The Mean Square Error in Kalman Filtering Sensor Selection is Approximately Supermodular

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Why does greedy sensor selection for Kalman filtering works when it shouldn’t?
Problem (KFSS)

*Select up to* $s$ *system outputs to estimate its internal states.*

\[
\begin{align*}
\text{minimize} & \quad \text{MSE}(S) \\
\text{subject to} & \quad |S| \leq s
\end{align*}
\]

- Why the MSE? KF
- NP-hard [Natarajan'95, Zhang’17, Ye’17]
Greedy KFSS

Definition
Select sensors/outputs one at a time by choosing the one that most improves estimation at each step.

\[
\textbf{function} \ \textsc{Greedy}(q) \\
\mathcal{G}_0 = \{\} \\
\text{for } j = 1, \ldots, q \\
\quad u = \arg\min_{v \in \mathcal{O} \setminus \mathcal{G}_{j-1}} \text{MSE} (\mathcal{G}_{j-1} \cup \{v\}) \\
\quad \mathcal{G}_j = \mathcal{G}_{j-1} \cup \{u\} \\
\text{end} \\
\text{end}
\]
Greedy KFSS

Definition
Select sensors/outputs one at a time by choosing the one that most improves estimation at each step.

function GREEDY(q)
    \( G_0 = \{ \} \)
    for \( j = 1, \ldots, q \)
        \( u = \arg\min_{v \in \mathcal{O} \setminus G_{j-1}} \text{MSE} (G_{j-1} \cup \{v\}) \)
        \( G_j = G_{j-1} \cup \{u\} \)
    end
end

▶ Low complexity
▶ Sequential
▶ Near-optimal for supermodular objectives
Problem (KFSS)

Select up to $s$ system outputs to estimate its internal states.

\[
\begin{align*}
\text{minimize} & \quad \text{MSE}(S) \\
\text{subject to} & \quad |S| \leq s
\end{align*}
\]

- Why the MSE? KF
- NP-hard [Natarajan’95, Zhang’17, Ye’17]
- Estimation MSE is not supermodular [Tzoumas’16, Olshevsky’16, Singh’17, Zhang’17]
  - Use a supermodular surrogate (e.g., log det) [Joshi’09, Shamaiah’10, Tzoumas’16]
Greedy KFSS

![Graphs showing smoothing and filtering count](graph.png)

\[
\frac{\text{MSE}(G) - \text{MSE}(S^*)}{\text{MSE}(\emptyset) - \text{MSE}(S^*)}
\]
Greedy KFSS is near-optimal

\[
\frac{\text{MSE}(G) - \text{MSE}(S^*)}{\text{MSE}(\emptyset) - \text{MSE}(S^*)}
\]
Kalman filtering sensor selection

(Approximate) supermodularity

Near-optimality of greedy KFSS

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Kalman filtering

\[ x_{k+1} = F x_k + w_k \]
\[ y_k = H x_k + v_k \]

\[ w_k \sim \mathcal{N}(0, \sigma_w^2 I) \quad v_k \sim \mathcal{N}(0, \sigma_v^2 I) \quad x_0 \sim \mathcal{N}(\bar{x}_0, \Pi_0) \]

Problem (Filtering)

Estimate \( x_k \) based on outputs up to time \( k \), i.e.,

\[ \hat{x}_k = \mathbb{E} [x_k | \{ y_j \}_{j \leq k}] \]

Solution (Kalman filter)

\[ \hat{x}_k = F \hat{x}_{k-1} + K_k [y_k - H F \hat{x}_{k-1}] \]
Kalman filtering

\[ x_{k+1} = F x_k + w_k \]
\[ y_k = H x_k + v_k \]
\[ w_k \sim \mathcal{N}(0, \sigma_w^2 I) \quad v_k \sim \mathcal{N}(0, \sigma_v^2 I) \quad x_0 \sim \mathcal{N}(\bar{x}_0, \Pi_0) \]

Problem (Filtering)

Estimate \( x_k \) based on outputs in \( S \subseteq \mathcal{O} \) up to time \( k \), i.e.,

\[ \hat{x}_k = \mathbb{E} [x_k \mid \{ (y_j)_S \}_{j \leq k}] \]

Solution (Kalman filter)

\[ \hat{x}_k(S) = F \hat{x}_{k-1}(S) + K_k \left[ (y_k)_S - H_S F \hat{x}_{k-1}(S) \right] \]
Problem (KF sensor selection)

Find $S \subseteq \mathcal{O}$, $|S| \leq s$, that minimizes the estimation MSE

$$\text{minimize} \quad \sum_{j=0}^{m-1} \theta_j \text{MSE}_{\ell+j}(S)$$

- Myopic sensor selection: $m = 1$
- Final estimation MSE: $\theta_j = 0$ for $j < m - 1$ and $\theta_{m-1} = 1$
- Exponentially weighted error: $\theta_j = \rho^{m-1-j}$, $\rho < 1$
Problem (KF sensor selection)

