Channel Coding at Low Capacity

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Motivation

✦ Narrowband communication: NB-IoT, eMCT [17-18]

By 2021, there will be 1.5 billion IoT devices

↓

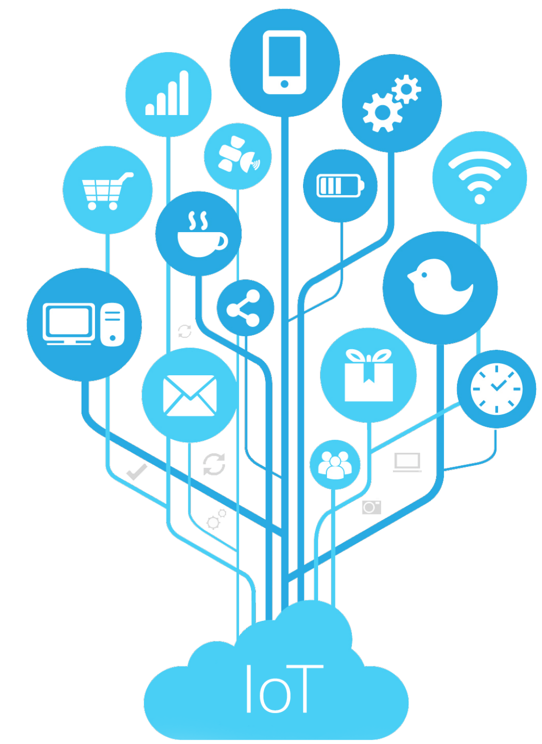
This infers a 170 dB of coupling loss

↓

Which leads to having a -13dB of the effective SNR

↓

This can be translated to a very small capacity of 0.03

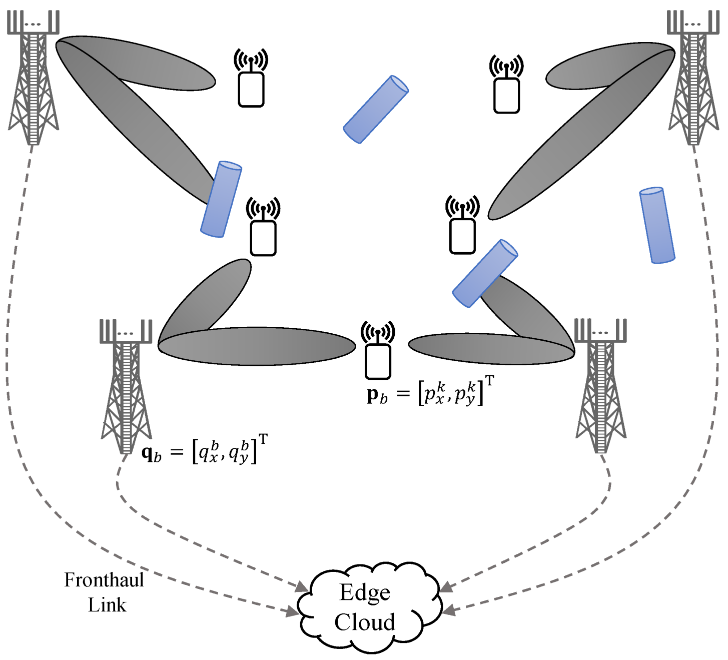


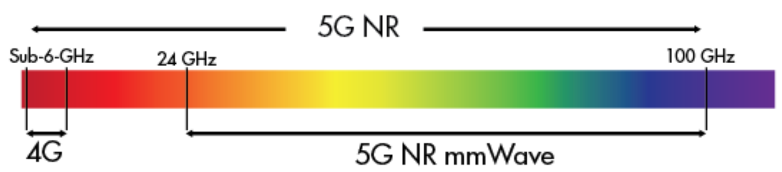
Motivation

- ✦ Wideband communication: mmWave [17-18]

$$C = B\log(1+\frac{P}{N_0 B})$$

$$B \to \infty \implies SNR \to 0 \implies C \to 0$$





Low Capacity Regime

The capacity C is small with respect to the blocklength n [2]

