Geometry and Computing

Volume 13

Geometric shapes belong to our every-day life, and modeling and optimization of such forms determine biological and industrial success. Similar to the digital revolution in image processing, which turned digital cameras and online video downloads into consumer products, nowadays we encounter a strong industrial need and scientific research on geometry processing technologies for 3D shapes.

Several disciplines are involved, many with their origins in mathematics, revived with computational emphasis within computer science, and motivated by applications in the sciences and engineering. Just to mention one example, the renewed interest in discrete differential geometry is motivated by the need for a theoretical foundation for geometry processing algorithms, which cannot be found in classical differential geometry.

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Differential Geometry and Lie Groups

A Second Course

Springer

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To my daughter Mia, my wife Anne,

my son Philippe, and my daughter Sylvie.

To my parents Howard and Jane.

Preface

This book is written for a wide audience ranging from upper-undergraduate to advanced graduate students in mathematics, physics, and more broadly engineering students, especially in computer science. Basically, it covers topics which belong to a second course in differential geometry. The reader is expected to be familiar with the theory of manifolds and with some elements of Riemannian geometry, including connections, geodesics, and curvature. Some familiarity with the material presented in the following books is more than sufficient: Tu [105] (the first three chapters), Warner [109] (the first chapter and parts of Chapters 2–4), Do Carmo [37] (the first four chapters), Gallot, Hulin, Lafontaine [48] (the first two chapters and parts of Chapter 3), O'Neill [84] (Chapters 1 and 3), and Gallier and Quaintance [47], which contains all the preliminaries needed to read this book.

The goal of differential geometry is to study the geometry and the topology of manifolds using techniques involving differentiation in one way or another. The pillars of differential geometry are:

(1) Riemannian metrics,
(2) Connections,
(3) Geodesics, and
(4) Curvature.

There are many good books covering the above topics, and we also provided our own account (Gallier and Quaintance [47]). One of the goals of differential geometry is also to be able to generalize "calculus on $\mathbb{R}^n$" to spaces more general than $\mathbb{R}^n$, namely manifolds. We would like to differentiate functions $f\colon M \to \mathbb{R}$ defined on a manifold, optimize functions (find their minima or maxima), and also to integrate such functions, as well as compute areas and volumes of subspaces of our manifold.

The generalization of the notion of derivative of a function defined on a manifold is the notion of tangent map, and the notions of gradient and Hessian are easily generalized to manifolds equipped with a connection (or a Riemannian metric, which yields the Levi-Civita connection). However, the problem of defining the integral of a function whose domain is a manifold remains.

One of the main discoveries made at the beginning of the twentieth century by Poincaré and Élie Cartan is that the "right" approach to integration is to integrate *differential forms*, and not functions. To integrate a function f, we integrate the form $f\omega$, where ω is a *volume form* on the manifold M. The formalism of differential forms takes care of the process of the change of variables quite automatically and allows for a very clean statement of *Stokes' theorem*.

The theory of differential forms is one of the main tools in geometry and topology. This theory has a surprisingly large range of applications, and it also provides a relatively easy access to more advanced theories such as cohomology. For all these reasons, it is really an indispensable theory, and anyone with more than a passable interest in geometry should be familiar with it.

In this book, we discuss the following topics:

(1) Differential forms, including vector-valued differential forms and differential forms on Lie groups.
(2) An introduction to de Rham cohomology.
(3) Distributions and the Frobenius theorem.
(4) Integration on manifolds, starting with orientability, volume forms, and ending with Stokes' theorem on regular domains.
(5) Integration on Lie groups.
(6) Spherical harmonics and an introduction to the representations of compact Lie groups.
(7) Operators on Riemannian manifolds: Hodge Laplacian, Laplace–Beltrami Laplacian, and Bochner Laplacian.
(8) Fiber bundles, vector bundles, principal bundles, and metrics on bundles.
(9) Connections and curvature in vector bundles, culminating with an introduction to Pontrjagin classes, Chern classes, and the Euler class.
(10) Clifford algebras, Clifford groups, and the groups $\mathbf{Pin}(n)$, $\mathbf{Spin}(n)$, $\mathbf{Pin}(p, q)$, and $\mathbf{Spin}(p, q)$.

Topics (3)–(7) have more of an analytic than a geometric flavor. Topics (8) and (9) belong to the core of a second course on differential geometry. Clifford algebras and Clifford groups constitute a more algebraic topic. These can be viewed as a generalization of the quaternions. The groups $\mathbf{Spin}(n)$ are important because they are the universal covers of the groups $\mathbf{SO}(n)$.

Since this book is already quite long, we resolved ourselves, not without regrets, to omit many proofs. We feel that it is more important to motivate, demystify, and explain the reasons for introducing various concepts and to clarify the relationship between these notions rather than spelling out every proof in full detail. Whenever we omit a proof, we provide precise pointers to the literature.

We must acknowledge our debt to our main sources of inspiration: Bott and Tu [12], Bröcker and tom Dieck [18], Cartan [21], Chern [22], Chevalley [24], Dieudonné [31–33], do Carmo [37], Gallot, Hulin, Lafontaine [48], Hirzebruch [57], Knapp [66], Madsen and Tornehave [75], Milnor and Stasheff [78], Morimoto [81], Morita [82], Petersen [86], and Warner [109].

The chapters or sections marked with the symbol ⊛ contain material that is typically more specialized or more advanced, and they can be omitted upon first (or second) reading.

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