Provider-Customer Coalitional Games

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Abstract— **Efficacy of proliferation of commercial wireless networks can be substantially enhanced through large scale cooperation among different providers. If a group of providers cooperate by allowing customers to be served by the resources of the whole group rather than just those of their own providers, they have the potential to utilize their resources more efficiently and enhance the quality of service they can offer. This in turn can result in higher profits for the providers. Such cooperation can, however, be successfully implemented if providers in a coalition judiciously allocate the resources, such as spectrum and base stations, accesspoints, etc., in a way that the individuals payoffs are commensurate to the resources they offer to the coalition. Initially, we assume that providers do not share their payoffs. We formulate this problem as a nontransferable payoff coalitional game and show that there exists a cooperation strategy that leaves no incentive for any subset of providers to split from the grand coalition, i.e., the *core* is nonempty. To compute this cooperation strategy and the corresponding payoffs, we subsequently relate this game and its core to an exchange market setting, and its equilibrium which can be computed by several practically efficient algorithms. Next, we investigate cooperation in a scenario, where customers are also decision makers and decide which provider to subscribe to, based on whether there is cooperation. We then formulate a coalitional game in this setting and show that it has a nonempty core. Finally, we extend previous results to the cases, where individuals assume more general payoff sharing relations, and their benefits are modeled as "vector payoff functions", comprised of mixed transferable and nontransferable components.**

I. Introduction

We have witnessed a significant growth in commercial wireless services in the past few years, and the trend is likely to continue in the foreseeable future. This growth has been in part fueled by demand for new services such as network games and multimedia transmissions. These services are taxing the available transmission resources which are either limited (e.g., spectrum, transmission energy), or costly (e.g., infrastructure). Cooperation among service providers has the potential to substantially improve the resource utilization, and should therefore facilitate the proliferation of wireless services.

To serve its customers, each provider uses (i) wireless spectrum that it acquires either directly from central regulators such as the FCC or in secondary markets from other providers that have already licensed this spectrum from the regulators, and (ii) infrastructure such as base station, access points, mesh points (which we refer to as service units) that it deploys in its coverage area. Cooperation between providers entails pooling and sharing some of these resources to ultimately better serve each others customers. This pooling and sharing of resources can improve coverage and throughput, which can in turn lead to higher customer satisfaction and higher revenues for the providers.

We now describe the benefits of cooperation among service providers. When different providers cooperate, their resources such as spectrum and infrastructure are likely to be optimally utilized. For example, if a provider's resources exceed traffic demands of its customers, it can use the underutilized portion to serve customers of other providers in its coalition, and enhance its profit. Similarly, even when its resources are congested, owing to poor propagation quality in the spectrum it owns, or temporary demand overload, it can deliver the desired quality of service to its customers using the resources of its collaborators. Such sharing turns out to be mutually beneficial as different providers are unlikely to experience poor quality of transmission and overload at the same time. Similarly, providers can augment their coverage areas by utilizing each others service units. Thus, overall, the providers can substantially enhance their net payoffs by cooperating.

The success of this setup, however, is contingent on whether providers, as selfish entities, find the cooperation worthwhile. More specifically, a provider expects to receive a payoff commensurate to the resources such as service units and channels it offers the coalition, and the wealth it generates. Design of rational cooperation strategies is imperative to motivate providers to participate in such cooperation. In particular, the cooperation strategy of each provider involves the determination of which providers to cooperate with and how to cooperate with providers (i.e., the allocations of the service units and the spectrum to the customers). Different choices for these decision variables determine the individual payoffs and the efficacy of cooperation. Also, collaborating providers may be able to share their profits. But again, some providers may only be willing to collaborate so as to enhance individual profits, but may not be willing to share the profits, due to lack of trust, the nontransferable nature of the profit, or others. In general, providers' total benefit could be a function of different types of payoffs, and they may be willing to share some types but not the rest. In the end, the payoff allocations also depend on whether and how providers share their payoffs resulting from the cooperation. Finally, cooperation among providers could have negative effects on the customer base of some providers. A successful cooperation strategy may as well be required to guarantee that such potential downside of cooperation does not outweigh its upside.

