Planar Robot Kinematics

The mathematical modeling of spatial linkages is quite involved. It is useful to start with planar robots because the kinematics of planar mechanisms is generally much simpler to analyze. Also, planar examples illustrate the basic problems encountered in robot design, analysis and control without having to get too deeply involved in the mathematics. However, while the examples we will discuss will involve kinematic chains that are planar, all the definitions and ideas presented in this section are general and extend to the most general spatial mechanisms.

Planar 3R manipulator

We will start with the example of the planar manipulator with three revolute joints. The manipulator is called a planar 3*R* manipulator. While there may not be any three degree of freedom (d.o.f.) industrial robots with this geometry, the planar 3*R* geometry can be found in many robot manipulators. For example, the shoulder swivel, elbow extension, and pitch of the Cincinnati Milacron T3 robot can be described as a planar 3*R* chain. Similarly, in a four d.o.f. SCARA manipulator, if we ignore the prismatic joint for lowering or raising the gripper, the other three joints form a planar 3*R* chain. Thus, it is instructive to study the planar 3*R* manipulator as an example.

In order to specify the geometry of the planar 3*R* robot, we require three parameters, l_1, l_2, and l_3. These are the three link lengths. In the figure, the three joint angles are labeled θ_1, θ_2, and θ_3. These are obviously variable. The precise definitions for the link lengths and joint angles are as follows. For each pair of adjacent axes we can define a common normal or the perpendicular between the axes.

- The *ith* common normal is the perpendicular between the axes for joint i and joint $i+1$.
- The *ith* link length is the length of the *ith* common normal, or the distance between the axes for joint i and joint $i+1$.
- The *ith* joint angle is the angle between the $(i-1)th$ common normal and *ith* common normal measured counter clockwise going from the $(i-1)th$ common normal to the *ith* common normal.

Note that there is some ambiguity as far as the link length of the most distal link and the joint angle of the most proximal link are concerned. We define the link length of the most distal link from the most distal joint axis to a reference point or a tool point on the end effector[1]. Generally, this is the center of the gripper or the end point of the tool. Since there is no zeroth common normal, we measure the first joint angle from a convenient reference line. Here, we have chosen this to be the x axis of a conveniently defined fixed coordinate system.

Another set of variables that is useful to define is the set of coordinates for the end effector. These coordinates define the position and orientation of the end effector. With a convenient choice of a reference point on the end effector, we can describe the position of the end effector using the coordinates of the reference point (x, y) and the orientation using the angle ϕ. The three end effector coordinates (x, y, ϕ) completely specify the position and orientation of the end effector.

[1]The reference point is often called the tool center point (TCP).

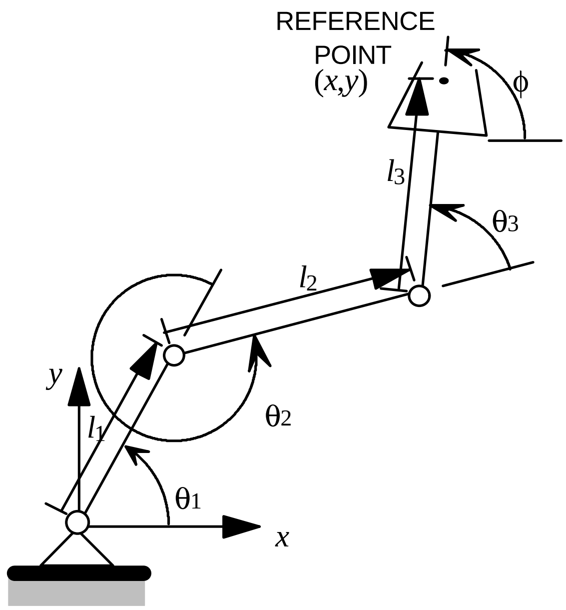

Figure 1 The joint variables and link lengths for a 3*R* planar manipulator

Direct Kinematics

From basic trigonometry, the position and orientation of the end effector can be written in terms of the joint coordinates in the following way:

$$
\begin{aligned}
x &= l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) + l_3 \cos(\theta_1 + \theta_2 + \theta_3) \\
y &= l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2) + l_3 \sin(\theta_1 + \theta_2 + \theta_3) \\
\phi &= (\theta_1 + \theta_2 + \theta_3)
\end{aligned}
\tag{1}
$$

Note that all the angles have been measured counter clockwise and the link lengths are assumed to be positive going from one joint axis to the immediately distal joint axis.

Equation (1) is a set of three nonlinear equations[2] that describe the relationship between end effector coordinates and joint coordinates. Notice that we have explicit equations for the end effector coordinates in terms of joint coordinates. However, to find the joint coordinates for a given set of end effector coordinates (x, y, ϕ), one needs to solve the nonlinear equations for θ_1, θ_2, and θ_3.

