**ABSTRACT**

Incremental combinations were first introduced as a solution to the convergence stagnation issue in parallel-independent combinations. Since then, this topology has been shown to enhance performance to the point of a combination of LMS filters outperforming the APA with lower computational complexity. In order to better understand and improve LMS structure, the present work develops mean and mean-square transient models for the incremental combination of two LMS filters, that show to be a generalization of the data reuse LMS (DR-LMS). By formulating the optimal supervisor and deriving its constraints, the previously proposed adaptive combiner is redesigned to improve the combination’s overall performance.

**INTRODUCTION**

Combination of AFs:
- Definition: set of AFs combined by a supervisor.
- Used when the accurate design of a single filter is difficult or the resulting algorithm’s complexity is too high.
- (Parallel) Independent components $\Rightarrow$ convergence stagnation

- Solutions: transfer of coefficients, feedback, incremental combination

Incremental combination:
- Fast convergence
- Supervisor design remains an open question

**DATA REUSE**

\[
\begin{align*}
(m_{n+1}, d_n(i)) &= (u_n, d(i)) \quad (\text{data buffering}) \\
(m_{n+1}, d_n(i)) &= (u_n, d(i)) \quad (\text{data sharing})
\end{align*}
\]

Parallel combination with coefficients feedback
\[
\begin{align*}
\tilde{u}_{n+1} &= \delta(i - \delta) u_{n+1} + (1 - \delta(i - \delta)) u_{n+1} \\
w_{n+1} &= \tilde{u}_{n+1} + \mu \tilde{e}_n [d(i) - u_n w_{n+1}] \\
w_n &= \sum_{i=1}^{N} m_n(i) u_n
\end{align*}
\]

**MEAN PERFORMANCE**

Derivations are carried on for $N = 2$ LMS filters assuming $u(i)$ arises from a real zero-mean i.d.d. process.

**GLOBAL COEFFICIENTS ERROR RECURRENCE**

\[
\begin{align*}
\tilde{w}_i &= w_{i-1} - \mu [p(i) - \mu'(i)] [u_i]_T \epsilon(i) \\
\tilde{w}_i &= w_i' - w_i \\
\tilde{e}_i &= \tilde{e}_i(i) \\
p(i) &= \eta_i \mu_1 \mu_2
\end{align*}
\]

A.1 (Data independence assumptions)
\[\{u_i, \sigma\} \text{ i.i.d. and independent of } \{v(i), i \geq j\}. \text{ Therefore, } \{u_i, \epsilon_i\}, \{d(i), d_j\}, \{u_i, d_j\} \text{ are independent for } i > j.\]

A.2 (Supervisor separation principle)
\[E[\tilde{e}_i(i)] = E[\tilde{e}_i(i)] E[w_i] \text{ and } E[\tilde{e}_i(i)] E[\tilde{e}_i(i)] = E[w_i] E[\tilde{e}_i(i)] \]

**MEAN-SQUARE PERFORMANCE**

\[
\begin{align*}
\text{MSD}(i) = E[\tilde{e}_i(i)]^2 = \text{Tr}(K_i) \\
\text{EMSE}(i) = E[\epsilon_i(i)]^2 = \text{E}[\tilde{u}_i(i)] \sigma_n^2 \text{MSD}(i) \\
\text{MSE}(i) = E[\epsilon_i(i)]^2 = \text{E}[\epsilon_i(i)]^2 = \text{EMSE}(i) + \sigma_n^4
\end{align*}
\]

**COMBINATION OF ADAPTIVE FILTERS**

Adaptive filters

LMS
\[
w_{n,i} = w_{n,i-1} + \mu_n u_{n,i} \epsilon_n(i)
\]

$w_{n,i} \to M \times 1$ coefficient vector of the $n^{th}$ component at iteration $i$

$\mu_n \to n^{th}$ component step size

$\epsilon_n(i) = d_n(i) - u_n w_{n,i-1} \to$ output estimation error

$u_n \to 1 \times M$ input regressor $\Rightarrow \tilde{u}_n(i)^2 = \sigma_1^2$

$d(i) \to u \tilde{w} + \epsilon(i) \to$ desired signal

$\tilde{w} \to M \times 1$ vector that models the unknown system

$\epsilon(i) \to$ i.i.d. measurement noise $\Rightarrow E[\epsilon(i)^2] = \sigma_n^2$

**SUPERVISOR ANALYSIS**

Optimal supervisor
\[\nabla_{\theta(i)} \text{ MSD}(i) = 0 \Leftrightarrow \eta_i \mu_1 = \eta_i \mu_2 \]

\[\eta_n(i) = \frac{1}{\mu_n} \frac{3(M + 2)\sigma_n^2 \text{MSD}(i) + 2M\sigma_n^2}{n = 1, 2}
\]

Supervisor constraint
\[\lim_{i \to \infty} E[\tilde{w}_i(i)] \leq \frac{2}{(M + 2)^2} < 1 \]

Adaptive supervisor

Determined supervisory: $\eta(i) = \frac{\alpha}{1 + \epsilon(i)^2}$

Filtered error supervisory:
\[\eta(i) = \alpha \cdot E[\tilde{e}_i(i)]^2
\]

\[\eta_n(i) = \frac{1}{3(M + 2)\sigma_n^2}
\]

**SIMULATIONS**

\[
\begin{align*}
\sigma_u^2 &= 1 \\
\sigma^2 &= 10^{-3}
\end{align*}
\]

Step sizes: $\mu_1 = 0.05$ and $\mu_2 = 0.005$

Small step sizes: $\mu_1 = 0.005$ and $\mu_2 = 0.003$

Ensemble averages: 200 independent realizations.

**MEAN-SQUARE MODEL VALIDATION**

Deterministic supervisor

Convex constraint: $\eta_1(i) = \eta_2(i)$ and $\eta_1(i) = 1$

No constraint: $\eta_1(i) = \eta_2(i)$ and $\eta_2(i) = 1$

**COMBINATION/SUPERVISOR COMPARISON**

Normalized convex

Convex with transfers of coefficients

Normalized convex with coefficients feedback

Convexly constrained incremental

Incremental with new adaptive supervisor