Tight Bounds for Single-Pass Streaming Complexity of the Set Cover Problem

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Joint work with Sanjeev Khanna (Penn) and Yang Li (Penn)
The Set Cover Problem

- **Input:** A collection of $m$ sets $S_1, \ldots, S_m$ from a universe $[n]$.

- **Goal:** Choose a smallest subset $C$ of the sets from $S_1, \ldots, S_m$ such that $C$ covers $[n]$, i.e., $\bigcup_{i \in C} S_i = [n]$.

The sets maybe weighted in general. We use $\text{OPT}$ to denote the optimal solution size/weight.

Approximation vs Estimation:
- $\alpha$-approximation: output a set cover of size at most $\alpha \cdot \text{OPT}$ plus a certificate of coverage for each element $e \in [n]$.
- $\alpha$-estimation: output an estimate for the size of minimum set cover in range $[\text{OPT}, \alpha \cdot \text{OPT}]$. 
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- This talk: space complexity of approximating the set cover problem in the streaming model.
The Streaming Set Cover Problem

Model:

- The input sets $S_1, \ldots, S_m$ are presented one by one in a stream.
- The streaming algorithm has a small space to maintain a summary of the input sets.
- At the end, the algorithm outputs an exact/approximate set cover using this summary.
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Introduced originally by [SG09] and further studied in several recent works [ER14, DIMV14, IMV15, CW16, HPIMV16].
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Remark. We are not concerned with poly-time computability in this model.
# State of the Art for Single-Pass Algorithms

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**Single-pass Algorithms:**

- $o(m)$ space regime is settled by the results of [ER14].
- However, sublinear space regime, that is, what can be done in $o(mn)$ space is wide open.
  - For example, is $O(1)$ approximation possible in $o(mn)$ space?
  - In general, what is the space-approximation tradeoff in this regime?
Our First Result

A tight space-approximation tradeoff for single-pass streaming algorithms:

Theorem

For any $\alpha = o(\sqrt{n})$, $\tilde{\Theta}(mn/\alpha)$ space is both sufficient and necessary for $\alpha$-approximating the set cover problem.
\(\alpha\)-Approximation in \(\tilde{O}(mn/\alpha)\) space

A simple algorithm for (weighted) set cover:

1. Guess \text{OPT} and ignore sets with weight > \text{OPT}.
\( \alpha \)-Approximation in \( \tilde{O}(mn/\alpha) \) space

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1. **Guess** OPT and ignore sets with weight \( > \) OPT.
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   (at most \( \alpha \) sets would be included with total weight \( \leq \alpha \cdot \text{OPT} \))
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3. Store all remaining sets over the new universe.
   (each remaining set contains < \(n/\alpha\) elements and hence they can all be stored in \(O(mn/\alpha)\) space)
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4. Solve the store set cover instance optimally to cover the elements remained uncovered by the prune step.
\( \alpha \)-Approximation in \( \tilde{O}(mn/\alpha) \) space

A simple algorithm for (weighted) set cover:

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4. Solve the store set cover instance optimally to cover the elements remained uncovered by the prune step.

Our lower bound shows that this simple algorithm is essentially the best possible in terms of space requirement!
Approximation vs Estimation

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However, our $\Omega(mn/\alpha)$ lower bound strongly relies on the fact that we are solving the approximation problem and not simply estimating the value of the optimal set cover.
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However, our $\Omega(mn/\alpha)$ lower bound strongly relies on the fact that we are solving the approximation problem and not simply estimating the value of the optimal set cover.

Question: Can it be that estimation is strictly easier than approximation?
Our Second Result

Estimation is indeed distinctly easier!

Theorem

For any $\alpha = o(\sqrt{n})$, there exists a randomized $\alpha$-estimation $\tilde{O}(mn/\alpha^2)$ space algorithm for the streaming set cover problem.

Works in general for any covering integer program, and in particular for weighted set-cover or set multi-cover problem.
Our Third Result

The factor $\alpha$ gap between space requirements of approximation versus estimation algorithms for streaming set cover is tight.

**Theorem**

For any $\alpha = o(\sqrt{n})$, any randomized algorithm that $\alpha$-estimates the set cover problem requires $\tilde{\Omega}(mn/\alpha^2)$ space.

This lower bound holds even for random arrival streams.
$\Omega(mn/\alpha)$ Space is Necessary to Compute an $\alpha$-Approximate Set Cover
Communication Complexity

We use communication complexity paradigm to prove our lower bound.
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One-way Two-player Communication Model:

- Alice gets a private input $X$ and Bob gets a private input $Y$.
- Their goal is to compute a function $P(X, Y)$.
- Alice is allowed to send a single message $M$ to Bob.
- Bob uses the message $M$ plus his input to compute $f(M, Y) \approx P(X, Y)$. 

Communication Complexity $CC(P)$: the minimum length of a message for any protocol that solves $P$ with probability at least $2/3$. 