As seen earlier, there are two types of coordinates that are useful for describing the configuration of the system. If we focus our attention on the task and the end effector, we would prefer to use Cartesian coordinates or end effector coordinates. The set of all such coordinates is generally referred to as the *Cartesian space* or *end effector space*[3]**.** The other set of coordinates is the so called joint coordinates that is useful for describing the configuration of the mechanical lnkage. The set of all such coordinates is generally called the *joint space.*

[2]The third equation is linear but collectively, the equations are nonlinear.

[3]Since each member of this set is an *n*-tuple, we can think of it as a vector and the space is really a vector space. But we shall not need this abstraction here.

In robotics, it is often necessary to be able to “map” joint coordinates to end effector coordinates. This map or the procedure used to obtain end effector coordinates from joint coordinates is called direct kinematics.

For example, for the 3-*R* manipulator, the procedure reduces to simply substituting the values for the joint angles in the equations

$$x = l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) + l_3 \cos(\theta_1 + \theta_2 + \theta_3)$$
$$y = l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2) + l_3 \sin(\theta_1 + \theta_2 + \theta_3)$$
$$\phi = (\theta_1 + \theta_2 + \theta_3)$$

and determining the Cartesian coordinates, x, y, and ϕ. For the other examples of open chains discussed so far (*R*-*P*, *P*-*P*) the process is even simpler (since the equations are simpler). In fact, for all serial chains (spatial chains included), the direct kinematics procedure is fairly straight forward.

On the other hand, the same procedure becomes more complicated if the mechanism contains one or more closed loops. In addition, the direct kinematics may yield more than one solution or no solution in such cases. For example, in the planar parallel manipulator in Figure 3, the joint positions or coordinates are the lengths of the three telescoping links (q_1, q_2, q_3) and the end effector coordinates (x, y, ϕ) are the position and orientation of the floating triangle. It can be shown that depending on the value of (q_1, q_2, q_3), the number of (real) solutions for (x, y, ϕ) can be anywhere from zero to six. For the Stewart Platform in Figure 4, this number has been shown to be anywhere from zero to forty.

Inverse kinematics

The analysis or procedure that is used to compute the joint coordinates for a given set of end effector coordinates is called inverse kinematics. Basically, this procedure involves solving a set of equations. However the equations are, in general, nonlinear and complex, and therefore, the inverse kinematics analysis can become quite involved. Also, as mentioned earlier, even if it is possible to solve the nonlinear equations, uniqueness is not guaranteed. There may not (and in general, will not) be a unique set of joint coordinates for the given end effector coordinates.

The inverse kinematics analysis for a planar 3-*R* manipulator appears to be complicated but we can derive analytical solutions. Recall that the direct kinematics equations (1) are:

$$x = l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) + l_3 \cos(\theta_1 + \theta_2 + \theta_3) \quad (2)$$

$$y = l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2) + l_3 \sin(\theta_1 + \theta_2 + \theta_3) \quad (3)$$

$$\phi = (\theta_1 + \theta_2 + \theta_3) \quad (4)$$

We assume that we are given the Cartesian coordinates, x, y, and ϕ and we want to find analytical expressions for the joint angles θ_1, θ_2, and θ_3 in terms of the Cartesian coordinates.

Substituting (4) into (2) and (3) we can eliminate θ_3 so that we have two equations in θ_1 and θ_2:

$$x - l_3 \cos(\phi) = l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) \tag{5}$$

$$y - l_3 \sin(\phi) = l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2) \tag{6}$$

where the unknowns have been grouped on the right hand side; the left hand side depends only on the end effector or Cartesian coordinates and are therefore known.

Rename the left hand sides, $x' = x - l_3 \cos\phi$, $y' = y - l_3 \sin\phi$, for convenience. We regroup terms in (5) and (6), square both sides in each equation and add them:

$$\left(x' - l_1 \cos\theta_1\right)^2 = \left[l_2 \cos(\theta_1 + \theta_2)\right]^2$$

$$+$$

$$\left(y' - l_1 \sin\theta_1\right)^2 = \left[l_2 \sin(\theta_1 + \theta_2)\right]^2$$

After rearranging the terms we get a single nonlinear equation in θ_1:

$$(-2l_1 x')\cos\theta_1 + (-2l_1 y')\sin\theta_1 + \left(x'^2 + y'^2 + l_1^2 - l_2^2\right) = 0 \tag{7}$$

Notice that we started with three nonlinear equations in three unknowns in (2-4). We reduced the problem to solving two nonlinear equations in two unknowns (5-6). And now we have simplified it further to solving a single nonlinear equation in one unknown (7).

