Network Optimization with Heuristic Rational Agents

Ceyhun Eksin
Dept. of Electrical and Systems Engineering
University of Pennsylvania
ceksin@seas.upenn.edu

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Network optimization
- Agents strive to minimize global cost by selecting local variables.

Distributed network optimization
- Local information is determined by the network.
- Solution method for estimation and detection problems in WSNs.

Reliability, computation complexity, power constraints etc.

Emergence of global behavior in social and biological networks
- Social and biological agents are prone to errors in decision-making.
- Thus, heuristic rationality.
A Network Optimization Formulation

- Network: \( G = (V, E) \) where
  - Vertices \( i \in V \) denote agents.
  - Edges \( (i, j) \in E \) denote connections between agents.
  - Neighborhood of agent \( i \), \( n(i) = \{ j : (j, i) \in E \} \)

- Variables: \( x = \{ x_1, x_2, \ldots, x_N \} \) where \( |V| = N \) and \( x_i \in X_i \subseteq \mathbb{R}^n \)

- Individual cost functions \( f_i(x_i, x_{n(i)}) \)
  \[
  f_i(x_i, x_{n(i)}) := f_{0i}(x_i) + \sum_{j \in n(i)} f_{ij}(x_i, x_j)
  \]
  where \( f_{ij}(x_i, x_j) = f_{ji}(x_j, x_i) \), for all \( i \), and \( j \in n(i) \).

- Then the global cost function
  \[
  f(x) = \sum_{i \in V} f_i(x_i, x_{n(i)})
  \]
Minimization requires simultaneous selection of $x_i$ for all $i \in V$

$$x^* = \arg\min_{x_1, x_2, \ldots, x_N} f(x)$$

Optimal value $x^*$ yields optimal cost, $p^* = f(x^*)$

Distributed Optimization:

- Observe neighborhood information $\{x_{n(i)}\}$

$$\tilde{x}_i = \arg\min_{x_i} f_i(x_i, x_{n(i)})$$

Selecting $\tilde{x}_i$ is too cunning $\Rightarrow$ heuristic rationality
Definition
Consider network agent $i$ associated with variable $x_i$ and denote as $x_{n(i)}(t)$ the values of neighboring variables at time $t$. We say that a probabilistic rule $x_i(t) \in \mathcal{X}_i$ is heuristic rational if and only if its expectation is a rational action,

$$
\mathbb{E}[x_i(t) \mid x_{n(i)}(t)] = \tilde{x}_i(t) = \arg\min_{x_i \in \mathcal{X}_i} f_i(x_i, x_{n(i)}(t)).
$$

- Due to heuristic actions
- Modeling/perception errors
Related efforts to lift unrealistic assumptions

- Asynchronous updates [Tsitsiklis et al., 1986]
- Time-varying links [Jadbabaie et al., 2003]
- Unreliable communication links [Kar and Moura, 2010], [Nedic et al., 2010], [Rabbat et al., 2005]
- Random communication noise [Kar and Moura, 2009]
- Key assumption: Agents are rational.
Opinion propagation in social networks

- $x_i \in [-1, 1]$ is the opinion of a social agent
- Stubborn agents $i \in S$ have fixed extreme opinions $x_i \in \{-1, 1\}$
- Penalty for disagreement: $f_{ij}(x_i, x_j) = (1/2)(x_i - x_j)^2$

\[
f_i(x_i, x_{n(i)}) = \frac{1}{2} \sum_{j \in n(i)} (x_i - x_j)^2
\]

- Rational action $\Rightarrow \tilde{x}_i(t) = \frac{1}{\#(n(i))} \sum_{j \in n(i)} x_j(t)$
- Averaging is not done exactly but rather guessed
- Agents only consider opinions of a random subset $\tilde{n}_i(t) \subseteq n_i$

\[
x_i(t) = \frac{1}{\#(\tilde{n}_i(t))} \sum_{j \in \tilde{n}_i(t)} x_j(t)
\]

- Voter models
Field Estimation with WSN

- WSN deployed to estimate spatially varying field
- $x_i$ is sensor i’s estimate value for its own location
- $y_i$ independent local noisy observation for sensor i’s location

\[ P(y \mid x) = e^{U(y \mid x)} \Rightarrow \prod_{i} P(y_i \mid x_i) = \prod_{i} e^{U(y_i \mid x_i)} \]

- Maximum a posteriori (MAP) estimator

\[ \tilde{x} = \arg\max_{x} \log(P(y \mid x)) + \log(P(x)) \]

- Markov spatial dependence \( P(x_i \mid x_j, j \neq i) = P(x_i \mid x_{n(i)}) \)
For some energy function $U(x) = \sum_{i,j \in n(i)} u_{ij}(x_i - x_j)$

$$\log P(x_i | x_{n(i)}) = \sum_{j \in n(i)} u_{ij}(x_i - x_j)$$

Define local cost $f_i(x_i, x_{n(i)}) = U(y_i | x_i) + \sum_{j \in n(i)} u_{ij}(x_i - x_j)$.

