Non-asymptotic Coded Slotted ALOHA

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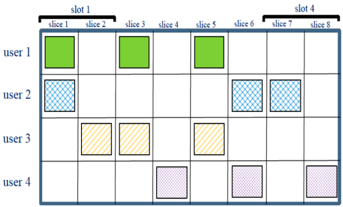
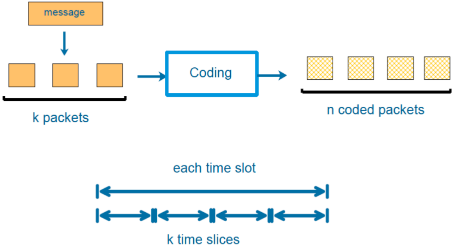
July 7, 2019

Non-asymptotic Analysis

- Multi-access channels
- Random access channels
- Non-asymptotic analysis $\equiv$ Number of users is moderate finite

Coded Slotted ALOHA (CSA)





- N: numer of users. M: number of slots. N_a: number of active users. $G = \frac{N_a}{M}$: channel load.
- Decoding k out of n coded packets is enough to recover the original message.

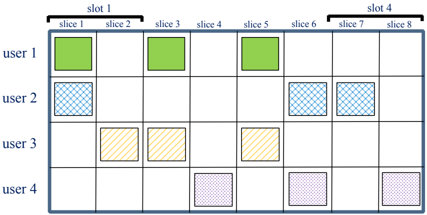
[1] G. Liva, “Graph-based analysis and optimization of contention resolution diversity slotted aloha,” 2011.

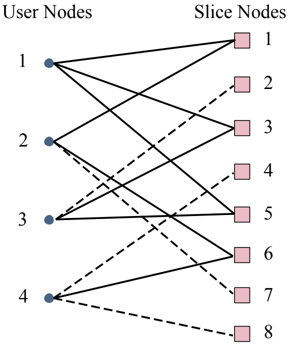
[2] M. Berioli, G. Cocco, G. Liva, and A. Munari, “Modern random access protocals,” 2016.

[3] A. G. i. Amat and G. Liva, “Finite length analysis of irregular repetition slotted aloha in the waterfall region,” 2018.

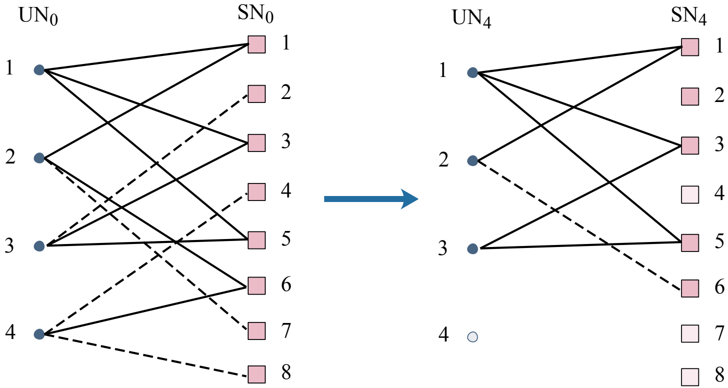
Coded Slotted ALOHA (CSA)

- Decoding: A successive interference cancelation (SIC) procedure.
- $N_a = 4, M = 4, k = 2, n = 3, G = \frac{N_a}{M}$





Coded Slotted ALOHA (CSA)



Gaps

$G = \frac{N_a}{M} \in (0, 1)$

1. Does the decoding process get stuck at some point? —The decoding stops when there is no singleton left
P_B: probability of decoding fails

