ECE 250 / CPS 250
Computer Architecture

Basics of Logic Design
Boolean Algebra, Logic Gates

Benjamin Lee

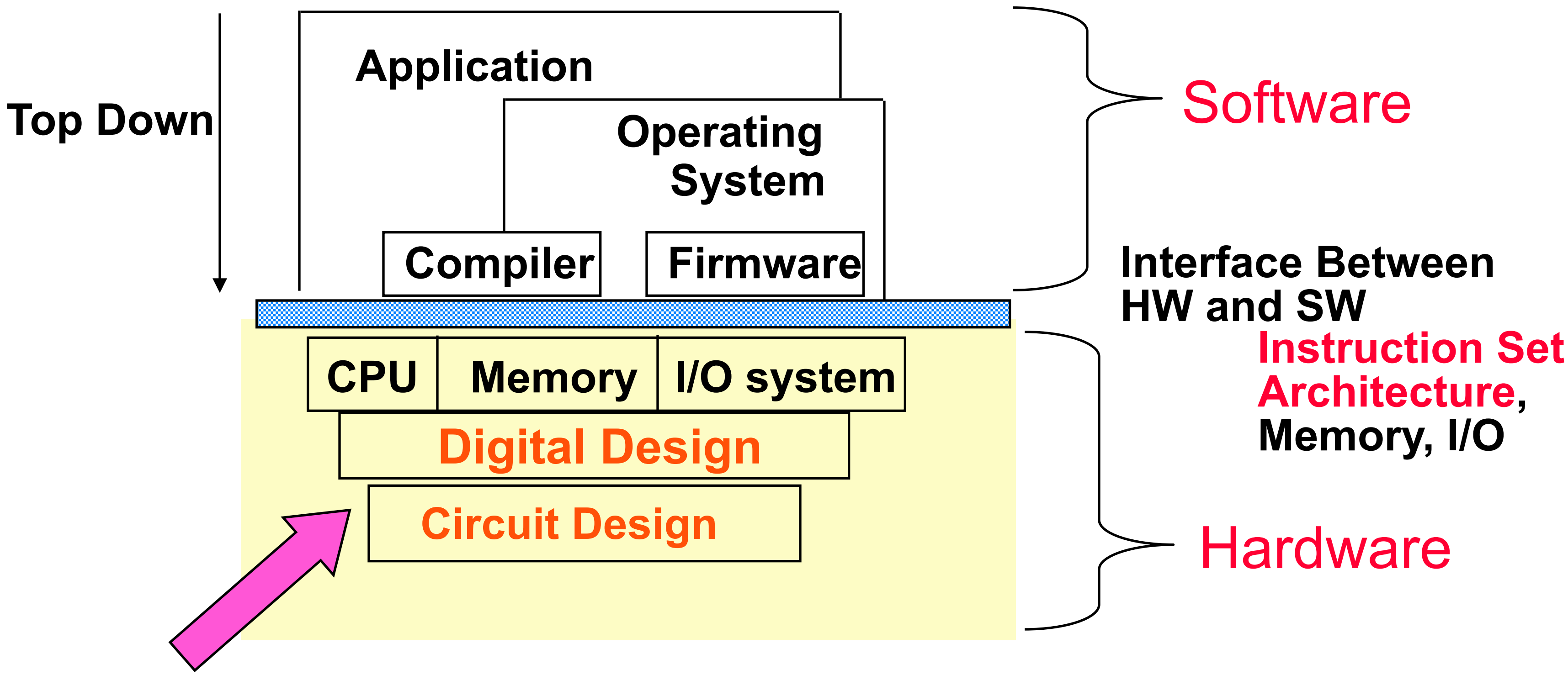
Slides based on those from

Andrew Hilton (Duke), Alvy Lebeck (Duke)
Benjamin Lee (Duke), and Amir Roth (Penn)

Reading

- **Appendix B (parts 1,2,3,5,6,7,8,9,10)**
- **This material is covered in MUCH greater depth in ECE/CS 350 – please take ECE/CS 350 if you want to learn enough digital design to build your own processor**

What We've Done, Where We're Going



(Almost) Bottom UP to CPU

Computer = Machine That Manipulates Bits

- Everything is in binary (bunches of 0s and 1s)
  - Instructions, numbers, memory locations, etc.
- Computer is a machine that operates on bits
  - Executing instructions → operating on bits

- Computers physically made of transistors
  - Electrically controlled switches
- We can use transistors to build logic
  - E.g., if this bit is a 0 and that bit is a 1, then set some other bit to be a 1
  - E.g., if the first 5 bits of the instruction are 10010 then set this other bit to 1 (to tell the adder to subtract instead of add)

How Many Transistors Are We Talking About?

<u>**Pentium III**</u>

- **Processor Core 9.5 Million Transistors**
- **Total: 28 Million Transistors**

<u>**Pentium 4**</u>

- **Total: 42 Million Transistors**

<u>**Core2 Duo (two processor cores)**</u>

- **Total: 290 Million Transistors**

<u>**Core2 Duo Extreme (4 processor cores, 8MB cache)**</u>

- **Total: 590 Million Transistors**

<u>**Core i7 with 6-cores**</u>

- **Total: 2.27 Billion Transistors**

How do they design such a thing? Carefully!

Abstraction!

- Use of abstraction (key to design of any large system)
  - Put a few (2-8) transistors into a logic gate (or, and, xor, …)
  - Combine gates into logical functions (add, select,….)
  - Combine adders, shifters, etc., together into modules
    - Units with well-defined interfaces for large tasks: e.g., decode
  - Combine a dozen of those into a core…
  - Stick 4 cores on a chip…

You are here:

- Use of abstraction (key to design of any large system)
  - Put a few (2-8) transistors into a logic gate
  - Combine gates into logical functions (add, select,….)
  - Combine adders, muxes, etc together into modules
    Units with well-defined interfaces for large tasks: e.g., decode
  - Combine a dozen of those into a core…
  - Stick 4 cores on a chip…

Boolean Algebra

- **First step to logic: Boolean Algebra**
  - **Manipulation of True / False (1/0)**
  - **After all: everything is just 1s and 0s**

- **Given inputs (variables): A, B, C, P, Q…**
  - **Compute outputs using logical operators, such as:**
- **NOT:** !A **(=** ~A **=** $\overline{A}$**)**
- **AND:** A&B **(=** A·B **=** A*B = AB **=** A∧B) = A&&B in C/C++
- **OR:** A | B **(=** A+B **=** A ∨ B**) =** A || B in C/C++
- **XOR:** A ^ B (= A ⊕ B)
- NAND, NOR, XNOR, Etc.