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Communication Complexity

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Connection to Streaming Complexity

Space needed by any streaming algorithm for a problem $P$ is at least the communication complexity of $P$. 

\[ A(s_1) \quad A(s_1 \circ s_2) \]

- Alice
- Bob

Stream:

\[ S_1 \quad S_2 \]
A Hard Input Distribution for Set Cover

Theorem

$CC(\alpha\text{-Approximate Set Cover}) = \Omega(mn/\alpha)$
A Hard Input Distribution for Set Cover

Theorem

\[ CC(\alpha\text{-Approximate Set Cover}) = \Omega(mn/\alpha) \]

- Alice and Bob each gets a collection of sets.
- Alice sends a single message to Bob and Bob outputs an \(\alpha\)-approximate set cover.
A Hard Input Distribution for Set Cover

Input Distribution $D$: [Diagram of a grid with $n$ dots]
A Hard Input Distribution for Set Cover

Input Distribution $D$: 

- **Alice**: near orthogonal sets of size $n/\alpha$. 

$[n]$
Input Distribution $\mathcal{D}$:

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![Diagram](image)
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![Diagram showing set $T$ and set $[n]$]
A Hard Input Distribution for Set Cover

Input Distribution $\mathcal{D}$:

- Alice: a collection of $m$ sets $S_1, \ldots, S_m$. 

$S_1, \ldots, S_m$
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The optimal set cover size is at most 3:
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Use $T, S_{i^*}$, and one more set for covering the special element.
Proof Sketch

Why $D$ is a hard distribution?

Claim

Solving set cover on $D$ is equivalent to identifying the special element.
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Why $\mathcal{D}$ is a hard distribution?

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Solving set cover on $\mathcal{D}$ is equivalent to identifying the special element.

Bob can identify the set $S_{i^*}$ with small communication.
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Why \( \mathcal{D} \) is a hard distribution?

**Claim**

*Solving set cover on \( \mathcal{D} \) is equivalent to identifying the special element.*

1. Bob can identify the set \( S_{i*} \) with small communication.
2. Bob knows using \( T \) and \( S_{i*} \) he can cover all but a single element, i.e., the special element \( e \).
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Solving set cover on $\mathcal{D}$ is equivalent to identifying the special element.

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2. Bob knows using $T$ and $S_i^*$ he can cover all but a single element, i.e., the special element $e$.
3. Bob’s task is then to identify the special element in $\overline{T}$.
   Identify = find a small enough subset of $\overline{T}$ that contains $e$.
   In other words, trap the special element $e$. 

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   - Identify = find a small enough subset of $\overline{T}$ that contains $e$.
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4. Bob can then cover the trap-set using sets other than $S_{i^*}$. 
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How small is \textit{small enough} for the trap-set size?
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Optimal set cover size is at most 3, hence Bob is allowed to use up to $3\alpha$ sets in the set cover.
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3. The near orthogonality of the sets implies that the trap-set has to be of size $< 3\alpha$. 
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Why $\mathcal{D}$ is a hard distribution?

Claim

Suppose Alice only has a single set, i.e., only $S_{i^*}$; then, trapping the special element requires full knowledge of Alice’s set.

Trap problem: the communication problem of trapping the special element, when Alice has a single set $S$ and Bob has a single set $A \cup \{e\}$. 
Intuitively, if Alice sends $o\left(\frac{n}{\alpha}\right)$ bits, only $o(1)$ fraction of the set $S$ is revealed to Bob. Since $A$ is chosen uniformly at random from $S$, Bob can only determine $o(1)$ fraction of $A$ that belongs to $S$. Consequently, Bob can only trap the special element by a set of size $(1 - o(1))|A| > \frac{3}{\alpha}$.

We formalize this using an information-theoretic argument and a novel reduction from the Index problem.
Proof Sketch

**Lemma**

\[ \text{CC(Trap)} = \Omega(n/\alpha) \]

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Claim

When $i^\ast$ is not known to Alice, trapping the special element requires $m$ times more communication:

$$\text{CC}(\alpha\text{-Approximate Set Cover}) \approx m \cdot \text{CC}(\text{Trap})$$
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- The index $i^*$ is unknown to Alice, hence Alice’s message essentially needs to solve Trap for most indices $i \in [m]$. 
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2. The sets are chosen independently, hence information sent for one set cannot be used for solving Trap on another set.

We formalize this using information complexity and a direct-sum style argument.
Summary

Hence,

\[ \text{CC}(\alpha\text{-Approximate Set Cover}) \approx \Omega\left(\frac{mn}{\alpha}\right) \]

Communication complexity is also a lower bound on the space complexity of the streaming algorithms:

**Theorem**

*For any \( \alpha = o(\sqrt{n}) \), \( \Omega\left(\frac{mn}{\alpha}\right) \) space is necessary for \( \alpha \)-approximating the set cover problem.*

Moreover, this space-approximation tradeoff is *tight*. 
$\tilde{O}(mn/\alpha^2)$ Space is Sufficient for $\alpha$-Estimating Set Cover
An $\alpha$-Estimation Algorithm in $\tilde{O}(mn/\alpha^2)$ Space

We show that,

Theorem

There exists a single-pass streaming that $\alpha$-estimates the weighted set cover problem in $\tilde{O}(mn/\alpha^2)$ space.