We present a coalitional game framework for cooperation

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among providers in a single-hop network using tools from cooperative game theory (section III). In particular, we first investigate providers' cooperative resource allocation in the scenario, where providers do not engage in payoff sharing. Using this framework, we show that there exists an operating point and corresponding payoffs that renders it optimal for all providers to cooperate. Specifically, if a group of providers leave the grand coalition, regardless of how they cooperate among each other, at least one provider or customer will be worse off (section IV). In the cooperative game theory terminology, this is equivalent to saying that the core of the game is nonempty.

To compute such an operating point, we next construct an "exchange market" setting, where service providers are considered to be agents in the market, and service units and channels are the goods (section V). Agents will then trade goods so as to maximize their own benefits. We show that in this setting, market equilibrium exists. Furthermore, we show that the allocation of goods in the economy given by the equilibrium can be translated to a cooperation strategy among providers with the corresponding payoffs in the core. As a result, we can compute an element of the core, by computing the market equilibrium, which is possible by using several available algorithms. This result is also of independent interest, as it links two different concepts in this context.

Next we study cooperation in a scenario, where customers are also decision makers. Particularly, customers can subscribe to the provider of their choice, and that choice can depend on providers' cooperation decisions (section VI). We propose a cooperation model and show that the core of this game is nonempty. Subsequently, we examine an algorithm to obtain a core element in this game.

Finally, we generalize our framework to accommodate a) more general payoff sharing rules, such as when there are groups of providers, and providers in each group would share their payoffs, while those in different groups would not, and b) vector payoff functions that are comprised of mixed transferable and nontransferable components of different types. We formulate two coalitional games using the above generalizations (section VII). Subsequently, we show that the previous results extend to these scenarios as well.

II. Related Work

Coalitional games have been used recently for modeling cooperation among nodes in the physical layer [1], [2], rate allocation in multiple access channels (MAC) [3], and studying cooperation between single antenna receivers and transmitters in an interference channel [4]. In this work, we instead focus on cooperative resource allocation by providers at the network and MAC layers.

Cooperation among providers in wireless networks has been previously studied in [5], using transferable payoff coalitional games. In the current work, we generalize formulations and results in [5], so as to consider (i) the scenario, where customers are also players in the game, (ii) more general payoff sharing rules, and (iii) vector payoff functions with components of various types. In order to obtain these results, we need different tools and analytical techniques, e.g., nontransferable payoff coalitional games.

III. System Model

Consider a network with a set of providers $\mathcal{N} = \{1, 2, \ldots, N\}$ and a set of customers $\mathcal{M} = \{1, 2, \ldots, M\}$. Each provider i, owns a set of service units $\mathcal{B}_i$ and has a set of subscribed customers $\mathcal{M}_i$. We assume that $\mathcal{B}_i \cap \mathcal{B}_j = \mathcal{M}_i \cap \mathcal{M}_j = \emptyset, i, j \in \mathcal{N}, i \neq j$. Providers can then use their service units to deliver service to their customers. We assume that service units and customers can communicate via single-hop links.

We assume that each service unit has access to a single channel. This assumption, however, causes no loss of generality. This is because in the case where service units have access to multiple channels with a radio available for every channel, each service unit channel combination can be used as a unit in $\mathcal{B}_i$, instead of a real service unit. We will see later how the case where service units have limited number of radios can also be captured by modifying the feasibility constraints. We also assume that no two service units in a vicinity have access to the same frequency band. As a result, communications of different service units with different customers do not interfere with each other. For instance, consider two service units k_1 and k_2. Then these service units either have access to different channels, or else, they are far apart and therefore do not cause any interference for each other. This interference model implies that links forming a matching can be scheduled simultaneously. Therefore, since the graph of the network is bipartite, the necessary and sufficient condition for a feasible schedule is that the fraction of time each node communicates is below 1 [6].