Equation (7) is of the type

$$P\cos\alpha + Q\sin\alpha + R = 0 \tag{8}$$

Equations of this type can be solved using a simple substitution as shown in Appendix 1. There are two solutions for θ_1 given by:

$$\theta_1 = \gamma + \sigma \cos^{-1}\left(\frac{-\left(x'^2 + y'^2 + l_1^2 - l_2^2\right)}{2l_1\sqrt{x'^2 + y'^2}}\right) \tag{9}$$

where,

$$\gamma = \operatorname{atan2}\left(\frac{-y'}{\sqrt{x'^2 + y'^2}}, \frac{-x'}{\sqrt{x'^2 + y'^2}}\right),$$

and $$\sigma = \pm 1.$$

Note that there are two solutions for θ_1, one corresponding to σ=+1, the other corresponding to σ=-1. Substituting any one of these solutions back into Equations (5) and (6) gives us:

$$\cos(\theta_1 + \theta_2) = \frac{x' - l_1\cos\theta_1}{l_2}$$

$$\sin(\theta_1 + \theta_2) = \frac{y' - l_1\sin\theta_1}{l_2}$$

This allows us to solve for θ_2 using the atan2 function.

$$\theta_2 = \operatorname{atan2}\left(\frac{y' - l_1 \sin\theta_1}{l_2}, \frac{x' - l_1 \cos\theta_1}{l_2}\right) - \theta_1 \tag{10}$$

Thus, for each solution for θ_1, there is one (unique) solution for θ_2. Finally, θ_3 can be easily determined from (c):

$$\theta_3 = \phi - (\theta_1 + \theta_2) \tag{11}$$

Equations (9-11) are the inverse kinematics solution for the 3-*R* manipulator. For a given end effector position and orientation, there are two different ways of reaching it, each corresponding to a different value of σ. These different configurations are shown in Figure 2.

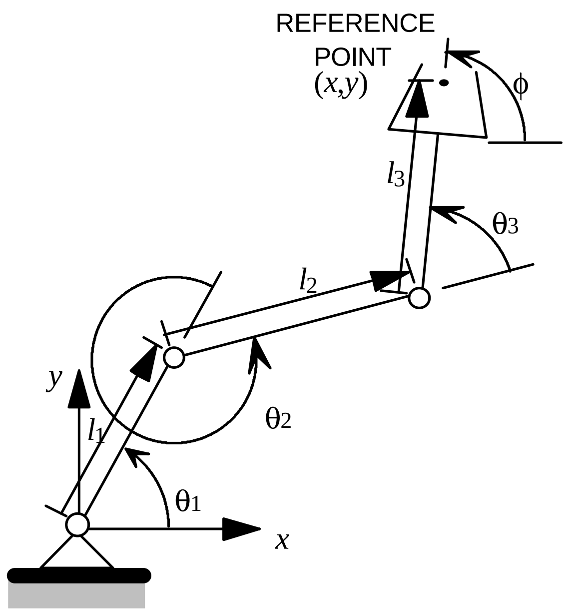

Figure 2 The two inverse kinematics solutions for the 3*R* manipulator: "elbow-up" configuration (σ=+1) and the "elbow-down" configuration (σ= -1)

Commanding a robot to move the end effector to a certain position and orientation is ambiguous because there are two configurations that the robot must choose from. From a practical point of view, if the joint limits are such that one configuration cannot be reached this ambiguity is automatically resolved[4].

Velocity analysis

When controlling a robot to go from one position to another, it is not just enough to determine the joint and end effector coordinates of the target position. It may be necessary to continuously control the trajectory or the path taken by the robot as it moves toward the target position. This is essential to avoid obstacles in the workspace. More importantly, there are tasks where the trajectory of the end effector is critical. For example, when welding, it is necessary to maintain the

[4]This is true of the human arm. If you consider planar movements, because the human elbow cannot be hyper extended, there is a unique solution for the inverse kinematics. Thus the central nervous system does not have to worry about which configuration to adopt for a reaching task.

tool at a desired orientation and a fixed distance away from the workpiece while moving uniformly[5] along a desired path. Thus one needs to control the velocity of the end effector or the tool during the motion. Since the control action occurs at the joints, it is only possible to control the joint velocities. Therefore, there is a need to be able to take the desired end effector velocities and calculate from them the joint velocities. All this requires a more detailed kinematic analysis, one that addresses velocities or the rate of change of coordinates in contrast to the previous section where we only looked at positions or coordinates.