Then we can write MAP estimator as

$$\tilde{x} = \arg\max \sum_{i \in V} f_i(x_i, x_{n(i)})$$

Communication errors, quantization and model mismatch.
Optimality

- Stochastic process \( \{F_k\}_{k \in \mathbb{N}} \) of optimality gaps
  \[
  F_k := f(x(t_k)) - p^*.
  \]
- Recursive application of heuristic rational rule generates \( x(t_k) \)
- The goal is to compare \( x(t_k) \) with the optimal configuration \( x^* \).
- Heuristic rational updates are analog to block coordinate descent.
- Random activation rule:
  - Activations are indexed by \( k \in \mathbb{N} \)
  - Activation time of \( k \)th activation is \( t_k \)
  - Agent that activates at time \( t_k \) is \( i_k \)
  - All agents have positive activation probability.
Assumptions

- **(A1) Strong convexity.** There exists a constant $m > 0$
  \[ f(y) \geq f(x) + \nabla f(x)^T (y - x) + \frac{m}{2} ||y - x||^2 \]

- **(A2) Lipschitz gradients.** There exists a constant $M > 0$
  \[ f(y) \leq f(x) + \nabla f(x)^T (y - x) + \frac{M}{2} ||y - x||^2 \]

- **(A3) Random activation.**
  - All agents are equally likely to become active.

- **(A4) Bounded variance.** $\mathbb{E} ||x_{i_k}(t_k) - \tilde{x}_{i_k}(t_k)||^2 \leq \sigma^2.$
Lemma

If assumptions (A1)-(A4) hold, the optimality gaps $F_{k+1}$ and $F_k$ satisfy

$$
\mathbb{E}[F_{k+1} \mid x(k)] \leq (1 - \beta)F_k + \frac{\sigma^2 M}{2}
$$

where we defined the condition number $\beta := m/(MN)$ and used the shorthand notation $x(k) := x(t_k)$.

Convergence behavior depends on $x(k + 1)$ position relative to $x^*$

- When $F_k > \sigma^2 M/2\beta$, rationality dominates.
- When $F_k \leq \sigma^2 M/2\beta$, randomness dominates.
Theorem

Define the best optimality gap by time $t_k$ as $F_{k_{\text{best}}} := \min_{l \in [0,k]} F_l$. If assumptions (A1)-(A4) hold, then

$$\lim_{k \to \infty} F_{k_{\text{best}}} \leq \frac{M\sigma^2}{2\beta}, \quad \text{a.s.}$$

- For almost all realizations, near optimality is achieved as $k$ grows.
- Near optimality is achieved infinitely often.
- Near optimality region grows with $\sigma^2 \uparrow$, $M \uparrow$ and $\beta \downarrow$. 
What is the process’ behavior between visits to near optimality?

Starting at $F_k = (1 + \rho)M\sigma^2/2\beta$, let $L = \min \{ l > 0 : F_{k+l} < F_k \}$.

Worst optimality gap reached during excursion:

$$F_k^\dagger := \max(F_{k+l} : 0 \leq l \leq L).$$
(A5) Bounded Increments. There exists a $\kappa < \infty$ such that

$$P(|F_{k+1} - F_k| \leq \kappa | F_k) = 1 \text{ for all } k.$$ 

Theorem

- If assumptions (A1)-(A5) hold, then, for arbitrary given constant $\gamma$,

$$P(F_k \geq \gamma | F_k) \leq e^{-c(\gamma - F_k)},$$

with $c = 2\rho M \sigma^2 / [ (\rho M \sigma^2)^2 + \kappa^2 ]$.