Truth Tables

- **Can represent as Truth Table: shows outputs for all inputs**

a	NOT(a)
0	1
1	0

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Truth Tables

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a	NOT(a)
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a	b	AND(a,b)
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0	1	0
1	0	0
1	1	1

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Truth Tables

- Can represent as truth table: shows outputs for all inputs

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0	1
1	0

a	b	AND(a,b)
0	0	0
0	1	0
1	0	0
1	1	1

a	b	OR(a,b)
0	0	0
0	1	1
1	0	1
1	1	1

a	b	XOR(a,b)
0	0	0
0	1	1
1	0	1
1	1	0

a	b	XNOR(a,b)
0	0	1
0	1	0
1	0	0
1	1	1

a	b	NOR(a,b)
0	0	1
0	1	0
1	0	0
1	1	0

Any Inputs, Any Outputs

- Can have any # of inputs, any # of outputs
- Can have arbitrary functions:

a	b	c	f_1	f_2
0	0	0	0	1
0	0	1	1	1
0	1	0	1	0
0	1	1	0	0
1	0	0	1	0
1	1	0	0	1
1	1	1	1	1

Let's Write a Truth Table for a Function…

- **Example:**
  - **(A & B) | !C**

Start with Empty TT

- **Column Per Input**
- **Column Per Output**

A	B	C	Output

Let’s write a Truth Table for a function…

- **Example:**
  (A & B) | !C

Start with Empty TT

Column Per Input

Column Per Output

Fill in Inputs

Counting in Binary

A	B	C	Output
0	0	0	

Let's write a Truth Table for a function…

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Let's write a Truth Table for a function…

- **Example:**
  (A & B) | !C

Start with Empty TT

Column Per Input

Column Per Output

Fill in Inputs

Counting in Binary

Compute Output

(0 & 0) | !0 = 0 | 1 = 1

A	B	C	Output
0	0	0	1
0	0	1	
0	1	0	
0	1	1	
1	0	0	
1	0	1	
1	1	0	
1	1	1	

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- **Example:**
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Column Per Input

Column Per Output

Fill in Inputs

Counting in Binary

Compute Output

(0 & 0) | !1 = 0 | 0 = 0

A	B	C	Output
0	0	0	1
0	0	1	0
0	1	0	
0	1	1	
1	0	0	
1	0	1	
1	1	0	
1	1	1	

Let's write a Truth Table for a function…

- **Example:**

  (A & B) | !C

Start with Empty TT

Column Per Input

Column Per Output

Fill in Inputs

Counting in Binary

Compute Output

(0 & 1) | !0 = 0 | 1 = 1

A	B	C	Output
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	
1	0	0	
1	0	1	
1	1	0	
1	1	1	

Let's write a Truth Table for a function…

- **Example:**
  (A & B) | !C

Start with Empty TT

  Column Per Input

  Column Per Output

Fill in Inputs

  Counting in Binary

Compute Output

A	B	C	Output
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0	0	1	0
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0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	1
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You try one…

- **Try one yourself:**
  - **(!A | B) & !C**

You try one…

- **Try one yourself:**
  (!A | B) & !C

Answer:

A	B	C	Output
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	0

Suppose I turn it around…

- **Given a Truth Table, find the formula?**

Hmmm..

A	B	C	Output
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	0
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1	1	1	1

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- **Given a Truth Table, find the formula?**

Hmmm …

Could write down every "true" case

Then OR together:

```
(!A & !B & !C)  |
(!A & !B & C)   |
(!A & B & !C)  |
(A & B &!C)  |
(A & B &C)
```

A	B	C	Output
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	0
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```

A	B	C	Output
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

Suppose I turn it around…

- **This approach: "sum of products"**
  - Works every time
  - Result is right…
  - But really ugly

```
(!A & !B & !C)  |
(!A & !B & C)   |
(!A & B & !C)   |
(A & B &!C)  |
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```

A	B	C	Output
0	**0**	**0**	**1**
0	**0**	**1**	**1**
0	**1**	**0**	**1**
0	1	1	0
1	0	0	0
1	0	1	0
1	**1**	**0**	**1**
1	**1**	**1**	**1**

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```
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(!A & B & !C) |
(A & B &!C) |
(A & B &C)
```

Could just be (A & B) here ?

A	B	C	Output
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

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```
(!A & !B & !C) |
(!A & !B & C)  |
(!A & B & !C) |
(A&B)
```

A	B	C	Output
0	0	0	1
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0	1	0	1
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Could just be (!A & !B) here

A	B	C	Output
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```
(!A & !B) |
(!A & B & !C) |
(A&B)
```

Could just be (!A & !B) here

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0	0	1	1
0	1	0	1
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1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

Suppose I turn it around…

- **This approach: "sum of products"**
  - Works every time
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  - But really ugly

```
(!A & !B) |
(!A & B & !C) |
(A&B)
```

Looks nicer…

Can we do better?

A	B	C	Output
0	**0**	**0**	**1**
0	**0**	**1**	**1**
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

Suppose I turn it around…

- **This approach: "sum of products"**
  - Works every time
  - Result is right…
  - But really ugly

```
(!A & !B) |
(!A & B & !C) |
(A&B)
```

This has a lot in common:
!A & (something)

A	B	C	Output
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

Suppose I turn it around…

- **This approach: "sum of products"**
  - Works every time
  - Result is right…
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```
(!A & !(B & C)) |
(A & B)
```

A	B	C	Output
0	0	0	1
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0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

Just did some of these by intuition.. but

- Somewhat intuitive approach to simplifying
- This is math, so there are formal rules
  - Just like "regular" algebra

Boolean Function Simplification

- **Boolean expressions can be simplified by using the following rules (bitwise logical):**
  - `A & A = A` `A | A = A`
  - `A & 0 = 0` `A | 0 = A`
  - `A & 1 = A` `A | 1 = 1`
  - `A & !A = 0` `A | !A = 1`
  - `!!A = A`
  - **& and | are both commutative and associative**
  - **& and | can be distributed:** `A & (B | C) = (A & B) | (A & C)`
  - **& and | can be subsumed:** `A | (A & B) = A`

DeMorgan's Laws

- **Two (less obvious) Laws of Boolean Algebra:**
  - **Let's push negations inside, flipping & and |**

```
!(A & B) = (!A) | (!B)
```

```
!(A | B) = (!A) & (!B)
```

  - **You should try this at home – build truth tables for both the left and right sides and see that they're the same**

Suppose I turn it around…

- **One more simplification on early example:**

```
(!A & !(B & C)) |
(A & B)

=

(!A & (!B | !C)) |
(A & B)
```

A	B	C	Output
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

Simplification Example:

```
! (!A | !(A & (B | C)))
```

Simplification Example:

```
! (!A | !(A & (B | C)))
```

DeMorgan's

```
!!A & !! (A & (B | C))
```

Simplification Example:

```
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DeMorgan's

```
!!A & !! (A & (B | C))
```

Double Negation Elimination

```
A & (A & (B | C))
```

Simplification Example:

```
! (!A | !(A & (B | C)))
```

DeMorgan's

```
!!A & !! (A & (B | C))
```

Double Negation Elimination

```
A & (A & (B | C))
```

Associativity of &

```
(A & A) & (B | C)
```

Simplification Example:

`! (!A | !(A & (B | C)))`

DeMorgan's

`!!A & !! (A & (B | C))`

Double Negation Elimination

`A & (A & (B | C))`

Associativity of &

`(A & A) & (B | C)`

A & A = A

`A & (B | C)`

You try this:

Come up with a formula for this Truth Table

Simplify as much as possible

A	B	C	Output
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	1
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Sum of Products:

```
(!A & !B & !C)  |
(!A & B & !C)   |
(A & !B & C)    |
(A & B & C)
```

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You try this:

Simplify:

```
(!A & !B & !C) | (!A & B & !C)
```

You try this:

```
Simplify:
   (!A & !B & !C) | (!A & B & !C)
Regroup (associative/commutative):
   ((!A & !C) & !B) | ((!A & !C) & B)
```

You try this:

```
Simplify:
   (!A & !B & !C) | (!A & B & !C)
Regroup (associative/commutative):
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Un-distribute:
   (!A & !C) & (!B | B)
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You try this:

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Simplify:
   (!A & !B & !C) | (!A & B & !C)
Regroup (associative/commutative):
   ((!A & !C) & !B) | ((!A & !C) & B)
Un-distribute:
   (!A & !C) & (!B | B)
OR identities:
   (!A & !C) & true  = (!A & !C)
```

