Robot Geometry and Kinematics

Outline

Industrial (conventional) robot arms
- Basic definitions for understanding 3-D geometry, kinematics
- Examples
- Classification by geometry
- Relationship between geometry and functionality
- Forward and inverse kinematics
- Implications for control

Other types of robots
- Parallel manipulators
- End effectors

See Notes: Slides available from website
Definitions

- **Degrees of freedom of a system**
  The number of independent variables (or coordinates) required to completely specify the configuration of the system.
  - Point on a plane
  - Point in 3-D space
  - Line on a plane
  - 2 planar links connected by a pin joint
  - Human shoulder
  - Car

- **Kinematic chain**
  A system of rigid bodies connected together by joints. A chain is called closed if it forms a closed loop. A chain that is not closed is called an open chain.

- **Serial chain**
  If each link of an open chain except the first and the last link is connected to two other links it is called a serial chain.

Definitions (continued)

- **Joints**
  Joints are connections between links
  - Revolute, rotary, or pin joint (R)
  - Prismatic, sliding, or telescoping joint (P)
  - Helical or screw joint (H)
  - Spherical or ball joint (S)

- **Planar kinematic chain**
  All the links are constrained to move in or parallel to the same plane.
  A planar chain can only allow prismatic and revolute joints. In fact, the axes of the revolute joints must be perpendicular to the plane of the chain while the axes of the prismatic joints must be parallel to or lie in the plane of the chain.

- **Connectivity of a joint**
  The number of degrees of freedom of a rigid body connected to a fixed rigid body through the joint.
The Planar 3-R manipulator

- Planar kinematic chain
- All joints are revolute with connectivity = 1
- What is the number of degrees of freedom?

Connectivity of a joint
The number of degrees of freedom of a rigid body connected to a fixed rigid body through the joint
- spherical joint (3)
- revolute joint (1)
- prismatic joint (1)
- helical joint (1)

Mobility of a chain
Number of degrees of freedom of the chain
- Serial chain
  \[ M = \sum_{i=1}^{n} f_i \]
- Examples
  - 3-R chain
    - \( M=3 \)
Connectivity, Mobility, and Degrees of Freedom

- Mobility of a chain
  - Number of degrees of freedom of the chain
    - General expression
      \[ M = 6(n - g) + \sum_{i=1}^{g} f_i \]
      
      - \( n \) number of moving links
      - \( g \) number of joints between the links
      - \( f_i \) connectivity of joint \( i \)

- Special case of planar geometry
  \[ M = 3(n - g) + \sum_{i=1}^{g} f_i \]

Examples: Mobility (Degrees of Freedom)

The Adept 1850 Palletizer

\[ M = 4 \]

The G365 Gantry robot manipulator

(CRS Robotics)

\[ M = 3 \]
Examples: Mobility (Degrees of Freedom)

Planar serial chain
- number of moving links, \( n = 3 \)
- number of joints, \( g = 3 \)
- connectivity, \( f_i = 1 \)

Planar parallel manipulator
- number of moving links, \( n = 7 \)
- number of joints, \( g = 9 \)
- connectivity, \( f_i = 1 \)

\[
M = 3(n - g) + \sum_{i=1}^{3} f_i
\]

Ingersoll Rand machine tool (Stewart Platform)
- number of moving links, \( n = 19 \)
- number of joints, \( g = 24 \)
- connectivity, \( f_i = 1 \)

\[
M = 6(n - g) + \sum_{i=1}^{6} f_i
\]

\[
M = 6(19 - 24) + 36 = 6
\]
Kinematic modeling

- Link
- Actuated joint
- End effector (EE)
  - Reference point on EE
- Joint coordinates $\theta_1, \theta_2, \theta_3$
- End effector coordinates $x, y, \phi$
- Link lengths ($l_i$)

RP manipulator
Kinematic transformations

Direct kinematics
- Joint coordinates to end effector coordinates
  Why is it useful?
  - Sensors are located at the joints. DK algorithm is used to figure out where the robot is in 3-D space.
  - Robot “thinks” in joint coordinates. Programmer/engineer thinks in “world coordinates” or end effector coordinates.