Find $S \subseteq \mathcal{O}$, $|S| \leq s$, that minimizes the estimation MSE

$$\minimize_{|S| \leq s} \sum_{j=0}^{m-1} \theta_j \text{Tr} [P_{\ell+j}(S)]$$

where

$$P_k(S) = \left( F P_{k-1}(S) F^T + \sigma_w^2 I + \sigma_v^{-2} \sum_{i \in S} h_i h_i^T \right)^{-1}_{P_k|k-1}$$

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(Approximate) supermodularity

Near-optimality of greedy KFSS
Definition (Supermodularity)

For $A \subseteq B \subseteq O$ and $u \in O \setminus B$

$$f(A) - f(A \cup \{u\}) \geq f(B) - f(B \cup \{u\})$$

"diminishing returns"
Greedy supermodular minimization

Theorem ([NWF’78])

Let $S^*$ be the optimal solution of the problem

$$\text{minimize} \quad f(S)$$

$|S| \leq s$

and $G$ be its greedy solution. If $f$ is (i) monotone decreasing and (ii) supermodular, then

$$\frac{f(G) - f(S^*)}{f(\emptyset) - f(S^*)} \leq e^{-1} \approx 0.37.$$
Theorem ([NWF’78])

If \( f \) is (i) monotone decreasing and (ii) supermodular, then

\[
\frac{f(G) - f(S^*)}{f(\emptyset) - f(S^*)} \leq e^{-1} \approx 0.37.
\]
\( \alpha \)-supermodularity

**Definition (Supermodularity)**

For \( \mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{O} \) and \( u \in \mathcal{O} \setminus \mathcal{B} \)

\[
f(\mathcal{A} \cup \{u\}) - f(\mathcal{A}) \leq f(\mathcal{B} \cup \{u\}) - f(\mathcal{B})
\]

If \( \alpha \geq 1 \):
\( f \) is supermodular

If \( \alpha < 1 \):
\( f \) is approximately supermodular
Definition (α-supermodularity)

For \( \mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{O} \), \( u \in \mathcal{O} \setminus \mathcal{B} \), and \( \alpha \geq 0 \)

\[
f(\mathcal{A} \cup \{u\}) - f(\mathcal{A}) \leq \alpha \left[ f(\mathcal{B} \cup \{u\}) - f(\mathcal{B}) \right]
\]

- If \( \alpha \geq 1 \): \( f \) is supermodular
- If \( \alpha < 1 \): \( f \) is approximately supermodular
**Theorem ([Chamon-Ribeiro’16])**

Let $S^*$ be the solution of the problem

$$\text{minimize} \quad f(S) \quad \text{subject to} \quad |S| \leq s$$

and $G_q$ be the $q$-th iteration of a greedy solution. If $f$ is 
(i) monotone decreasing and (ii) $\alpha$-supermodular, then

$$\frac{f(G_q) - f(S^*)}{f(\emptyset) - f(S^*)} \leq e^{-\alpha q/s}.$$
Theorem ([Chamon-Ribeiro’16])

If $f$ is (i) monotone decreasing and (ii) $\alpha$-supermodular, then

$$\frac{f(G_q) - f(S^*)}{f(\emptyset) - f(S^*)} \leq e^{-\alpha q/s}.$$  

- For $q = s$ and $\alpha = 1$, we recover the classical $e^{-1}$ result
- If $\alpha < 1$, then $e^{-1}$ is recovered for $q = \alpha^{-1}s$
What is $\alpha$ for KFSS? Combinatorial problem

$$\alpha = \min_{A \subseteq B \subseteq O} \frac{\text{MSE}(A) - \text{MSE}(A \cup \{u\})}{\text{MSE}(B) - \text{MSE}(B \cup \{u\})}$$
Theorem ([Chamon-Pappas-Ribeiro’17])

The objective of KFSS is $\alpha$-supermodular with

$$\alpha \geq \min_{\ell \leq k \leq \ell + m - 1} \frac{\lambda_{\min} [P_k(O)]}{\lambda_{\max} [P_{k|k-1}]}$$

$$P_k(O) = (P_{k|k-1} + \sigma_v^{-2} H^T H)^{-1}$$

$$P_{k|k-1} = FP_{k-1}F^T + \sigma_w^2 I$$

$\sigma_v^2 \gg \sigma_w^2$ and small $\kappa(F) \Rightarrow \alpha \approx 1$
\( n = 100 \) states and \( H = I \)
Kalman filtering sensor selection

(Approximate) supermodularity

Near-optimality of greedy KFSS
\( n = 100 \) states and \( H = I \)
Conclusion

Why does greedy KFSS work so well?

- The MSE in KFSS is not supermodular, but almost
- Greedy KFSS is efficient and has a guaranteed near-optimal performance
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More details: http://www.seas.upenn.edu/~luizf