You try this:

Come up with a formula for this Truth Table

Simplify as much as possible

Sum of Products:

```
(!A & !C) |
(A & !B & C)     |
(A & B & C)
```

A	B	C	Output
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Come up with a formula for this Truth Table

Simplify as much as possible

Sum of Products:

```
(!A & !C) |
(A & C)
```

A	B	C	Output
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0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	0
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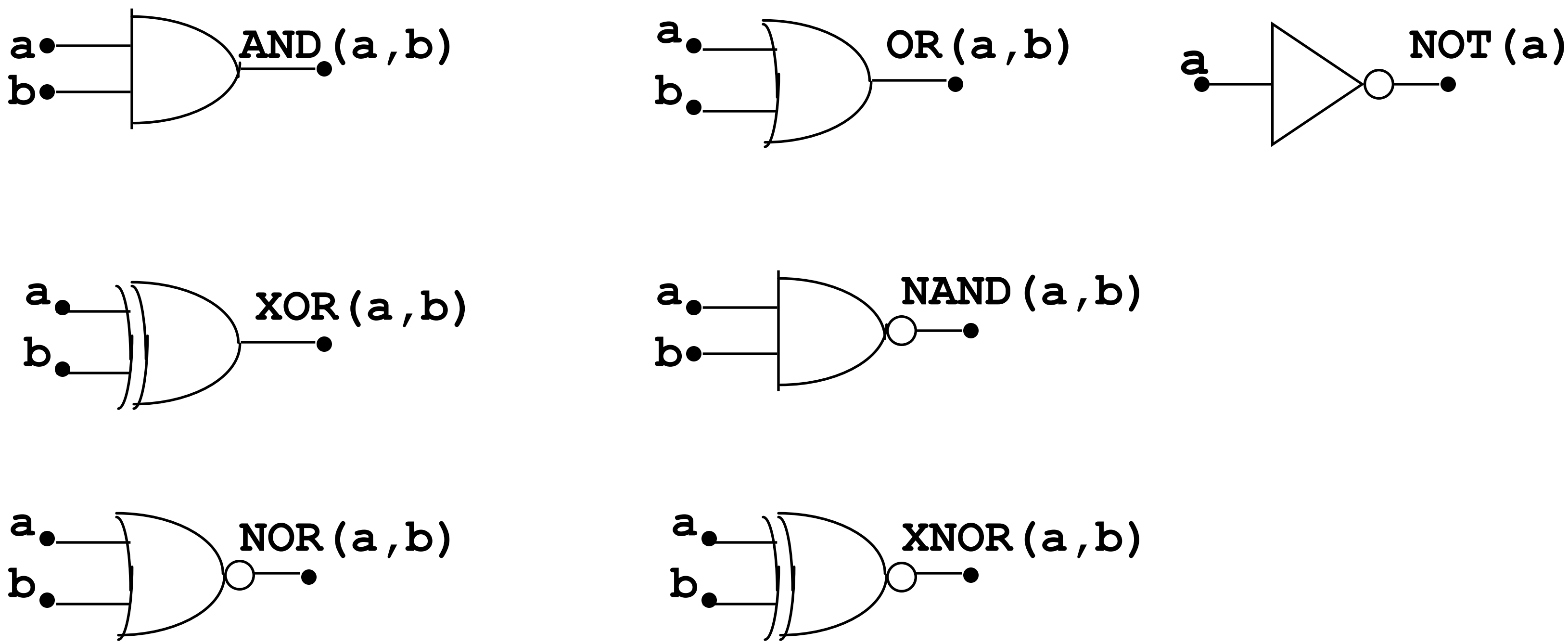
Applying the Theory

- Lots of good theory
- Can reason about complex Boolean expressions
- But why is this useful?   (fun party trick)

Boolean Gates

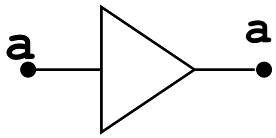
- **Gates are electronic devices that implement simple Boolean functions (building blocks of hardware)**

Examples



Guide to Remembering your Gates

- **This one looks like it just points its input where to go**
  - **It just produces its input as its output**
  - **Called a buffer**



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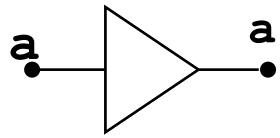
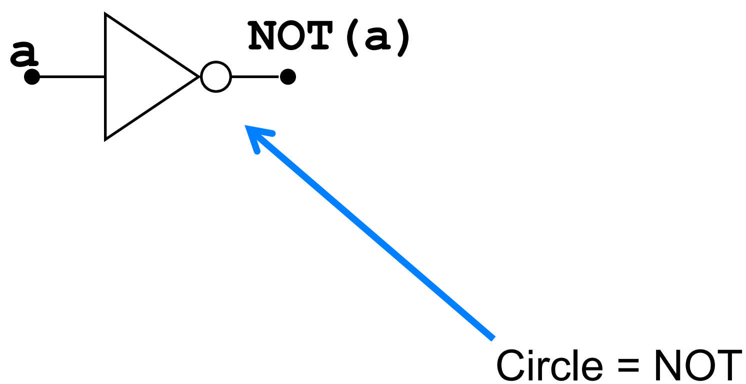
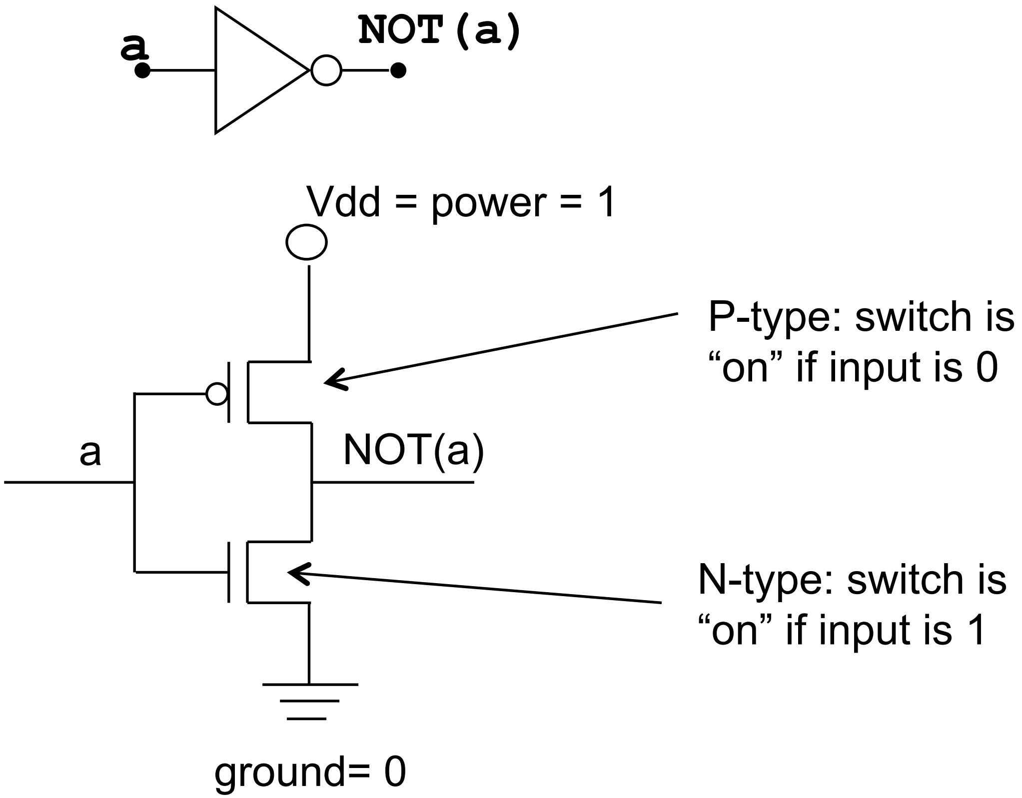