Inverse kinematics
- End effector coordinates
  Why is it useful?
  - Given a desired position and orientation of the EE, we want to be able to get the robot to move to the desired goal. IK algorithm used to obtain the joint coordinates.
  - Essential for control.
Direct kinematics

Transform joint coordinates to end effector coordinates

\[ x = l_1 \cos \theta_1 + l_2 \cos(\theta_1 + \theta_2) + l_3 \cos(\theta_1 + \theta_2 + \theta_3) \]
\[ y = l_1 \sin \theta_1 + l_2 \sin(\theta_1 + \theta_2) + l_3 \sin(\theta_1 + \theta_2 + \theta_3) \]
\[ \phi = \theta_1 + \theta_2 + \theta_3 \]

RP manipulator
Direct kinematics
Transform joint coordinates to end effector coordinates

\[ 
\begin{align*}
    x &= l_2 \cos \theta_2 + l_3 \cos(\theta_2 + \theta_3) + l_4 \cos(\theta_2 + \theta_3 + \theta_4) \\
    y &= l_2 \sin \theta_2 + l_3 \sin(\theta_2 + \theta_3) + l_4 \sin(\theta_2 + \theta_3 + \theta_4) \\
    z &= d_1 \\
    \phi &= \theta_2 + \theta_3 + \theta_4 
\end{align*} \]

Adept palletizer: PRRR manipulator

Kinematics of “truly 3-D” manipulators is difficult
- PUMA 560
- Cincinnati Milacron T-3

Why?
Inverse kinematics

Transform end effector coordinates to joint coordinates

Given $x, y, \phi$, solve for $\theta_1, \theta_2, \theta_3$

\[
\begin{align*}
  x &= l_1 \cos \theta_1 + l_2 \cos(\theta_1 + \theta_2) + l_3 \cos(\theta_1 + \theta_2 + \theta_3) \\
  y &= l_1 \sin \theta_1 + l_2 \sin(\theta_1 + \theta_2) + l_3 \sin(\theta_1 + \theta_2 + \theta_3) \\
  \phi &= (\theta_1 + \theta_2 + \theta_3)
\end{align*}
\]

Given $x, y, \phi$, solve for $\theta_1, \theta_2, \theta_3$
Inverse kinematics

Given \( x, y \), solve for \( \theta_1, \theta_2 \)

- Solution no. 1
  
  \[
  d_2 = \pm \sqrt{x^2 + y^2} \\
  \theta_1 = \tan^{-1}\left(\frac{y}{x}\right)
  \]

- Solution no. 2
  
  \[
  d_2 = -\sqrt{x^2 + y^2} \\
  \theta_1 = \tan^{-1}\left(\frac{y}{x}\right)
  \]
Inverse kinematics is more difficult!

Solving nonlinear equations
Equations can be very complicated
Equations involve trigonometric functions
  - Trig functions are transcendental
  - Inverse trig functions have multiple solutions
  \[ \tan(x) = \tan(x + k\pi), k = \ldots -2, -1, 0, 1, 2, \ldots \]

We use the \( \text{atan2} \) function to solve inverse kinematics problems.

Example
\[ s = \sin x, \quad c = \cos x \]
\[ s = \frac{1}{2}, \quad c = \frac{\sqrt{3}}{2} \]
Find \( x \)

Define \( \text{atan2} \) function with two arguments

\[
\begin{align*}
\sin^{-1}\left(\frac{1}{2}\right) &= 30^\circ, 150^\circ \\
\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) &= 30^\circ, -30^\circ
\end{align*}
\]
\[ \text{atan2}\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = 30^\circ \]

Inverse kinematics for the 3R manipulator

\[
x = l_1 \cos \theta_1 + l_2 \cos(\theta_1 + \theta_2) + l_3 \cos(\theta_1 + \theta_2 + \theta_3) \\
y = l_1 \sin \theta_1 + l_2 \sin(\theta_1 + \theta_2) + l_3 \sin(\theta_1 + \theta_2 + \theta_3) \\
\phi = (\theta_1 + \theta_2 + \theta_3)
\]

Given \( x, y, \phi \), solve for \( \theta_1, \theta_2, \theta_3 \)
Inverse kinematics: Step 1

Given \( x, y, \phi \), solve for \( \theta_1, \theta_2, \theta_3 \)

\[
x = l_1 \cos \theta_1 + l_2 \cos(\theta_1 + \theta_2) + l_3 \cos(\theta_1 + \theta_2 + \theta_3) \\
y = l_1 \sin \theta_1 + l_2 \sin(\theta_1 + \theta_2) + l_3 \sin(\theta_1 + \theta_2 + \theta_3) \\
\phi = (\theta_1 + \theta_2 + \theta_3)
\]