Asymptotically, when $N_a, M \to \infty$, there exists G^* such that

if $G < G^*$ the decoding is successful (almost surely) $P_B = 0$

if $G \geq G^*$ the decoding is unsuccessful (almost surely) $P_B = 1$

2. Non-asymptotically, for finite N_a, M, how does P_B behave?

Our Contributions

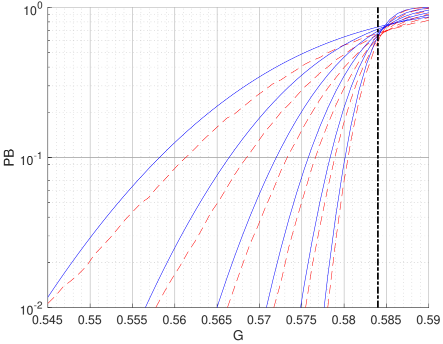
$G = \frac{N_a}{M} \in (0, 1)$

1. Does the decoding process get stuck at some point?

We compute G^* (analytically) for CSA

2. Non-asymptotically, for finite N_a, M, how does P_B behave?

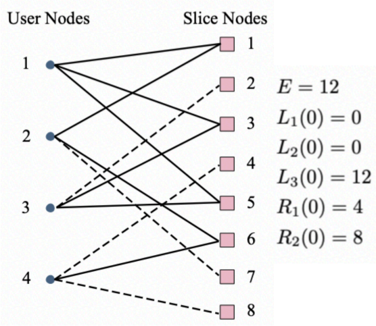
We obtain the non-asymptotic P_B (analytically) for CSA



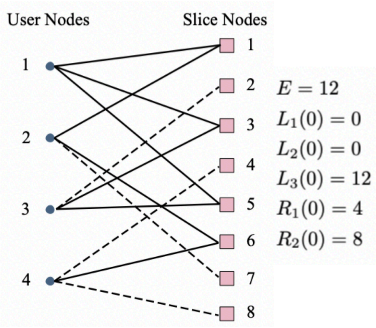
Our Approaches

Formulating the dynamics of decoding using sequential ODEs

- $E = nN_a$: Initial number of edges
- $t = q\Delta t$: Time where $\Delta t = \frac{1}{E}$ and q is the decoding step
- $L_i(t)$: The expected number of edges connected to a degree i user node at time t



- $l_i(t) = \frac{L_i(t)}{E} = L_i(t)\Delta t$
- $e(t) = \sum_i l_i(t)$
- $R_j(t)$: The expected number of edges connected to a degree j slice node at time t
- $r_j(t) = \frac{R_j(t)}{E} = R_j(t)\Delta t$



Asymptotic Evolution of l_i

$$\begin{cases} L_i(t+\Delta t) - L_i(t) = -i \cdot \dfrac{l_i(t)}{e(t)} + i \cdot \dfrac{l_{i+1}(t)}{e(t)}, \quad n-k < i < n \\ L_n(t+\Delta t) - L_n(t) = -n \cdot \dfrac{l_n(t)}{e(t)}. \end{cases}$$

If N_a is large, and $\Delta t \to 0$, then

$$\begin{cases} \dfrac{dl_i(t)}{dt} = i \cdot \dfrac{l_{i+1}(t) - l_i(t)}{e(t)}, \ n-k < i < n, \\ \dfrac{dl_n(t)}{dt} = -n \cdot \dfrac{l_n(t)}{e(t)}. \end{cases}$$

Asymptotic Evolution of r_j

By similar analysis, define $a(t)$: the expected number of removed edges at time t.

$$a(t) = (n-k+1)\cdot\frac{l_{n-k+1}(t)}{e(t)} + \sum_{i=n-k+2}^{n} 1\cdot\frac{l_i(t)}{e(t)}.$$

$$\begin{cases} \dfrac{dr_j(t)}{dt} = j\cdot(r_{j+1}(t) - r_j(t))\cdot\dfrac{a(t)-1}{e(t)}, & j \geq 2, \\ r_1(t) = e(t) - \sum_{j\geq 2} r_j(t) & \end{cases}$$

Density Evolution

For the left-hand side:

$$\begin{cases} \dfrac{dl_i(t)}{dt} = i \cdot \dfrac{l_{i+1}(t) - l_i(t)}{e(t)}, \; n-k < i < n, \\ \dfrac{dl_n(t)}{dt} = -n \cdot \dfrac{l_n(t)}{e(t)}. \end{cases}$$

For the right-hand side:

$$\begin{cases} \dfrac{dr_j(t)}{dt} = j \cdot (r_{j+1}(t) - r_j(t)) \cdot \dfrac{a(t)-1}{e(t)}, \; j \geq 2, \\ r_1(t) = e(t) - \displaystyle\sum_{j \geq 2} r_j(t). \end{cases}$$

where

$$a(t) = (n-k+1) \cdot \frac{l_{n-k+1}(t)}{e(t)} + \sum_{i=n-k+2}^{n} 1 \cdot \frac{l_i(t)}{e(t)},$$

