

Solutions to Homework Set 4, March 4 1999

Problem 1

(a)

00	→	00000
01	→	01101
10	→	10110
11	→	11011

Recall that for linear block codes, the minimum Hamming distance equals the minimum Hamming weight. The minimum Hamming weight of non-zero codewords equals 3, thus $d_{\min}=3$.

[Note that this code is indeed a linear (block) code. Linearity follows from the fact that the code is a systematic code for which a G matrix can be found easily; first part of the 2×5 G matrix is a 2×2 identity matrix and the other three columns of G can be chosen to always produce the desired three parity bits from the message word.]

(b) Since $d_{\min}=3$, at most 1 error can be corrected.

(c)

$$\begin{aligned}
 P\{\text{decoding error}\} &= P\{\text{more than 1 bit error out of 5 bits}\} \\
 &= 1 - (P\{0 \text{ error/5 bits}\} + P\{1 \text{ error/5 bits}\}) \\
 &= 1 - \left[(1-10^{-3})^5 + \binom{5}{1} 10^{-3} (1-10^{-3})^4 \right] \\
 &= 1 - (1-10^{-3})^4 \left((1-10^{-3}) + (5 \times 10^{-3}) \right) \\
 &\approx 1 - (1 - 4 \times 10^{-3} + 6 \times 10^{-6})(1 + 4 \times 10^{-3}) \\
 &= 1 - (1 - 4 \times 10^{-3} + 6 \times 10^{-6} + 4 \times 10^{-3} - 16 \times 10^{-6} + 24 \times 10^{-9}) \\
 &\approx 1 - (1 - 10 \times 10^{-6}) \\
 &= 10^{-5}
 \end{aligned}$$

Problem 2

(a) Without loss of generality, assume even parity. The codewords then have even number of 1's, and thus $d_{\min}=2$. (Once again, note that an even-parity bit scheme always produces a linear block code; the $k \times (k+1)$ G matrix for this is easily seen to be the $k \times k$ identity matrix with one more column of all "1"s. For a linear block code, we can find $d_{\min}$ by finding the minimum Hamming weight of the non-zero codewords. For even parity, this is clearly 2).

[Note that if we assume odd parity, then the resulting code is *not* linear! However, this code is closely related to the even-parity one and in fact has exactly the same minimum distance and hence the same capabilities for error detection. For odd

parity, the all-zero data word does not produce the all-zero codeword. In fact the all-zero sequence is not a valid codeword. This shows immediately that the code is not linear, since a linear code must always have the all-zero sequence as one codeword (corresponding to the all-zero data word). It is easy to see that to get a codeword of the odd-parity code corresponding to some input word $\mathbf{d}$, we can start with the corresponding even-parity codeword and then flip the last (parity) bit, i.e. add (mod-2) the sequence $\mathbf{e}=00\dots01$ to that codeword. Thus if $\mathbf{G}$ is the matrix producing the even-parity codewords, we now have for the odd-parity code $\mathbf{c}=\mathbf{dG} + \mathbf{e}$ where the sum is a modulo-2 sum. This is *not* of the linear form. But it shows that the distance between any two codewords of the even-parity code is the same as that for the corresponding two codewords of the odd-parity code. Thus the two codes have exactly the same min. distance.]

- (b) The code can be broken down into 2 concatenated (4+1) parity codes, each having $d'_{\min}=2$. Since 2 different length-10 codewords can have the same first or last parts, $d_{\min}$ for the (10,8) code is still 2. Thus the code is still guaranteed to detect 1 error. The rate of the code is $8/10=0.8$.
- (c) Code (a) will never detect 2 errors ($P=0$).
 Code (b) will detect 2 errors if they are in separate parts (first 4 bits and its parity bit, and the other 4 bits plus parity bit) of its 10-bit codewords. Given 2 errors, they can both be in one part, or the other part, or 1 in each. Total number of 2-error patterns is $\binom{10}{2}=45$, total number of 2-error patterns with one error in each group of 5 bits is $5 \times 5 = 25$. Thus, 5/9 of 2-bit errors will be detected.

Problem 5

- (a)
- $$P(\text{error}) = P(\text{1 or more bits are in error}) = 1 - P(\text{no error})$$
- $$= 1 - (1 - p)^4 \approx 1 - (1 - 4p) = 4p$$
- (b)
- $$P(\text{error}) = P(\text{2 or more bits out of 7 are in error}) = 1 - P(\text{0 or 1 bit error})$$
- $$= 1 - [P(\text{0 error}) + P(\text{1 error})]$$
- $$= 1 - \left[\binom{7}{0} p^0 (1-p)^7 + \binom{7}{1} p^1 (1-p)^6 \right]$$
- $$\approx 1 - [(1 - 7p + 21p^2) + 7p(1 - 6p)]$$
- $$= 1 - (1 - 7p + 21p^2 + 7p - 42p^2)$$
- $$= 1 - (1 - 21p^2)$$
- $$= 21p^2$$

Here we ignored terms of order p^3 and higher since they are much smaller than the p^2 term. Plug in $p=10^{-3}$ to get numerical quantities.

- (c) The message data rate must be reduced by 4/7 in order to accommodate the use of the code in a fixed transmission-rate link. Thus if you were able to send 7 bits per sec. without coding, with coding the useful message data rate is only 4 bits per sec.

Problem 6

- (a) Recall that H(7,4) codewords are generated as:

$$c_i = d_i G_1 \quad (1)$$

$$G_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

where c_i and d_i are row vectors of the codeword and the data word, respectively. We can then add even parity to c_i by multiplying it with a new generator matrix:

$$c_i' = c_i G_2 \quad (2)$$

$$G_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

where c_i' is the row vector representing a new codeword after adding the parity bit. Note that the last column in G_2 always adds a bit "1" if the number of 1's in c_i is odd.

Now plug (1) into (2):

$$c_i' = d_i G_1 G_2 = d_i G$$

From the last equation it follows:

$$G = G_1 G_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}$$

- (b) First recall that minimum Hamming distance equals minimum Hamming weight for non-zero codewords, for linear block codes. For H(7,4) code, the minimum Hamming distance is 3. If an even parity is added to the codeword with Hamming weight of 3, the result is a modified codeword with Hamming weight of 4. Observe that this codeword still has the minimum Hamming weight for the new code. Therefore, the minimum Hamming weight for the modified code is 4, as is the minimum Hamming distance.

- (c) The new code is still guaranteed to correct only single bit errors. If 2 bits are in error, the Hamming distance between the received codeword and the transmitted codeword will be 2. There might exist another valid codeword whose Hamming distance to the received codeword is also 2. Therefore, the receiver cannot necessarily reconstruct the original codeword for double bit errors.

$H(8,4)$ code can detect 3-bit errors because the minimum Hamming distance is 4.

(Solutions of other 2 problems given out in class)