Two equations in two unknowns - \( \theta_1, \theta_2 \)

Inverse kinematics: Step 2

Rename the left hand sides,

\[
x' = x - l_3 \cos \phi, \quad y' = y - l_3 \sin \phi.
\]

Given \( x', y' \), solve for \( \theta_1, \theta_2 \).

\[
(-2l_1x') \cos \theta_1 + (-2l_1y') \sin \theta_1 + (x'^2 + y'^2 + l_i^2 - l_3^2) = 0
\]

One equation in one unknown - \( \theta_1 \)
Inverse kinematics: Step 3

Solve for \( \theta_1 \)

\[
(-2l_1x)\cos \theta_1 + (-2l_1y)\sin \theta_1 + \left(x'^2 + y'^2 + l_1^2 - l_2^2\right) = 0
\]

One equation in one unknown - \( \theta_1 \)

How do we solve equations of the type

\[
P \cos \alpha + Q \sin \alpha + R = 0
\]

Solutions to the equation

\[
P \cos \alpha + Q \sin \alpha + R = 0
\]

Define \( \gamma \) so that

\[
\gamma = \tan \left( \frac{Q}{\sqrt{P^2 + Q^2}} \right)
\]

so that we can rewrite the equation in the form:

\[
\cos \gamma \cos \alpha + \sin \gamma \sin \alpha + \frac{R}{\sqrt{P^2 + Q^2}} = 0
\]

\[
\cos(\alpha - \gamma) = \frac{-R}{\sqrt{P^2 + Q^2}} \rightarrow \alpha = \gamma + \sigma \cos^{-1}\left(\frac{-R}{\sqrt{P^2 + Q^2}}\right) \quad \sigma = \pm 1
\]

two solutions
Inverse kinematics: Step 3

\[ (-2l_1x')\cos \theta_1 + (-2l_1y')\sin \theta_1 + (x'^2 + y'^2 + l_1^2 - l_2^2) = 0 \]

\[ P \quad Q \quad R \]

\[ \gamma = \tan^{-2} \left( \frac{Q}{\sqrt{P^2 + Q^2}}, \frac{P}{\sqrt{P^2 + Q^2}} \right) \]

\[ \theta_1 = \gamma + \sigma \cos^{-1} \left( \frac{- (x'^2 + y'^2 + l_1^2 - l_2^2)}{2l_1 \sqrt{x'^2 + y'^2}} \right) \]

Inverse kinematics Step 4: Find \( \theta_2 \) and \( \theta_3 \)

\[ \theta_1 = \gamma + \sigma \cos^{-1} \left( \frac{- (x'^2 + y'^2 + l_1^2 - l_2^2)}{2l_1 \sqrt{x'^2 + y'^2}} \right) \]

\[ x - l_2 \cos \phi = l_1 \cos \theta_1 + l_2 \cos (\theta_1 + \theta_2) \]

\[ \theta_2 = \tan^{-2} \left( \frac{y' - l_1 \sin \theta_1}{l_2}, \frac{x' - l_1 \cos \theta_1}{l_2} \right) - \theta_1 \]

\[ \theta_3 = \phi - (\theta_1 + \theta_2) \]
Inverse Kinematics

There are two solutions to the inverse kinematics of a 3R manipulator

Except at the so-called “singular points”.

<table>
<thead>
<tr>
<th>Direct Kinematics</th>
<th>Inverse Kinematics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Serial chain geometry</td>
<td>Geometric modeling is difficult</td>
</tr>
<tr>
<td></td>
<td>Kinematic modeling is relatively simple</td>
</tr>
<tr>
<td></td>
<td>Difficult</td>
</tr>
<tr>
<td></td>
<td>Nonlinear problem with multiple solutions</td>
</tr>
<tr>
<td>Parallel chain geometry</td>
<td>Order of magnitude more difficult than</td>
</tr>
<tr>
<td></td>
<td>inverse kinematics for each chain</td>
</tr>
<tr>
<td></td>
<td>Comparable to serial chains (usually simpler)</td>
</tr>
</tbody>
</table>