✦ Formal definition

$C < n^{s-1}$, where $s \in [0, 1)$ is a tuning parameter

s can be specified depending on

- the application
- the channel under discussion

✦ Informal definition

$$\kappa := nC = o(n)$$

$$\kappa = \mathcal{O}(1) \quad \leftrightarrow \quad C = \mathcal{O}\left(\frac{1}{n}\right) \quad \leftrightarrow \quad s = 0$$

$$\kappa = \mathcal{O}(\sqrt{n}) \quad \leftrightarrow \quad C = \mathcal{O}\left(\frac{1}{\sqrt{n}}\right) \quad \leftrightarrow \quad s = \frac{1}{2}$$

Challenges

✦ In the moderate-capacity regime [1,4,7]

$$\log_2 M^*(n, p_e) = nC - \sqrt{nV}Q^{-1}(p_e) + \mathcal{O}(\log_2 n)$$

This is the best that can be achieved in the moderate-capacity regime.
(The last term depends on the type of the channel [8-11])

✦ In the low-capacity regime [2], however,

- using the above prediction leads to off numeral estimates since some neglected terms will become significant and also because the governing laws are different.
- Moreover, the current practical code designs are not efficient in this regime.

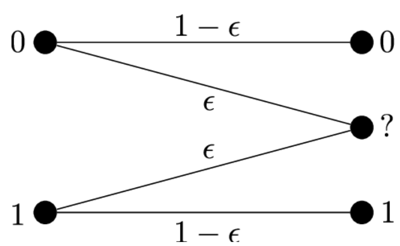
Our Contribution

- ✦ Deriving non-asymptotic laws for BEC and BSC in the low-capacity regime
- ✦ Proposing a provably efficient practical code design for the low-capacity regime

Binary Erasure Channel (BEC)

Let ϵ be the erasure probability

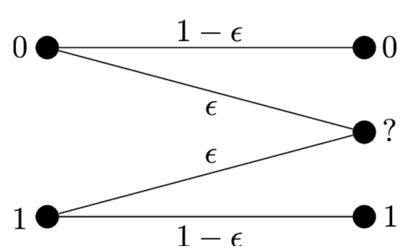
✦ In the <u>moderate-capacity</u> regime: e.g., $\epsilon = 0.3, 0.6$



output: ...1??0?1???00?1??0110??10???111?1??.....

Gaussian Convergence Laws $R = n(1-\epsilon) + \mathcal{O}\left(\sqrt{n}\right)$

✦ In the <u>low-capacity</u> regime: e.g., $\epsilon = 0.95, 0.99$



output: ...?????1????0??????????1??????1??.....

Rare information —> Poisson Convergence Laws [16] $P\{R < r\} = \mathcal{P}_{n(1-\epsilon)}(r)$

Binary Erasure Channel (BEC)

$$M^*(n, p_e) = \max\{M \mid \exists(M, p_e)\text{-code for } W^n\}$$

where n is the blocklength and p_e is the average probability of error

The raw achievability bound (RCU) for BEC [1,5,6]

$$p_e \leq \sum_{r=0}^{n} \binom{n}{r} \epsilon^{n-r}(1-\epsilon)^r 2^{-[r-\log_2(M-1)]^+}$$

The raw converse bound for BEC [1,5,6]

$$p_e \geq \sum_{r\ <\log_2 M} \binom{n}{r} \epsilon^{n-r}(1-\epsilon)^r \left(1 - \frac{2^r}{M}\right)$$

Binary Erasure Channel (BEC)

The raw achievability bound (RCU)

The raw converse bound

Using Gaussian convergence laws for the moderate-capacity regime, due to [1]

Using Poisson convergence laws for the low-capacity regime + a finer analysis

$$\log_2 M^*(n, p_e) = nC - \sqrt{nV} Q^{-1}(p_e) + \mathcal{O}(1)$$

$$C = 1 - \epsilon$$

$$V = \epsilon(1 - \epsilon)$$

Our result on the next page

Binary Erasure Channel (BEC)