- A circle always means negate (invert)

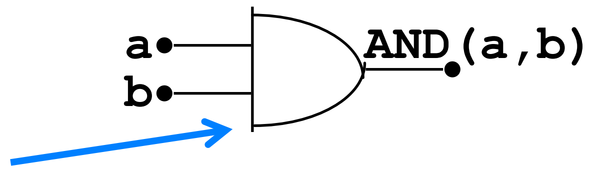


Brief Interlude: Building An Inverter



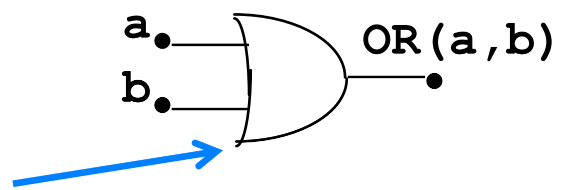
Guide to Remembering Your Gates

- **AND Gates have a straight edge, like an A (in AND)**



Straight like an A

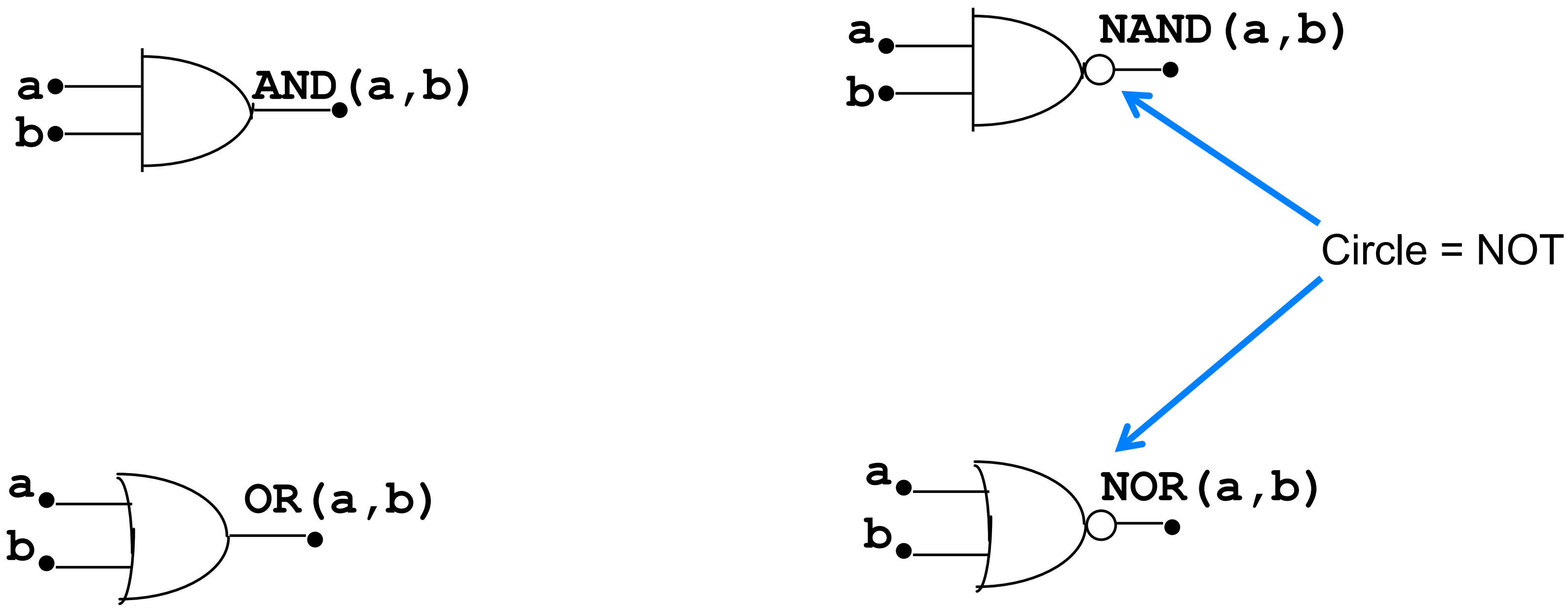
OR Gates have a curved edge, like an O (in OR)



Curved, like an O

Guide to Remembering Your Gates

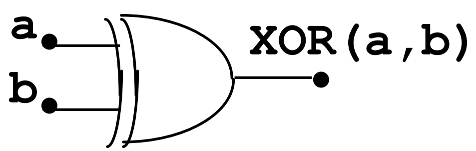
- **If we stick a circle on them…**



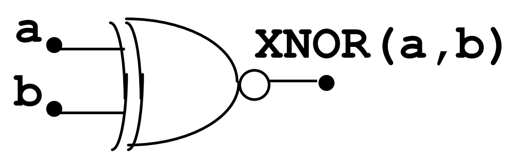
- **We get NAND (NOT-AND) and NOR (NOT-OR)**
  - **NAND(a,b) = NOT(AND(a,b))**

Guide to Remembering Your Gates

- XOR looks like OR (curved line)
  - But has two lines (like an X does)



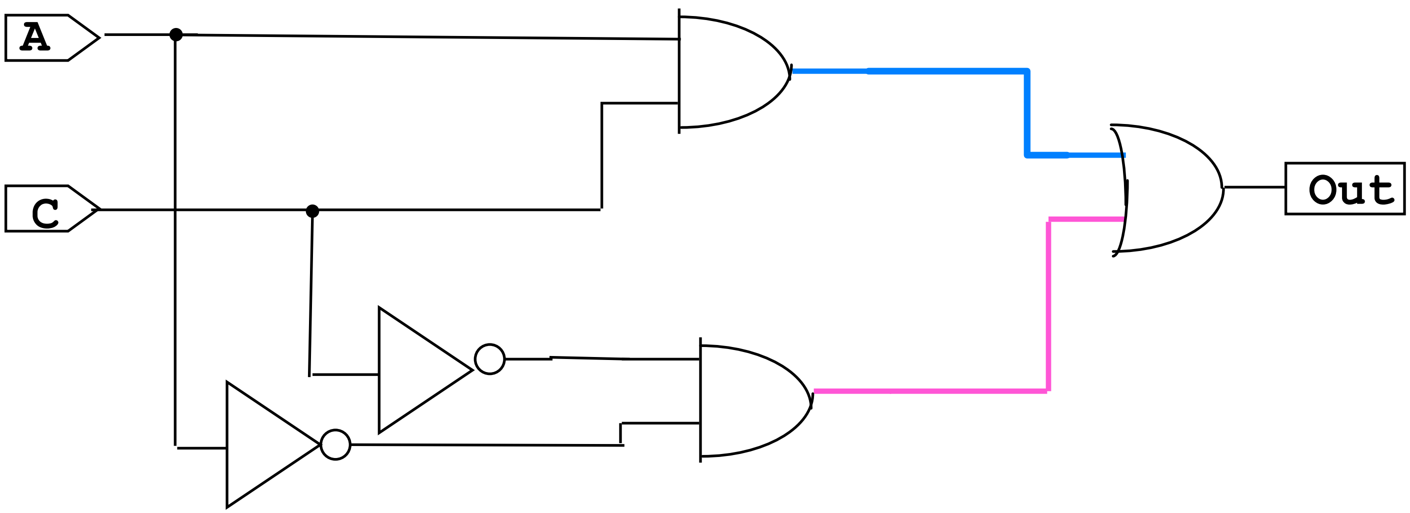
- Can put a dot for XNOR
  - XNOR is 1-bit "equals" by the way



