Geometry and Computing

Volume 12

Geometric shapes belong to our every-day life, and modeling and optimization of such forms determine biological and industrial success. Similar to the digital revolution in image processing, which turned digital cameras and online video downloads into consumer products, nowadays we encounter a strong industrial need and scientific research on geometry processing technologies for 3D shapes.

Several disciplines are involved, many with their origins in mathematics, revived with computational emphasis within computer science, and motivated by applications in the sciences and engineering. Just to mention one example, the renewed interest in discrete differential geometry is motivated by the need for a theoretical foundation for geometry processing algorithms, which cannot be found in classical differential geometry.

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Jean Gallier • Jocelyn Quaintance

Differential Geometry and Lie Groups

A Computational Perspective

Springer

Jean Gallier
Department of Computer and Information Science
University of Pennsylvania
Philadelphia, PA, USA

Jocelyn Quaintance
Department of Computer and Information Science
University of Pennsylvania
Philadelphia, PA, USA

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To my wife Anne, my son Philippe,

and my daughters Mia and Sylvie.

To my parents Howard and Jane.

Preface

This book is written for a wide audience ranging from upper undergraduate to advanced graduate students in mathematics, physics, and more broadly engineering students, especially in computer science. It covers manifolds, Riemannian geometry, and Lie groups, some central topics of mathematics. However, computer vision, robotics, and machine learning, to list just a few "hot" applied areas, are increasingly consumers of differential geometry tools, so this book is also written for professionals who wish to learn about the concepts and tools from differential geometry used to solve some of their problems.

Although there are many books covering differential geometry and Lie groups, most of them assume that the reader is already quite familiar with manifold theory, which is a severe obstacle for a reader who does not possess such a background. In this book, we only assume some modest background in calculus and linear algebra from the reader and basically develop manifold theory from scratch. Additional review chapters covering some basics of analysis, in particular the notion of derivative of a map between two normed vector spaces, and some basics of topology are provided for the reader who needs to firm up her/his background in these areas. This book is split into two parts:

1. The basic theory of manifolds and Lie groups (Part I: "Introduction to Differential Manifolds and Lie Groups").
2. Some of the fundamental topics of Riemannian geometry (Part II: "Riemannian Geometry, Lie Groups, and Homogeneous Spaces").

The culmination of the concepts and results presented in this book is the theory of *naturally reductive homogeneous manifolds and symmetric spaces*. It is remarkable that most familiar spaces are naturally reductive manifolds. Remarkably, they all arise from some suitable action of the rotation group $\mathbf{SO}(n)$, a Lie group, which emerges as the master player. The machinery of naturally reductive manifolds, and of symmetric spaces (which are even nicer!), makes it possible to compute explicitly *in terms of matrices* all the notions from differential geometry (Riemannian metrics, geodesics, etc.) that are needed to generalize optimization methods to Riemannian manifolds. Such methods are presented in Absil, Mahony, and Sepulchre [2], and there is even a software package (MANOPT) that implements some of these procedures.

The interplay between Lie groups, manifolds, and analysis yields a particularly effective tool. We tried to explain in some detail how these theories all come together to yield such a beautiful and useful tool.

We have also included chapters that present material having significant practical applications. These include

1. Chapter 9, on constructing manifolds from gluing data, which has applications in surface reconstruction from 3D meshes.
2. Chapter 22, on the "Log-Euclidean framework," which has applications in medical imaging.
3. Chapter 23, on homogeneous reductive spaces and symmetric spaces, which has applications in robotics, machine learning, and computer vision. For example, Stiefel and Grassmannian manifolds come up naturally. Furthermore, in these manifolds, it is possible to compute explicitly geodesics, Riemannian distances, gradients, and Hessians. This makes it possible to actually extend optimization methods such as gradient descent and Newton's method to these manifolds. A very good source on these topics is Absil, Mahony, and Sepulchre [2].

In the past five years, we have also come to realize that *Lie groups* and *homogeneous manifolds*, especially naturally reductive ones, are two of the most important topics for their role in applications. It is remarkable that most familiar spaces, spheres, projective spaces, Grassmannian and Stiefel manifolds, and symmetric positive definite matrices are naturally reductive manifolds. Remarkably, they all arise from some suitable action of the rotation group $\mathbf{SO}(n)$, a Lie group, who emerges as the master player. The machinery of naturally reductive manifolds, and of symmetric spaces (which are even nicer!), makes it possible to compute explicitly *in terms of matrices* all the notions from differential geometry (Riemannian metrics, geodesics, etc.) that are needed to generalize optimization methods to Riemannian manifolds.

Since we discuss many topics ranging from manifolds to Lie groups, this book is already quite big, so we resolved ourselves, not without regrets, to omit many proofs. The purist may be chagrined, but we feel that it is more important to motivate, demystify, and explain the reasons for introducing various concepts and to clarify the relationship between these notions rather than spelling out every proof in full detail. Whenever we omit a proof, we provide precise pointers to the literature. In some cases (such as the theorem of Hopf and Rinow), the proof is just too beautiful to be skipped, so we include it.

The motivations for writing these notes arose while the first author was coteaching a seminar on Special Topics in Machine Perception with Kostas Daniilidis in the Spring of 2004. In the Spring of 2005, the first author gave a version of his course *Advanced Geometric Methods in Computer Science* (CIS610), with the main goal of discussing statistics on diffusion tensors and shape statistics in medical imaging. This is when he realized that it was necessary to cover some material on Riemannian geometry but he ran out of time after presenting Lie groups and never got around to doing it! Then, in the Fall of 2006, the first author went on a wonderful and very productive sabbatical year in Nicholas Ayache's group (ACSEPIOS) at INRIA Sophia Antipolis, where he learned about the beautiful and exciting work of Vincent Arsigny, Olivier Clatz, Hervé Delingette, Pierre Fillard, Grégoire Malandin, Xavier Pennec, Maxime Sermesant, and, of course, Nicholas Ayache, on statistics on manifolds and Lie groups applied to medical imaging. This inspired him to write chapters on differential geometry, and after a few additions made during Fall 2007 and Spring 2008, notably on left-invariant metrics on Lie groups, the little set of notes from 2004 had grown into a preliminary version of this manuscript. The first author then joined forces with the second author in 2015, and with her invaluable assistance produced the present book as well as a second volume dealing with more advanced topics.

We must acknowledge our debt to two of our main sources of inspiration: Berger's *Panoramic View of Riemannian Geometry* [14] and Milnor's *Morse Theory* [81]. In our opinion, Milnor's book is still one of the best references on basic differential geometry. His exposition is remarkably clear and insightful, and his treatment of the variational approach to geodesics is unsurpassed. We borrowed heavily from Milnor [81]. Since Milnor's book is typeset in "ancient" typewritten format (1973!), readers might enjoy reading parts of it typeset in LaTeX. We hope that the readers of these notes will be well prepared to read standard differential geometry texts such as do Carmo [39], Gallot, Hulin, Lafontaine [49], and O'Neill [91], but also more advanced sources such as Sakai [100], Petersen [93], Jost [64], Knapp [68], and of course Milnor [81].

The chapters or sections marked with the symbol $\circledast$ contain material that is typically more specialized or more advanced, and they can be omitted upon first (or second) reading.

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Philadelphia, PA, USA — Jean Gallier
Philadelphia, PA, USA — Jocelyn Quaintance

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