Theorem. Consider transmission over BEC(ϵ) in low-capacity regime and let $\kappa = n(1-\epsilon)$. Then,

$$M_1 \leq M^*(n, p_e) \leq M_2,$$

where M_1 is the solution of

$$\mathfrak{P}_1(M_1) + \alpha\sqrt{\mathfrak{P}_1(M_1)} - p_e = 0, \tag{1}$$

and M_2 is the solution of

$$\mathfrak{P}_2(M_2) - \alpha\sqrt{\mathfrak{P}_2(M_2)} - \alpha\sqrt{\mathcal{P}_\kappa(\log_2 M_2)} - p_e = 0, \tag{2}$$

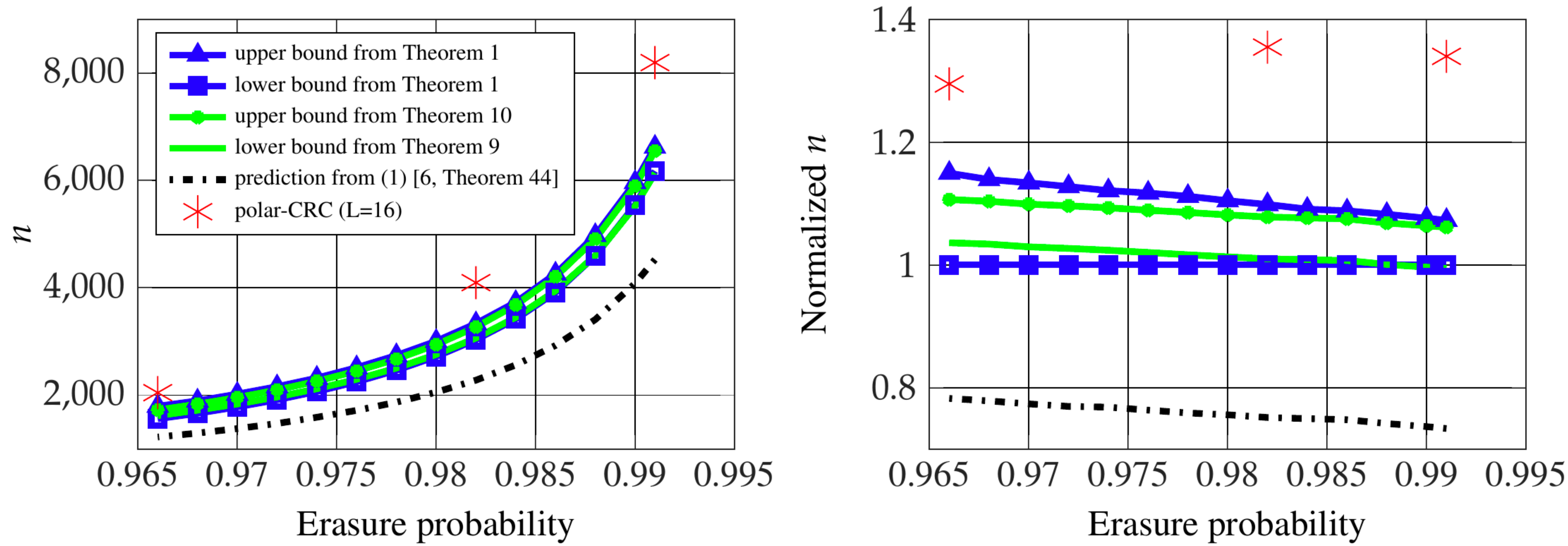
and

$$\mathfrak{P}_1(M_1) = \mathcal{P}_\kappa(\log_2 M_1) + M_1 e^{-\kappa/2}\left(1 - \mathcal{P}_{\kappa/2}(\log_2 M_1)\right),$$

$$\mathfrak{P}_2(M_2) = \mathcal{P}_\kappa(\log_2 M_2) - \frac{e^\kappa}{M_2}\mathcal{P}_{2\kappa}\left(\log_2 M_2\right),$$

$$\alpha = \frac{\sqrt{2}}{\epsilon^{3/2}}\left(1 + 2\sqrt{\frac{3}{\epsilon\kappa}}\right)\left(\sqrt{e} - 1\right)(1-\epsilon).$$