$$e(t) = \sum_{i=n-k+1}^{n} l_i(t) = \sum_{j \geq 1} r_j(t)$$

Change of Variable

- To eliminate $e(t)$:
  $t \longmapsto x = \exp\left(\int_0^t \frac{d\tau}{e(\tau)}\right) \qquad \Rightarrow \qquad \frac{dx}{x} = \frac{dt}{e(t)}.$
- To eliminate $a(t)$: $\frac{\lambda'(x)}{\lambda(x)} = \frac{a(x)-1}{x}, \qquad \lambda(1) = 1.$
- The density evolution is

$$\begin{cases} \dfrac{dl_i(x)}{dx} = i \cdot \dfrac{l_{i+1}(x) - l_i(x)}{x}, \; n-k < i < n, \\ \dfrac{dl_n(x)}{dx} = -n \cdot \dfrac{l_n(x)}{x}, \end{cases}$$

$$\frac{dr_j(x)}{dx} = j \cdot (r_{j+1}(x) - r_j(x)) \cdot \frac{\lambda'(x)}{\lambda(x)}, \qquad j \geq 2.$$

$$r_1(x) = e(x) - \sum_{j \geq 2} r_j(x).$$

Theorem 1

$$l_i(x) = \sum_{j=n-k+1}^{n} \frac{\alpha_j^{(i)}}{x^j}, \qquad i \in \{n-k+1, \cdots, n\}, \tag{1}$$

where $\{\alpha_j^{(i)}\}_{i,j}$, $n-k+1 \leq i \leq j \leq n$, is a finite 2-dimensional recursive sequence of integers which is (uniquely) determined by the following equations:

$$\begin{cases} \alpha_n^{(n)} = 1 \\ \alpha_i^{(i)} = \sum_{j=i+1}^{n} (-1)^{j-i+1} \binom{j-1}{i-1} \alpha_j^{(j)}, \quad n-k+1 \leq i < n, \\ \alpha_j^{(i)} = (-1)^{j-i} \binom{j-1}{i-1} \alpha_j^{(j)}, \qquad i < j \leq n. \end{cases} \tag{2}$$

Theorem 2

Suppose $G = \frac{N_a}{M}, R = \frac{k}{n}$

$$r_j(x) = \frac{1}{(j-1)!\lambda^j(x)} \left(\frac{G}{R}\right)^{j-1} \exp\left(-\frac{G}{R\lambda(x)}\right), j \geq 2 \qquad (3)$$

$$r_1(x) = \left[\sum_{j=n-k+1}^{n} \frac{\beta_j}{x^j}\right] - \frac{1}{\lambda(x)}\left(1 - \exp\left(-\frac{G}{R\lambda(x)}\right)\right) \qquad (4)$$

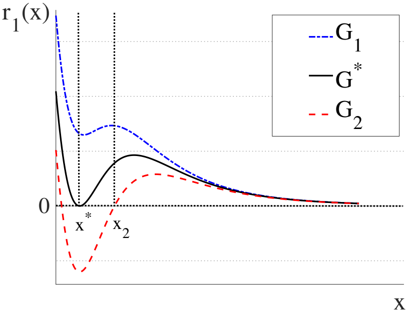
where

$$\beta_j := \alpha_j^{(j)}(-1)^j \sum_{i=n-k+1}^{j} (-1)^i \binom{j-1}{i-1}$$

$$\lambda(x) := \exp\left((n-k)\int_1^x \frac{\sum_{j=0}^{k-1} \alpha_{n-j}^{(n-k+1)} y^j}{\sum_{j=0}^{k-1} \beta_{n-j} y^{j+1}} dy\right)$$

Computation of G^*

$$G_1 < G^* < G_2$$



$$r_1\left(x; \tilde{G}(x)\right) = 0, \quad G(x) \geq G(x^*), x \in N(x^*; \delta)$$

$$\left.\frac{d\tilde{G}(x)}{dx}\right|_{x=x^*} = 0, \qquad G^* = \tilde{G}(x^*) \tag{5}$$

Theorem 3

$$\tilde{G}(x) = -R\lambda(x)\log\left(1 - e(x)\lambda(x)\right)$$

Define $h(x) = e(x)\lambda(x)$. Then $G^* = \tilde{G}(x^*)$, where x^* is the solution of the following algebraic equation:

$$\log(1 - h(x)) = \frac{1 - h(x)}{h(x)}\left(1 + \frac{xe'(x)}{(n-k)l_{n-k+1}(x)}\right). \qquad (6)$$