Consider the $3R$ manipulator as an example. By differentiating Equation (1) with respect to time, it is possible to obtain equations that relate the the different velocities.

$$\dot{x} = -l_1\dot{\theta}_1 s_1 - l_2\left(\dot{\theta}_1 + \dot{\theta}_2\right)s_{12} - l_3\left(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3\right)s_{123}$$
$$\dot{y} = l_1\dot{\theta}_1 c_1 + l_2\left(\dot{\theta}_1 + \dot{\theta}_2\right)c_{12} + l_3\left(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3\right)c_{123}$$
$$\dot{\phi} = \left(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3\right)$$

where we have used the short hand notation:

$$s_1 = \sin\theta_1, \quad s_{12} = \sin\left(\theta_1 + \theta_2\right), \quad s_{123} = \sin\left(\theta_1 + \theta_2 + \theta_3\right)$$
$$c_1 = \cos\theta_1, \quad c_{12} = \cos\left(\theta_1 + \theta_2\right), \quad c_{123} = \cos\left(\theta_1 + \theta_2 + \theta_3\right)$$

$\dot{\theta}_i$ denotes the joint speed for the ith joint or the time derivative of the ith joint angles, and $\dot{x}$, $\dot{y}$, and $\dot{\phi}$ are the time derivatives of the end effector coordinates. Rearranging the terms, we can write this equation in matrix form:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} -\left(l_1 s_1 + l_2 s_{12} + l_3 s_{123}\right) & -\left(l_2 s_{12} + l_3 s_{123}\right) & -l_3 s_{123} \\ \left(l_1 c_1 + l_2 c_{12} + l_3 c_{123}\right) & \left(l_2 c_{12} + l_3 c_{123}\right) & l_3 c_{123} \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} \tag{12}$$

The 3x3 matrix is called the Jacobian matrix[6] and we will denote it by the symbol **J**. If you look at the elements of the matrix they express the rate of change of the end effector coordinates with respect to the joint coordinates:

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial \theta_1} & \frac{\partial x}{\partial \theta_2} & \frac{\partial x}{\partial \theta_3} \\ \frac{\partial y}{\partial \theta_1} & \frac{\partial y}{\partial \theta_2} & \frac{\partial y}{\partial \theta_3} \\ \frac{\partial \phi}{\partial \theta_1} & \frac{\partial \phi}{\partial \theta_2} & \frac{\partial \phi}{\partial \theta_3} \end{bmatrix}$$

Given the rate at which the joints are changing, or the vector of joint velocities,

$$\dot{\mathbf{q}} = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix},$$

[5]In some cases, a weaving motion is required and the trajectory of the tool is more complicated.
[6]The name Jacobian comes from the terminology used in multi-dimensional calculus.

using Equation (12), we can obtain expressions for the end effector velocities,

$$\dot{\mathbf{p}} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\phi} \end{bmatrix}.$$

If the Jacobian matrix is non singular (its determinant is non zero and the matrix is invertible), then we can get the following expression for the joint velocities in terms of the end effector velocities:

$$\dot{\mathbf{q}} = \mathbf{J}^{-1}\dot{\mathbf{p}} \tag{12}$$

Thus if the task (for example, welding) is specified in terms of a desired end effector velocity, Equation (13) can be used to compute the desired joint velocity provided the Jacobian is non singular.

Naturally we want to determine the conditions under which the Jacobian becomes singular. This can be done by computing the determinant of **J** and setting it to zero. Fortunately, the expression for the determinant of the Jacobian, in this example, can be simplified using trigonometric identities to:

$$|\mathbf{J}| = l_1 l_2 \sin\theta_2 \tag{14}$$

This means that the Jacobian is singular only when θ_2 is either 0 or 180 degrees. Physically, this corresponds to the elbow being completely extended or completely flexed. Thus, as long we avoid going through this configuration, the robot will be able to follow any desired end effector velocity.

Appendix: Solution of the nonlinear equation (8)

$$P\cos\alpha + Q\sin\alpha + R = 0$$

Define γ so that

$$\cos\gamma = \frac{P}{\sqrt{P^2+Q^2}} \text{ and } \quad \sin\gamma = \frac{Q}{\sqrt{P^2+Q^2}}$$

Note that this is always possible. γ can be determined by using the atan2 function:

$$\gamma = \text{a}\tan 2\left(\frac{Q}{\sqrt{P^2+Q^2}}, \frac{P}{\sqrt{P^2+Q^2}}\right)$$

Now (8) can be rewritten as:

$$\cos(\alpha - \gamma) = \frac{-R}{\sqrt{P^2+Q^2}}$$

This gives us two solutions for α in terms of the known angle γ:

$$\alpha = \gamma + \sigma\cos^{-1}\left(\frac{-R}{\sqrt{P^2+Q^2}}\right), \qquad \sigma = \pm 1$$