- When $F_k > M \sigma^2 / 2\beta$, it behaves like a supermartingale.
- The worst optimality gap probability decreases exponentially.
- As $c$ increases, we have a tighter bound.
- An increase in $\rho$, $M$, $\sigma^2$, or $\kappa$ implies $c$ decreases.
Simulations

- Opinion Propagation
- Field estimation with WSN
- Herd foraging

- N agents on a $L \times W$ rectangular field.
- Agents activation times are independent with $\exp(1)$.
- Coordinates of agent $i$ is chosen uniformly at random.
- $n(i) = \{ j : \| r_i - r_j \| \leq d, j \neq i \}$ for some threshold $d > 0$. 
Opinion Propagation with Stubborn Agents

- \( N = 100, \ S = \{1, 2\} \) with \( x_1(t) = 1 \) and \( x_2(t) = -1 \) for all \( t \)
- \( x_i \in [-1, 1] \) for all \( i \in V/S \)
- **Heuristic Rationality**: Rational action is added \( U[-0.1, 0.1] \)

- Stubborn agents are influential among agents within their vicinity.
Optimality gap with heuristic rational agents

- Global and Individual optimality gap under varying m.s.e.

- Near optimality region is proportional to irrationality.
Opinion Propagation on Small-World Networks

- Edges are rewired with $p_r = 0.1$ reducing average path length.

- Random links enhance opinion propagation.
- Influence of stubborn agents is reduced.
Random field estimation

- Zero-mean Gaussian observations: \( U(y_i|x_i) = (x_i - y_i)^2 / 2\sigma^2 \).
- Gaussian MRF with quadratic field energy:
  \[
  U(x) = \sum_{i,j \in n(i)} u_{ij}(x_i - x_j) = \sum_{i,j \in n(i)} (x_i - x_j)^2 / 2\lambda
  \]
- Local cost for sensor \( i \) is
  \[
  f_i(x_i, x_{n(i)}) = \frac{1}{2\sigma^2} (x_i - y_i)^2 + \frac{1}{2\lambda} \sum_{j \in n(i)} (x_i - x_j)^2.
  \]
- Rational estimate \( \tilde{x}_i(t) = \frac{\lambda y_i + \sigma^2 \sum_{j \in n(i)} x_j(t)}{\lambda + N_i \sigma^2} \)
- Received signals are quantized \( x_{qj}(t) = x_j(t) + q_j(t) \)
Temperature field and WSN

- Temperature range $[0^\circ F, 255^\circ F]$.
- Two heat sources at $h_1 = (7, 14)$ and $h_2 = (12, 13)$
- Temperature at heat sources is $255^\circ F$.
- Temperature drops at rate $75^\circ F/m$ for $3m$ starting from heat source.

- $N = 270$ temperature sensors on $15m \times 20m$ field.
- Geometric graph with $d = 1m$. 
Temperature field estimation

- Noise power $\sigma^2 = 0.9 \times 10^3$ and smoothing coefficient $\lambda = 10^3$.
- The quantization level for estimates are integers in $[0, 255]$.

- Estimates become closer to field values over time.
Log-likelihood

As time progresses, sensor estimates get closer to field values.
Cohesive foraging in animal networks

- **Non-convex cost function**

\[
 f_i(x_i, x_{n(i)}) = \frac{h}{2} \| x_i - x_g \|^2 + \sum_{j \in n(i)} \frac{a}{2} \| x_i - x_j(t) \|^2 + \frac{bc}{2} e^{-\| x_i - x_j(t) \|^2 / c}
\]

where \( h, a, b, c \) are positive constants and food location is \( x_g \).

- **Heuristic rational** updates are on the average **local** optimal.
  - \( \tilde{x}_i \) is a solution to \( \nabla_{x_i} f_i(x_i, x_{n(i)}) = 0 \).
  - Uniformly distributed \([-\alpha, \alpha]\) noise is superimposed.
Cohesive foraging in animal networks

- $N=7$, fixed neighborhood structure, $\alpha = 0.1$.
- Animals are initially within (0,0) and (1,1).
- Network structure of seven animals at $t = 0$, $t = 15$, $t = 30$, $t = 45$
Cohesive foraging in animal networks

- Evolution of global and individual optimality gap over time.

- Global and individual costs decrease over time.

- Noise can coincidentally be helpful to individuals.
Conclusions

- Introduced a network optimization formulation
  - Each agent has local convex cost function $f_i$
  - Global cost $f = \sum_i f_i$
- Defined distributed update rule heuristic rational optimization
  - On the average optimal
- Random activation rule
  - Each agent has positive activation probability any given time
- Exemplified heuristic rational behavior in
  - Opinion propagation
  - Field estimation with WSN.
- Convergence to near optimality almost surely
- Outside near optimality, exponential bound on excursion probability