Boolean Functions, Gates and Circuits

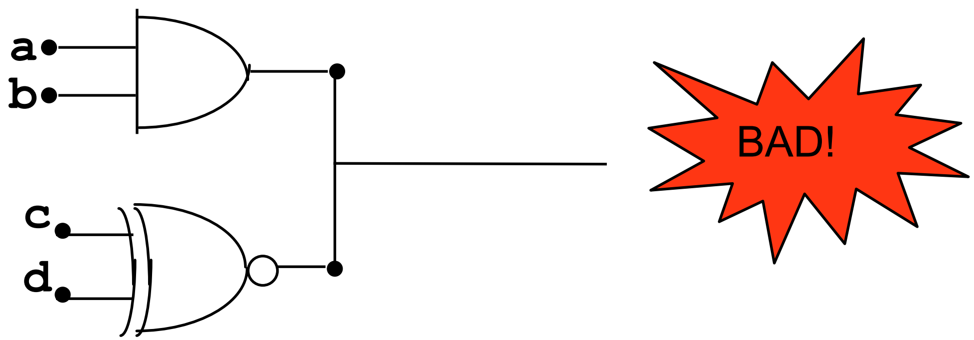
- **Circuits are made from a network of gates.**

`(!A & !C) | (A & C)`



A few more words about gates

- **Gates have inputs and outputs**
  - **If you try to hook up two outputs, bad things happen (your processor catches fire)**



  - **If you don't hook up an input, it behaves kind of randomly (also not good, but not set-your-chip-on-fire bad)**

Let’s Make a Useful Circuit

- **Pick between 2 inputs (called 2-to-1 MUX)**
  - **Short for multiplexor**

- **What might we do first?**

Let's Make a Useful Circuit

- Pick between 2 inputs (called 2-to-1 MUX)
  - Short for multiplexor

- What might we do first?
  - Make a truth table?
    - S is selector:
      - S=0, pick A
      - S=1, pick B

A	B	S	Output
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	1

Let's Make a Useful Circuit

- **Pick between 2 inputs (called 2-to-1 MUX)**
  - **Short for multiplexor**

- **What might we do first?**
  - **Make a truth table?**
    - **S is selector:**
      - **S=0, pick A**
      - **S=1, pick B**
- **Next: sum-of-products**

```
(!A & B & S) |
(A & !B & !S) |
(A & B & !S ) |
(A & B & S)
```

A	B	S	Output
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	1

Let's Make a Useful Circuit

- **Pick between 2 inputs (called 2-to-1 MUX)**
  - **Short for multiplexor**
- **What might we do first?**
  - **Make a truth table?**
    - **S is selector:**
      - **S=0, pick A**
      - **S=1, pick B**
- **Next: sum-of-products**
- **Simplify**

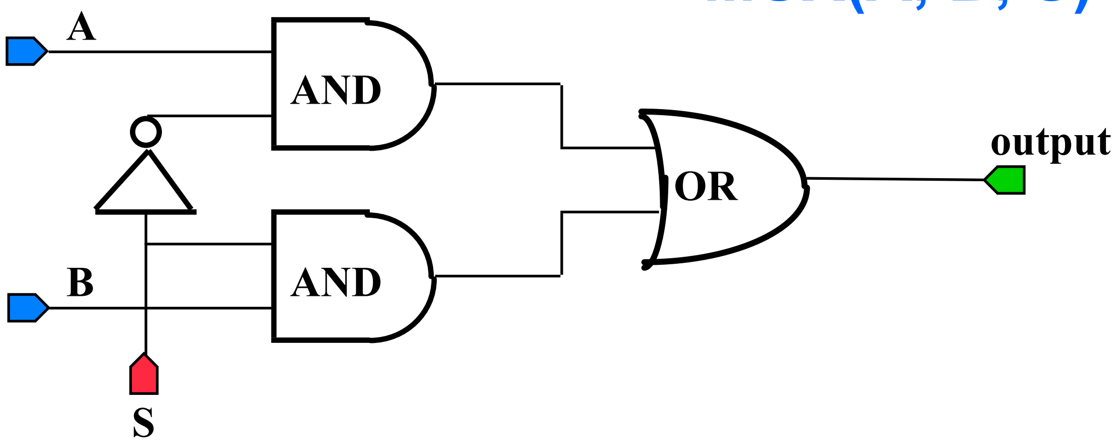
```
(A & !S) |
(B &  S)
```

A	B	S	Output
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	1

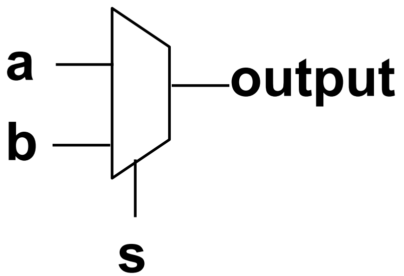
Circuit Example: 2x1 MUX

Draw it in gates:

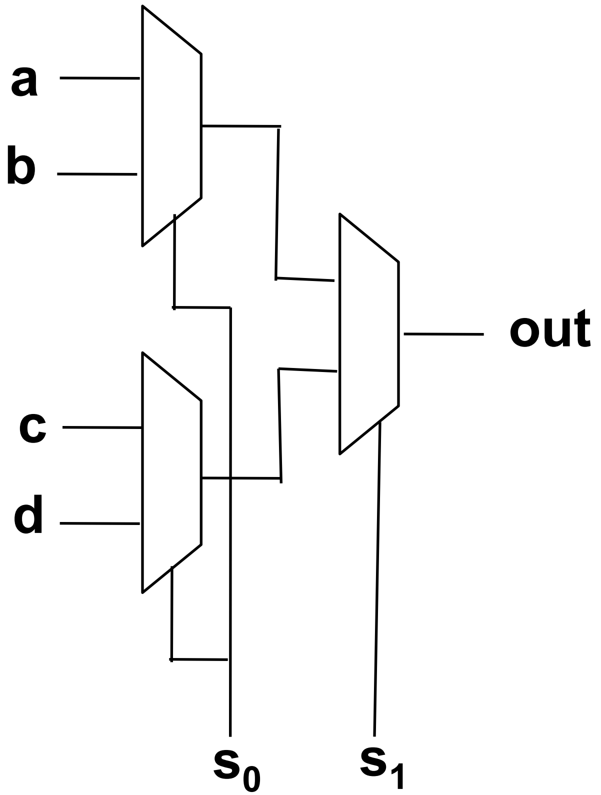
MUX(A, B, S) = (A & !S) | (B & S)

