Fall 2017 CIS 262

Automata, Computability and Complexity

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Homework 6

October 19, 2017; Due November 2, 2017, beginning of class

"B problems" must be turned in.

Problem B1 (80 pts). Let $D = (Q, \Sigma, \delta, q_0, F)$ be a deterministic finite automaton. Define the relations $\approx$ and $\sim$ on Σ^* as follows:

$$x \approx y \quad \text{if and only if,} \quad \text{for all} \quad p \in Q, \\ \delta^*(p, x) \in F \quad \text{iff} \quad \delta^*(p, y) \in F,$$

and

$$x \sim y \quad \text{if and only if,} \quad \text{for all} \quad p \in Q, \quad \delta^*(p, x) = \delta^*(p, y).$$

(1) Show that $\approx$ is a left-invariant equivalence relation and that $\sim$ is an equivalence relation that is both left and right invariant. (A relation R on Σ^* is *left invariant* iff uRv implies that $wuRwv$ for all $w \in \Sigma^*$, and R is *left and right invariant* iff uRv implies that $xuyRxvy$ for all $x, y \in \Sigma^*$.)

(2) Let n be the number of states in Q (the set of states of D). Show that $\approx$ has at most 2^n equivalence classes and that $\sim$ has at most n^n equivalence classes.

Hint. In the case of $\approx$, consider the function $f\colon \Sigma^* \to 2^Q$ given by

$$f(u) = \{p \in Q \mid \delta^*(p, u) \in F\}, \quad u \in \Sigma^*,$$

and show that $x \approx y$ iff $f(x) = f(y)$. In the case of $\sim$, let Q^Q be the set of all functions from Q to Q and consider the function $g\colon \Sigma^* \to Q^Q$ defined such that $g(u)$ is the function given by

$$g(u)(p) = \delta^*(p, u), \quad u \in \Sigma^*, \ p \in Q,$$

and show that $x \sim y$ iff $g(x) = g(y)$.

(3) Given any language $L \subseteq \Sigma^*$, define the relations λ_L and μ_L on Σ^* as follows:

$$u \, \lambda_L \, v \quad \text{iff,} \quad \text{for all} \quad z \in \Sigma^*, \quad zu \in L \quad \text{iff} \quad zv \in L,$$

and

$$u\,\mu_L\,v \quad \text{iff,} \quad \text{for all} \quad x, y \in \Sigma^*, \quad xuy \in L \quad \text{iff} \quad xvy \in L.$$

Prove that λ_L is left-invariant, and that μ_L is left and right-invariant. Prove that if L is regular, then both λ_L and μ_L have a finite number of equivalence classes.

Hint: Show that the number of classes of λ_L is at most the number of classes of $\approx$, and that the number of classes of μ_L is at most the number of classes of $\sim$.

Problem B2 (60 pts). (1) Prove that the intersection, $L_1 \cap L_2$, of two regular languages, L_1 and L_2, is regular, **using the Myhill-Nerode characterization** of regular languages.

(2) Let $h\colon \Sigma^* \to \Delta^*$ be a homomorphism, as defined on pages 31-33 of the slides on DFA's and NFA's. For any regular language, $L' \subseteq \Delta^*$, prove that

$$h^{-1}(L') = \{w \in \Sigma^* \mid h(w) \in L'\}$$

is regular, **using the Myhill-Nerode characterization** of regular languages.

Proceed as follows: Let $\simeq'$ be a right-invariant equivalence relation on Δ^* of finite index n, such that L' is the union of some of the equivalence classes of $\simeq'$. Let $\simeq$ be the relation on Σ^* defined by

$$u \simeq v \quad \text{iff} \quad h(u) \simeq' h(v).$$

Prove that $\simeq$ is a right-invariant equivalence relation of finite index m, with $m \leq n$, and that $h^{-1}(L')$ is the union of equivalence classes of $\simeq$.

To prove that that the index of $\simeq$ is at most the index of $\simeq'$, use h to define a function $\widehat{h}\colon (\Sigma^*/\simeq) \to (\Delta^*/\simeq')$ from the partition associated with $\simeq$ to the partition associated with $\simeq'$, and prove that $\widehat{h}$ is injective.

Prove that the number of states of any minimal DFA for $h^{-1}(L')$ is at most the number of states of any minimal DFA for L'. Can it be strictly smaller? If so, give an explicit example.

TOTAL: 140 points