These ideas can be further generalized to estimate optimal solution value of any covering integer program.
\(\alpha\)-Approximation in \(\tilde{O}(mn/\alpha)\) space

A simple algorithm for (weighted) set cover:

1. Guess OPT and ignore sets with weight \(>\) OPT.
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Element Sampling

How to save another factor $\alpha$ to achieve $O(mn/\alpha^2)$ when the goal is only estimating?
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Element Sampling:

- Sample each element with probability $1/\alpha$ and work with the sampled universe in the second phase of the algorithm.
- Store the sampled instance completely (after pruning).
  (each set has $\leq n/\alpha^2$ elements in the sampled universe and hence total space requirement is $O(mn/\alpha^2)$)

The hope is that the sampling procedure reduces the weight of the optimal set cover by a factor of at most $\alpha$. 
Element Sampling

Let

- $\mathcal{I}$ be an instance of the weighted set cover problem.
- $\mathcal{I}_\alpha$ be an instance obtained from $\mathcal{I}$ by sampling each element of the universe $[n]$ with probability $1/\alpha$. 

Clearly, $\text{OPT}(\mathcal{I}_\alpha) \leq \text{OPT}(\mathcal{I})$.

Ideally, we also want $\text{OPT}(\mathcal{I}_\alpha) \geq \text{OPT}(\mathcal{I})/\alpha$ with probability $\Omega(1)$.

This way, we can use $\text{OPT}(\mathcal{I}_\alpha)$ as a proxy for $\text{OPT}(\mathcal{I})$. But is this true?
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This is not true in general.

Consider the following instance $\mathcal{I}$ with $n$ sets:

- $S_1 = \{1\}$ with weight $W \gg n$.
- $S_i = \{i\}$ for $i > 1$ with weight 1.

Clearly,

- $\text{OPT}(\mathcal{I}) = (n - 1) + W$
- $\Pr \left[ \text{OPT}(\mathcal{I}_\alpha) \geq \text{OPT}(\mathcal{I})/\alpha \right] = o(1)$
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- $\text{OPT}(\mathcal{I}) = (n - 1) + W$
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The problem is existence of elements that are too expensive to cover.
Element Sampling Lemma

- For each element $e \in [n]$, define $\text{Cost}(e)$ to be the minimum weight of any set that covers $e$.
- Define $\text{Cost}(\mathcal{I}) := \max_{e \in [n]} \text{Cost}(e)$.

$\text{Cost}(\mathcal{I})$ is clearly a lower bound on $\text{OPT}(\mathcal{I})$. 

Lemma (Element Sampling Lemma)

For any instance $\mathcal{I}$, let $\mathcal{I}_\alpha$ be an instance obtained by sampling each element independently with probability $\ln(n) / \alpha$, then,

\[ \Pr[\text{OPT}(\mathcal{I}_\alpha) + \text{Cost}(\mathcal{I}) \geq \text{OPT}(\mathcal{I}_\alpha)] \geq \frac{1}{2} \]

Sepehr Assadi (Penn)
Element Sampling Lemma

- For each element $e \in [n]$, define $\text{Cost}(e)$ to be the minimum weight of any set that covers $e$.
- Define $\text{Cost}(\mathcal{I}) := \max_{e \in [n]} \text{Cost}(e)$.

$\text{Cost}(\mathcal{I})$ is clearly a lower bound on $\text{OPT}(\mathcal{I})$. 
**Element Sampling Lemma**

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**Lemma (Element Sampling Lemma)**

For any instance $\mathcal{I}$, let $\mathcal{I}_\alpha$ be an instance obtained by sampling each element independently with probability $\frac{\ln(n)}{\alpha}$, then,

\[
\Pr \left[ \text{OPT}(\mathcal{I}_\alpha) + \text{Cost}(\mathcal{I}) \geq \frac{\text{OPT}(\mathcal{I})}{\alpha} \right] \geq \frac{1}{2}
\]
Upper Bound Statement

**Theorem**

For any $\alpha = o(\sqrt{n})$, $\tilde{\Theta}(mn/\alpha^2)$ space is sufficient for $\alpha$-estimating the weighted set cover problem.

Moreover, this space-estimation tradeoff is tight.
Summary of Our Results

For the set cover problem in single-pass streams,
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\(\alpha\)-approximation:

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For the set cover problem in single-pass streams,

$\alpha$-approximation:

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Summary of Our Results

For the set cover problem in single-pass streams,

$\alpha$-approximation:
\[ \widetilde{\Theta}(mn/\alpha) \text{ space is necessary and sufficient.} \]

$\alpha$-estimation:
\[ \widetilde{\Theta}(mn/\alpha^2) \text{ space is necessary and sufficient.} \]

Our results resolve the space-complexity of set cover in single-pass streams.
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