Binary Erasure Channel (BEC)



At each ϵ we plotted the optimal n such that $M^*(n, 10^{-2}) = 2^{40}$

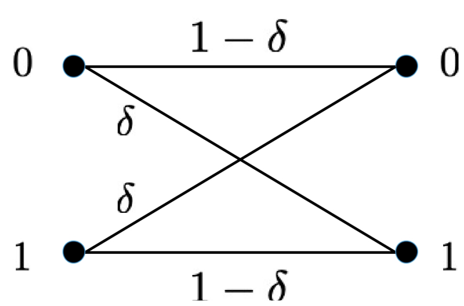
Green: The raw upper and lower bounds for the reality
Blue: Our prediction of the bounds
Black: The prediction from [1] (A single prediction for both upper and lower bounds)

The polar code is concatenated with cyclic redundancy check (CRC) code of length 6, and is decoded with the list-SC algorithm [3] with list size L=16.

Binary Symmetric Channel (BSC)

Let δ be the crossover probability

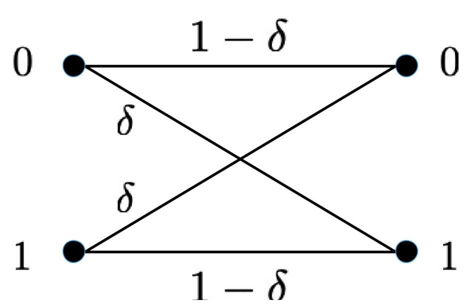
✦ In the <u>moderate-capacity</u> regime: e.g., $\delta = 0.2, 0.9$



output: ...1010011100011010110101000110110.....

Gaussian Convergence Laws

✦ In the <u>low-capacity</u> regime: e.g., $\delta = 0.46, 0.52$



output: ...0010101110010111000110011011111010.....

Again Gaussian Convergence Laws
but needs a finer analysis since some neglected terms are significant now

Binary Symmetric Channel (BSC)

The raw achievability bound (RCU) for BSC [1,5,6]

$$p_e \leq \sum_{r=0}^{n} \binom{n}{r} \delta^r (1-\delta)^{n-r} \min\left\{1, (M-1)S_n^r\right\}$$

$$S_n^r = \sum_{s=0}^{r} \binom{n}{s} 2^{-n}$$

The raw converse bound for BSC [1,5,6]

$$M \leq \frac{1}{\beta_{1-p_e}^n}$$

where β_α^n for a real $\alpha \in [0,1]$ is defined below based on values of β_ℓ where ℓ is an integer:

$$\beta_\alpha^n = (1-\lambda)\beta_L + \lambda\beta_{L+1},$$

$$\beta_\ell = \sum_{r=0}^{\ell} \binom{n}{r} 2^{-n},$$

such that $\lambda \in [0,1)$ and integer L satisfy the following:

$$\alpha = (1-\lambda)\alpha_L + \lambda\alpha_{L+1},$$

$$\alpha_\ell = \sum_{r=0}^{\ell-1} \binom{n}{r} \delta^r (1-\delta)^{n-r}.$$

Binary Symmetric Channel (BSC)

The raw achievability bound (RCU)

The raw converse bound

for the moderate-capacity regime, due to [1]

for the low-capacity regime + a finer analysis

$$\log_2 M^*(n, p_e) = nC - \sqrt{nV}Q^{-1}(p_e) + \frac{1}{2}\log_2 n + \mathcal{O}(1)$$