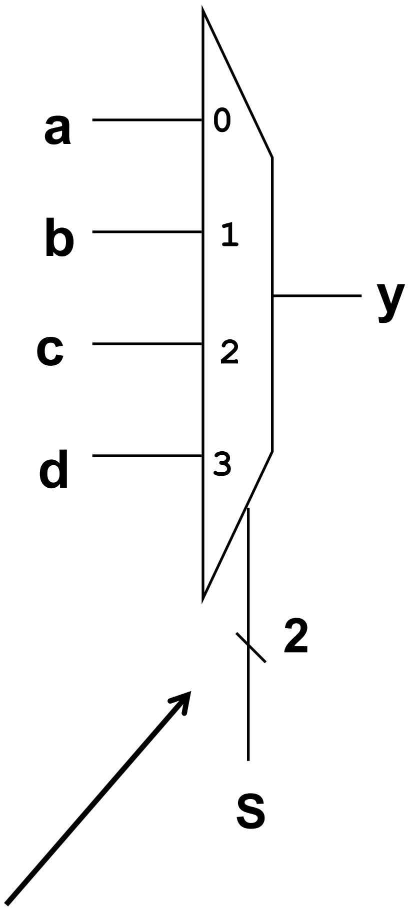


So common, we give it
its own symbol:



Example 4x1 MUX



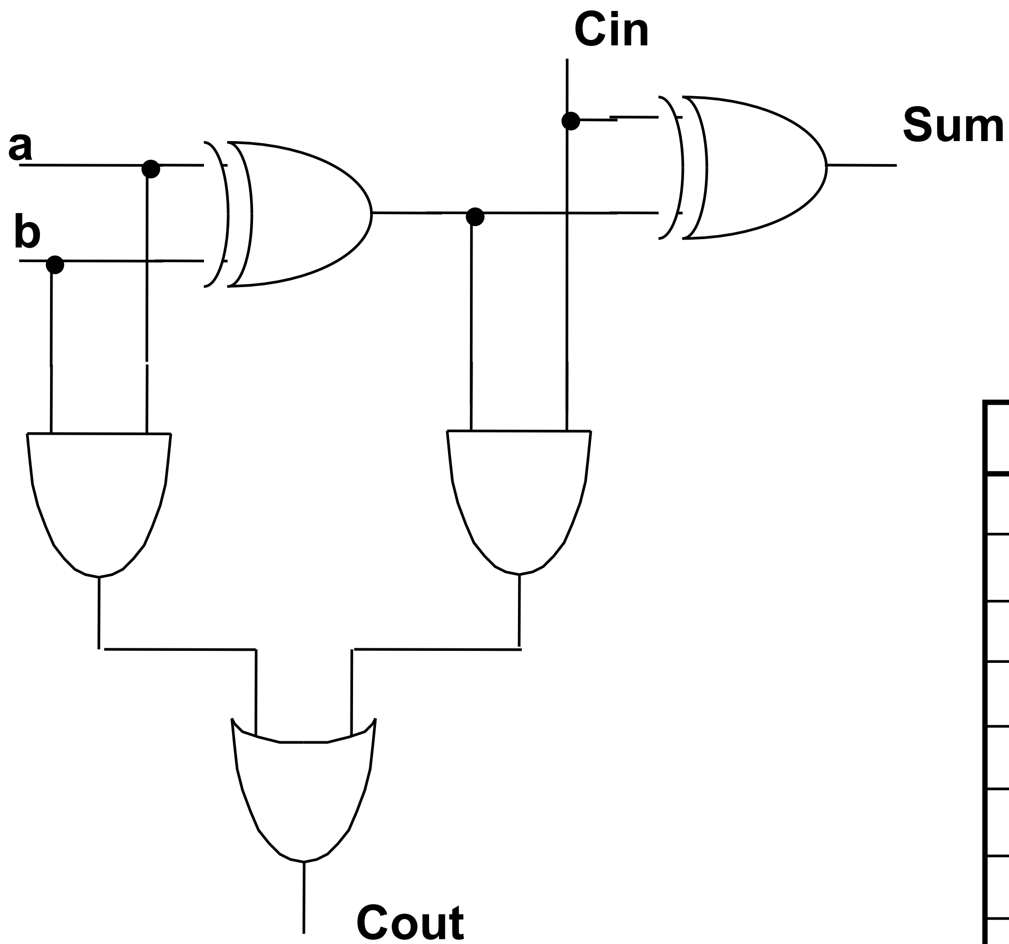


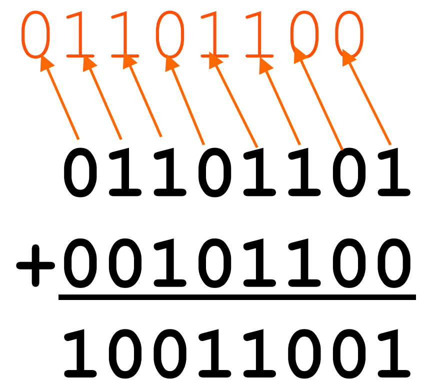
The / 2 on the wire means “2 bits”

Arithmetic and Logical Operations in ISA

- **What operations are there?**
- **How do we implement them?**
  - **Consider a 1-bit Adder**

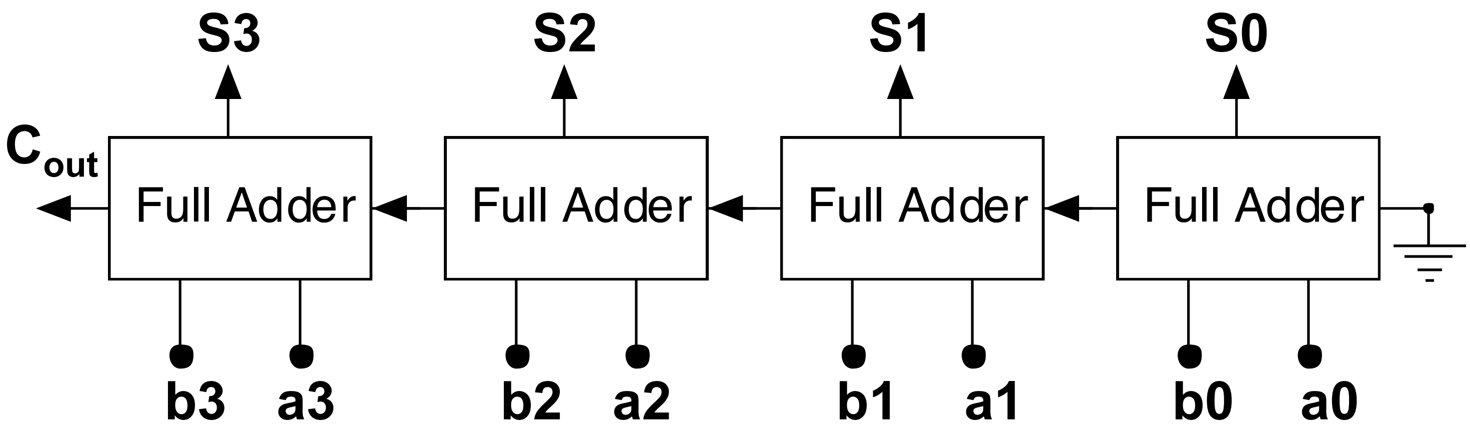
A 1-bit Full Adder





a	b	C_{in}	Sum	C_{out}
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

Example: 4-bit adder



Subtraction

- **How do we perform integer subtraction?**
- **What is the hardware?**
  - **Recall: hardware was why 2's complement was good idea**