Numerical Comparison for G^*

Let $N_a = 20000$ by averaging over 2000 trials
The Error is less than 0.01% even for moderate values of N_a

Parameters	Simulated G^*	Computed G^*
n = 5, k =2	0.737	0.7388
n = 5, k =3	0.582	0.5840
n = 6, k =2	0.724	0.7253
n = 6, k =3	0.669	0.6699
n = 8, k =2	0.659	0.6602
n = 8, k =5	0.545	0.5458
n = 12, k =4	0.636	0.6372
n = 12, k =10	0.266	0.2664
n = 25, k =4	0.459	0.4595

Scaling Law for Non-asymptotic Behavior

Inspired by a result from statistical physics:
Non-asymptotic P_B in the waterfal region:

$$\lim_{\substack{N_a \to \infty \\ \text{s.t. } N_a^{1/\mu}(G^* - G) = z}} P_B(N_a, G) = f(z). \tag{7}$$

In our problem, f is the Q-function

$$P_B(N_a, G) = Q\left(\frac{\sqrt{N_a}}{\alpha}\left(G^* - \beta N_a^{-2/3} - G\right)\right). \tag{8}$$

[4] A. Amraoui, A. Montanari, T. Richardson, and R. Urbanke, "Finite-length scaling for iteratively decoded LDPC ensembles," 2009.

Covariance Evolution

$$z = (z_0, z_1, \cdots, z_d) = (r_1, r_2, l_{n-k+1}, \cdots, l_n)$$

$\delta^{(z_i z_j)}(x)$: the normalized covariance between the corresponding node-based quantities of z_i and z_j at time x,

$\hat{f}^{(z_i z_j)}(x)$: the covariance between the corresponding edge-based quantities of z_i and z_j at time x,

$\hat{f}^{(z_i)}$: the expected change of the corresponding edge-based quantity of z_i.

$$\frac{d\delta^{(z_i z_j)}(x)}{dx} = \frac{e(x)}{x}\left[\frac{\hat{f}^{(z_i z_j)}(x)}{n} + \sum_{k=0}^{d} \delta^{(z_i z_k)}(x)\frac{\partial \hat{f}^{(z_j)}(x)}{\partial z_k} + \frac{\partial \hat{f}^{(z_i)}(x)}{\partial z_k}\delta^{(z_k z_j)}(x)\right],$$

Theorem 4: Computation of α and β

The probability of error P_B is

$$P_B = Q\left(\frac{\sqrt{N_a}}{\alpha}\left(G^* - \beta N_a^{-2/3} - G\right)\right)$$

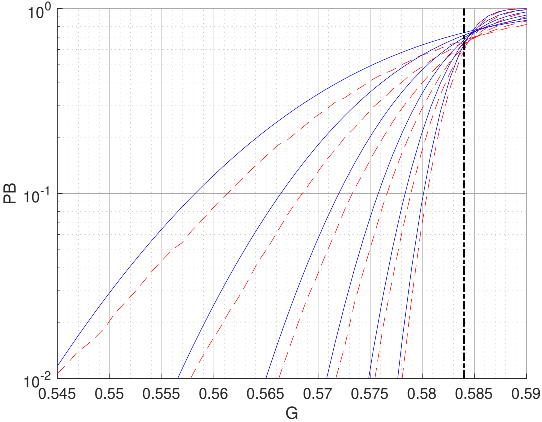
where

$$\alpha = -\sqrt{\frac{\delta^{(r_1 r_1)}(x)}{n}}\left(\frac{\partial r_1(x;G)}{\partial G}\right)^{-1}\Bigg|_{x=x^*,G=G^*},$$

$$\beta = -\left(\frac{\hat{f}^{(r_1 r_1)}(x)}{n}\right)^{2/3}\left[\sum_{k=1}^{d}\frac{\partial \hat{f}^{(r_1)}(x)}{\partial z_k}\hat{f}^{(z_k)}(x)\right]^{-1/3} \times\left(\frac{\partial r_1(x;G)}{\partial G}\right)^{-1}\Bigg|_{x=x^*,G=G^*}.$$

Simulation Results for P_B

$N_a = 1000, 2000, 4000, 8000, 16000, 32000.$
$n = 5$, $k = 3$, over 2×10^5 trials
$\alpha = 0.42362$, $\beta = 0.8629$, $G^* = 0.5840$



Thank you!