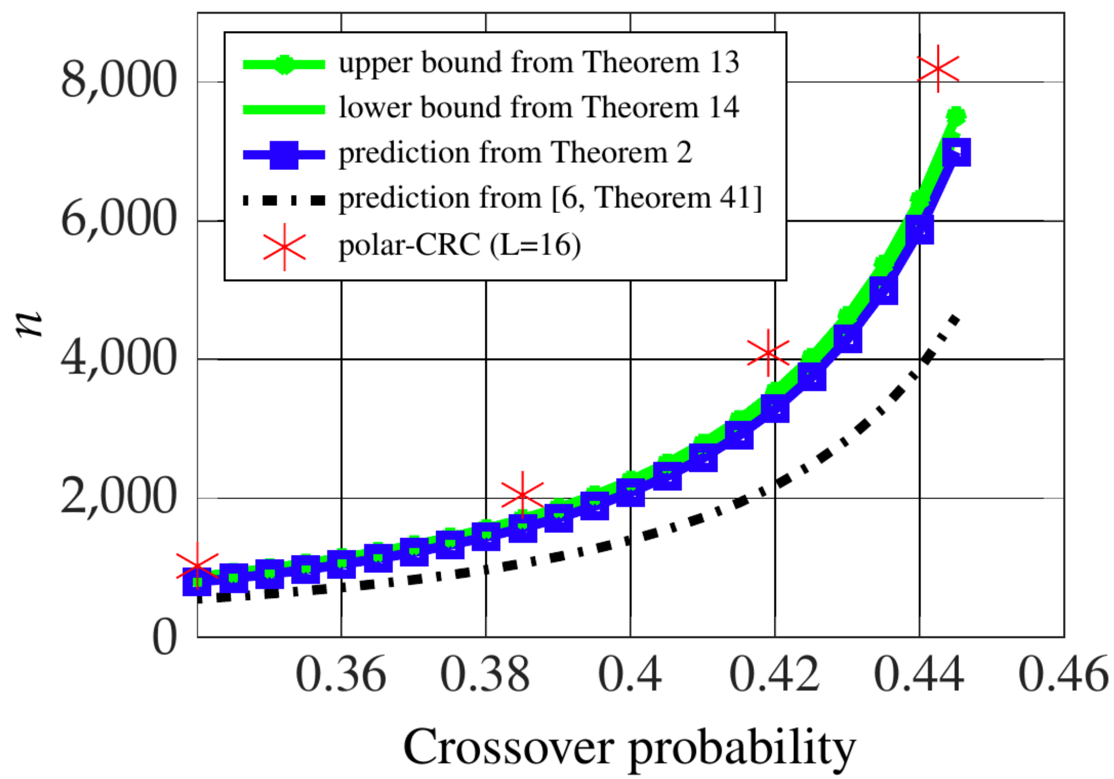
$$C = 1 - h_2(\delta)$$

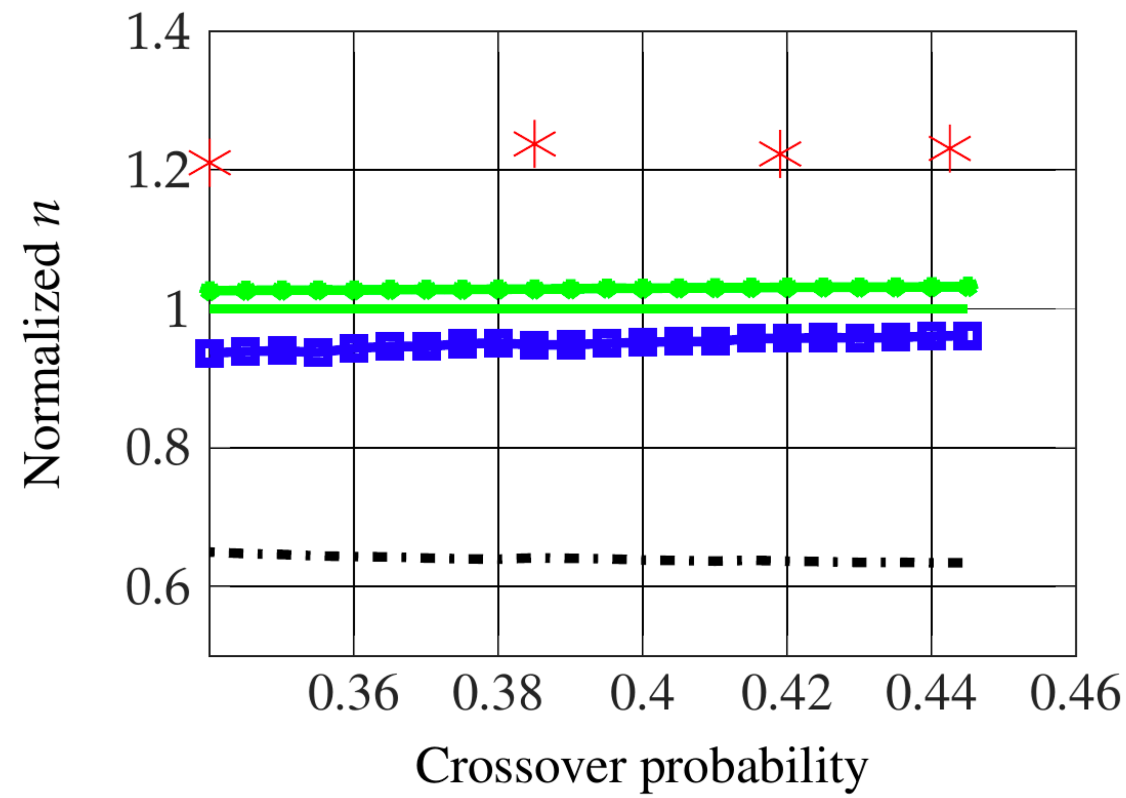
$$V = \delta(1-\delta)\log_2^2((1-\delta)/\delta)$$

Theorem. $$\log_2 M^*(n, p_e) = \kappa - 2\sqrt{\frac{2\delta(1-\delta)}{\ln 2}} \cdot \sqrt{\kappa}\, Q^{-1}(p_e) + \frac{1}{2}\log_2 \kappa - \log_2 p_e + \mathcal{O}(\log\log\kappa)$$

$$\kappa = n(1 - h_2(\delta))$$

Binary Symmetric Channel (BSC)





At each δ we plotted the optimal n such that $M^*(n, 10^{-2}) = 2^{40}$

Green: The raw upper and lower bounds for the reality
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The polar code is concatenated with cyclic redundancy check (CRC) code of length 6, and is decoded with the list-SC algorithm [3] with list size L=16.

Practical Code Design

In summary:

- ✦ Current standards use repetition + some off-the-shelf moderate-rate codes like Turbo codes/ LDPC.
- ✦ Using iterative codes + repetition results in mediocre performances.
- ✦ Polar codes implicitly apply the optimal repetition length.