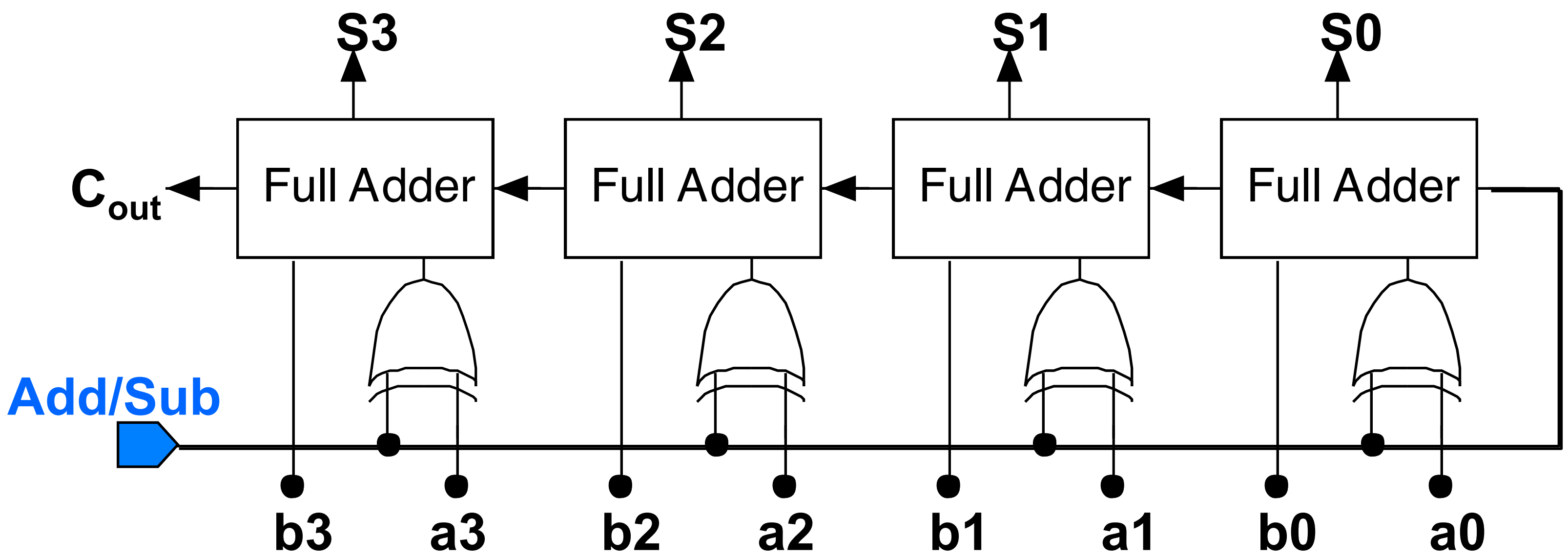
- **Remember: Subtraction is just addition**

  X – Y =

  X + (-Y) =

  X + (~Y +1)

