final Exam

December 6, 2017

Problem 1 (10 pts). Let Σ be an alphabet.

(1) What is an ambiguous context-free grammar? What is an inherently ambiguous context-free language?

(2) Is the following context-free grammar ambiguous, and if so demonstrate why?

$$
\begin{aligned}
E &\longrightarrow E + E \\
E &\longrightarrow E * E \\
E &\longrightarrow (E) \\
E &\longrightarrow a.
\end{aligned}
$$

Problem 2 (5pts). Given any trim DFA $D = (Q, \Sigma, \delta, q_0, F)$ accepting a language $L = L(D)$, if D is a minimal DFA, then prove that its Myhill-Nerode equivalence relation $\simeq_D$ is equal to ρ_L.

Problem 3 (10 pts).

Consider the following DFA D_0 with start state A and final state D given by the following transition table:

	a	b
A	B	A
B	C	A
C	D	A
D	D	A

Reverse all the arrows of D_0, obtaining the following NFA N (wihout ϵ-transitions) with start state D and final state A given by the following table:

	a	b
A	$\emptyset$	$\{A,B,C,D\}$
B	$\{A\}$	$\emptyset$
C	$\{B\}$	$\emptyset$
D	$\{C,D\}$	$\emptyset$

This NFA accepts the language $\{aaa\}\{a,b\}^*$.

(1) Use the subset construction to convert N to a DFA D (with 5 states).

(2) Prove that D is a minimal DFA.

Problem 4 (10 pts). Given any context-free grammar $G = (V, \Sigma, P, S')$, with special starting production $S' \longrightarrow S$ where S' only appears in this production, the set of characteristic strings C_G is defined by

$$C_G = \{\alpha\beta \in V^* \mid S' \underset{rm}{\Longrightarrow}^* \alpha B v \underset{rm}{\Longrightarrow} \alpha\beta v,\\ \alpha, \beta \in V^*, v \in \Sigma^*, B \to \beta \in P\}.$$

Consider the grammar G with nonterminal set $\{S, A, C\}$ and terminal set $\{a, b, c\}$ given by the following productions:

$$\begin{aligned} S' &\longrightarrow S \\ S &\longrightarrow AC \\ A &\longrightarrow aAb \\ A &\longrightarrow ab \\ C &\longrightarrow c. \end{aligned}$$

Describe all *rightmost* derivations and the set C_G.

Problem 5 (20 pts).

(i) Give a context-free grammar for the language

$$L_1 = \{a^m b^n c^p \mid n \neq p,\ m, n, p \geq 1\}.$$

(ii) Prove that the language L is not regular.

Problem 6 (10 pts).

(i) Give a context-free grammar for the language

$$L_2 = \{a^m b^n \mid n < 3m,\ m > 0,\ n \geq 0\}.$$

Problem 7 (10 pts).

Prove that if the language $L_1 = \{a^n b^n c^n \mid n \geq 1\}$ is not context-free (which is indeed the case), then the language $L_2 = \{w \mid w \in \{a, b, c\}^*, \#(a) = \#(b) = \#(c)\}$ is not context-free either.

Problem 8 (10 pts). (1) Prove that the following sets are not computable ($\varphi_1, \varphi_2, \ldots, \varphi_i$, $\ldots$ is an enumeration of the partial computable functions):

$$
\begin{aligned}
A &= \{i \in \mathbb{N} \mid \varphi_i = \varphi_a * \varphi_b\} \\
B &= \{\langle i, j, k \rangle \in \mathbb{N} \mid \varphi_i = \varphi_j * \varphi_k\} \\
C &= \{i \in \mathbb{N} \mid \varphi_i(i) = a\}
\end{aligned}
$$

where a and b are two fixed natural numbers.

(2) Prove that C is listable.

Problem 9 (5 pts). Define the sets K, K_0 and TOTAL. For each one, state whether it is computable, computably enumerable, or not computably enumerable.

Problem 10 (5 pts).

Let $f\colon \mathbb{N} \to \mathbb{N}$ be a total computable function. Prove that if f is a bijection, then its inverse f^{-1} is also (total) computable.

Problem 11 (10 pts). Recall that the **Clique Problem** for undirected graphs is this: Given an undirected graph $G = (V, E)$ and an integer $K \geq 2$, is there a set C of nodes with $|C| \geq K$ such that for all $v_i, v_j \in C$, there is *some* edge $\{v_i, v_j\} \in E$? Equivalently, does G contain a complete subgraph with at least K nodes?

Give a polynomial reduction from the **Clique Problem** for undirected graphs to the **Satisfiability Problem**.

Assuming that the graph $G = (V, E)$ has n nodes and that the budget is an integer K such that $2 \leq K \leq n$, create nK boolean variables x_{ik} with intended meaning that $x_{ik} = \mathbf{T}$ if node v_i is chosen as the kth element of a clique C, with $1 \leq k \leq K$, and write clauses asserting that K nodes are chosen to belong to a clique C.