The next slides argue these results in detail.

Practical Code Design

Consider the largest (optimal) repetition size r_β which satisfies

$$\frac{n}{r_\beta} C(W^{r_\beta}) \geq \beta n C(W)$$

Theorem. If $W = \mathrm{BEC}(\epsilon)$, then for the largest repetition size, r_β, we have

$$\frac{n(1-\epsilon)\ell}{2\left(1-\frac{\beta}{\ell}\right)} \cdot \left(\frac{\beta}{\ell}\right)^2 \leq \frac{n}{r_\beta} \leq \frac{n(1-\epsilon)\ell}{2\left(1-\frac{\beta}{\ell}\right)},$$

where $\ell = -\frac{\ln \epsilon}{1-\epsilon}$. Equivalently, assuming $\kappa = n(1-\epsilon)$, we have

$$\frac{\kappa}{2\left(1-\beta\right)} \cdot \beta^2 (1 + \mathcal{O}(1-\epsilon)) \leq \frac{n}{r_\beta} \leq \frac{\kappa}{2\left(1-\beta\right)} (1 + \mathcal{O}(1-\epsilon)).$$

Practical Code Design

The previous result about BEC leads to the following theorem

Theorem. Among all BMS channels with the same capacity, BEC has the largest repetition length r_β. Hence, for any BMS channel with capacity C and $\kappa = nC$, we have