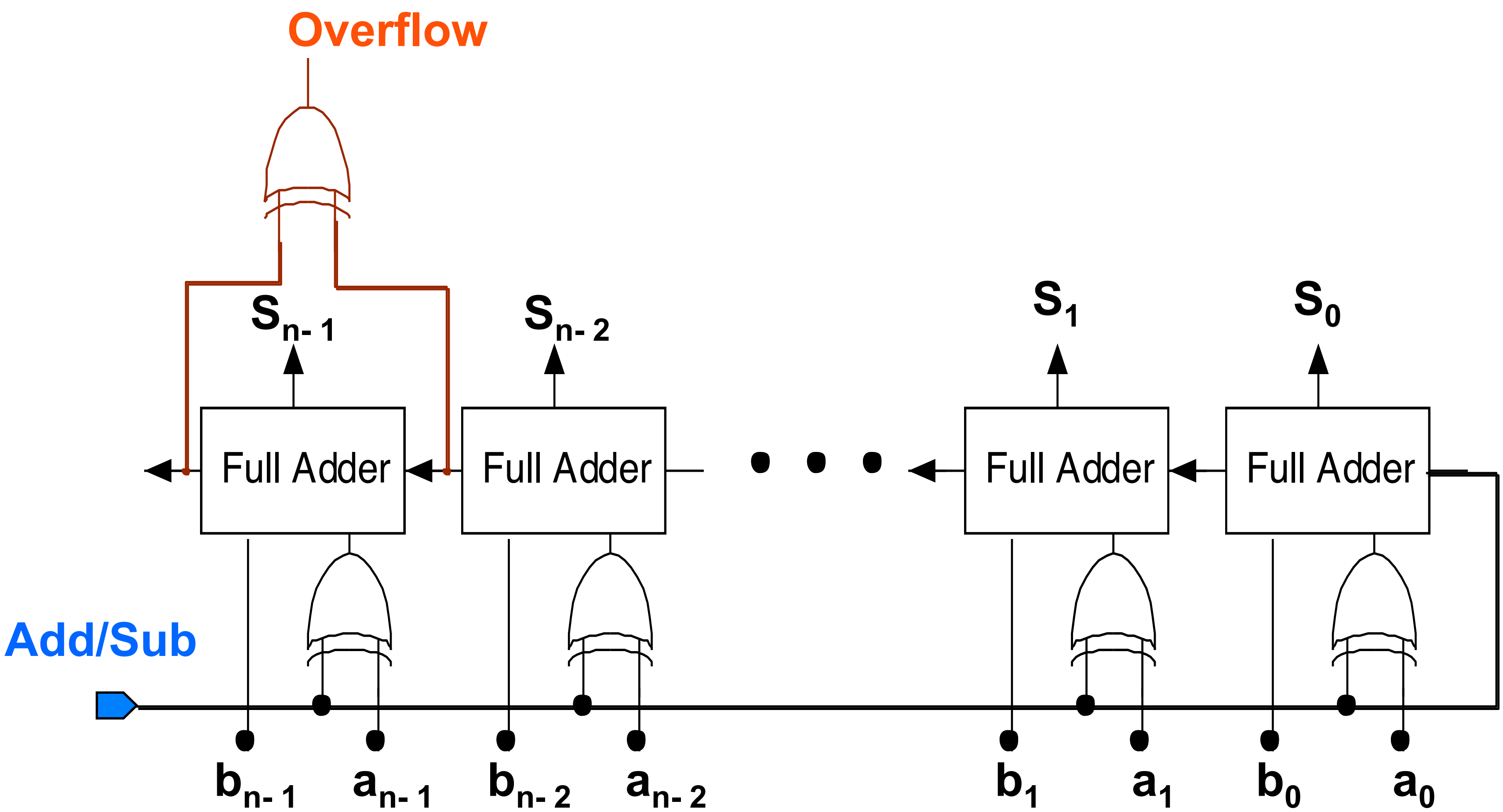
Example: Adder/Subtractor



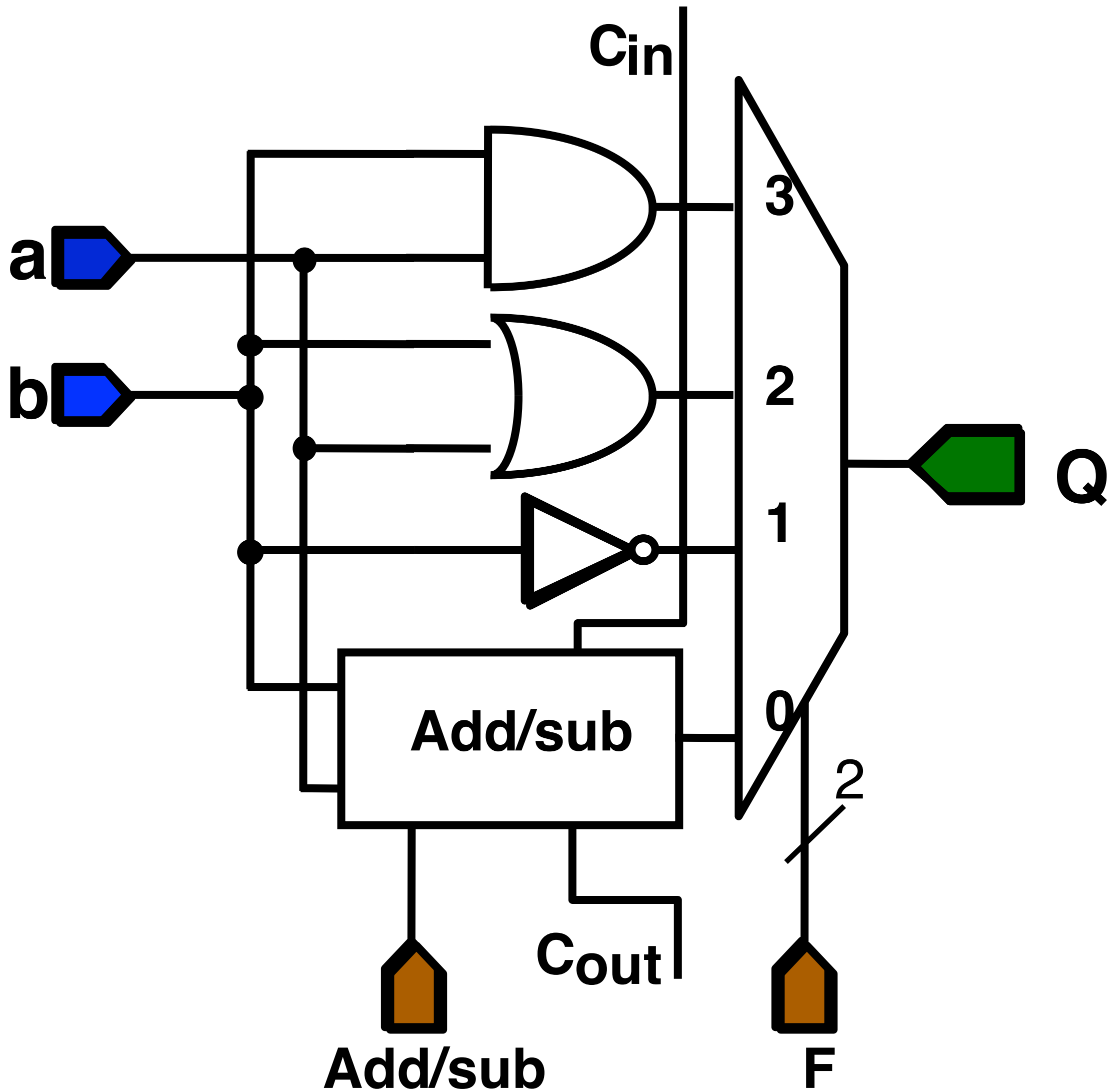
Overflow

- **We can detect unsigned overflow by looking at CO**
- **How would we detect signed overflow?**
  - **If adding positive numbers and result “is” negative**
  - **If adding negative numbers and result “is” positive**
  - **At most significant bit of adder, check if CI != CO**
  - **Can check with XOR gate**

Add/Subtract With Overflow Detection

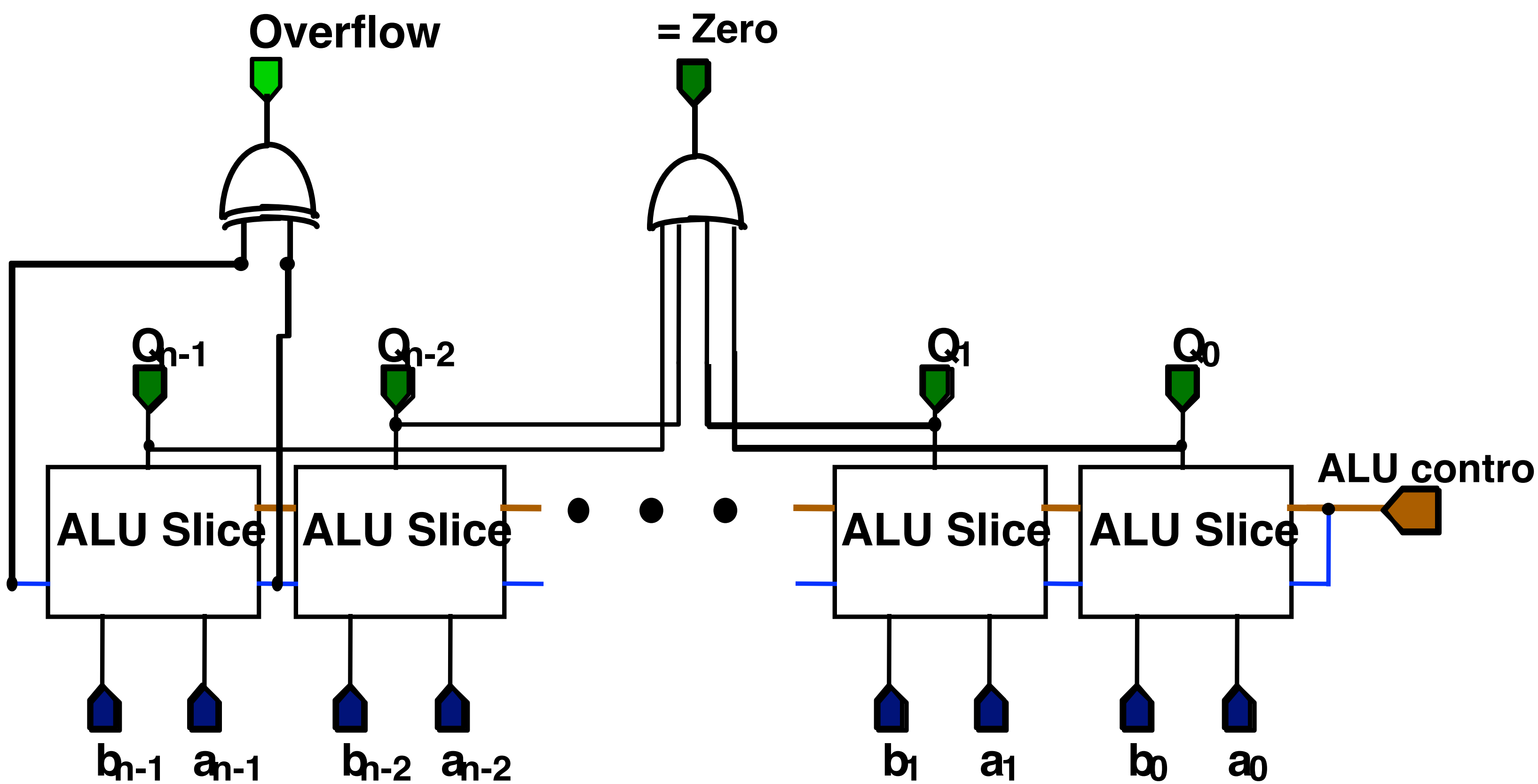


ALU Slice



A	F	Q
0	0	a + b
1	0	a - b
-	1	NOT b
-	2	a OR b
-	3	a AND b

The ALU



Overflow
= Zero
Qn-1
Qn-2
Q1
Q0
ALU control
ALU Slice
ALU Slice
ALU Slice
ALU Slice
bn-1
an-1
bn-2
an-2
b1
a1
b0
a0

Summary

- **Boolean Algebra & functions**
- **Logic gates (AND, OR, NOT, etc)**
- **Multiplexors**
- **Adder**
- **Arithmetic Logic Unit (ALU)**