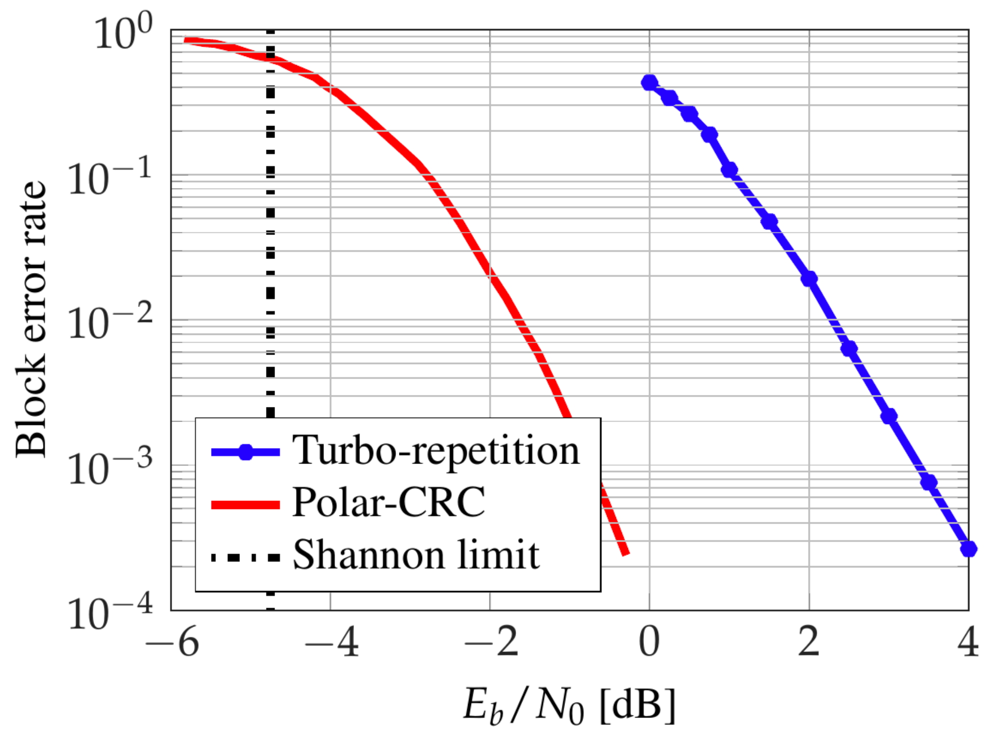
$$\frac{n}{r_\beta} \geq \frac{\kappa}{2(1-\beta)}\beta^2(1+\mathcal{O}(1-C)).$$

Remark. we can conclude that for any BMS channel with low capacity, in order to have the total rate loss of order $\mathcal{O}(1)$, the repetition size should be at most $\mathcal{O}(n/\kappa^2)$.

Now the following theorem shows that polar codes automatically perform a repetition coding which matches the optimal repetition size $\mathcal{O}(n/\kappa^2)$.

Theorem. Consider using a polar code of length $n = 2^m$ for transmission over a BMS channel W. Let $m_0 = \log_2(4\kappa^2)$ where $\kappa = nC(W)$. Then any synthetic channel $W_n^{(i)}$ whose Bhattacharyya value is less than $\frac{1}{2}$ has at least m_0 plus operations in the beginning. As a result, the polar code constructed for W is equivalent to the concatenation of a polar code of length (at most) 2^{m_0} followed by 2^{m-m_0} repetitions.

Practical Code Design



- Comparison for low-capacity BAWGN. The number of information bits is k=40.
- The polar-CRC has length 8192, is constructed using 6 CRC bits, and is decoded using the SC-list decoder with L =16.
- The Turbo-repetition has an underlying (120,40) Turbo code which is repeated 68 times (total length = 8160) and is decoded with 6 iterations.
- The Shannon limit for this setting is -4.75dB.

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Thank you!