The instantaneous rates the customers receive depend on the current quality of the channels accessed by the service units (which in case of secondary access channels also includes the current actions of the channels' primary users) and the current positions of the customers, which can be random. We therefore assume that when customer j is served by service unit k, j receives a rate r_{kj}, a random variable which is a function of the state of channel k and position of customer j. Let Ω_{kj} be the state space of r_{kj}. We assume that $|\Omega_{kj}|$ is finite. This assumption is motivated by the fact that feasible service rates in any practical communication system belong to a finite set. Thus, we assume that each channel has a finite number of states. Also, we can partition the service region in such a way that the service rates received by the customers inside a member of the partition do not depend on the locations of the customers. Let $\Omega = \prod_{\substack{k \in \mathcal{B}_\mathcal{N} \\ j \in \mathcal{M}_\mathcal{N}}} \Omega_{kj}$ and $P(\omega)$ be the probability of an outcome $\omega \in \Omega$.

Definition 3.1: A coalition $\mathcal{S} \subseteq \mathcal{N}$, is a group of providers who cooperate. For a coalition $\mathcal{S}$, let $\mathcal{B}_\mathcal{S}$ and $\mathcal{M}_\mathcal{S}$ denote the set of service units and customers associated with providers in $\mathcal{S}$. The term *grand coalition* refers to the coalition $\mathcal{N}$.

When providers cooperate, they allow their service units to serve customers subscribed to other providers. Such cooperation has the potential to enhance the quality of service to customers, which in turn can increase providers' payoffs.

A service unit owned by provider i can serve a customer j only if both are associated with the same provider, or the providers associated with them are in a coalition. Let $\alpha_{kj} \in [0,1]$ be the fraction of time service unit k serves customer j. α_{kj}s are determined by the allocation scheme. Let $y_{kj}^{\mathcal{S}}$ denote the rate customer j receives from service unit k when $j \in \mathcal{M}_{\mathcal{S}}$. Then, if $k \in \mathcal{B}_{\mathcal{S}}$, and the outcome is ω, $y_{kj}^{\mathcal{S}}(\omega) = \alpha_{kj}(\omega) r_{kj}(\omega)$. Else, $y_{kj}^{\mathcal{S}}(\omega) = 0, \forall \omega \in \Omega$. Now consider a coalition $\mathcal{S}$ and a provider i in $\mathcal{S}$. Define the rate vector $\mathbf{y}_i^{\mathcal{S}}(\omega) = (y_{kj}^{\mathcal{S}}(\omega), k \in \mathcal{B}_{\mathcal{N}}, j \in \mathcal{M}_i)$, and $\mathbf{y}_i(\omega) = \mathbf{y}_i^{\mathcal{N}}(\omega)$. We assume that when provider i is in coalition $\mathcal{S}$, the outcome is ω, and $\{\alpha_{kj}, k \in \mathcal{B}_{\mathcal{S}}, j \in \mathcal{M}_S\}$ is the allocation, provider i receives a payoff equal to $f_i(\mathbf{y}_i^{\mathcal{S}}(\omega))$, which is the difference between the revenue i receives from, and the costs (e.g., power consumption) it incurs by serving its customers. $f_i(.)$s are assumed to be concave functions. Then, the expected payoff provider $i \in \mathcal{S}$ earns will equal $\sum_{\omega \in \Omega} \mathbb{P}(\omega) f_i(\mathbf{y}_i^{\mathcal{S}}(\omega))$.

For a coalition $\mathcal{S}$, the allocation $\{\alpha_{kj}, k \in \mathcal{B}_{\mathcal{S}}, j \in \mathcal{M}_S\}$ is feasible if it satisfies the following conditions.

1) $\sum_{j \in \mathcal{M}_{\mathcal{S}}} \alpha_{kj}(\omega) \leq 1, k \in \mathcal{B}_{\mathcal{S}}, \omega \in \Omega$
2) $\sum_{k \in \mathcal{B}_{\mathcal{S}}} \alpha_{kj}(\omega) \leq 1, j \in \mathcal{M}_{\mathcal{S}}, \omega \in \Omega$
3) $\alpha_{kj} \geq 0, k \in \mathcal{B}_{\mathcal{S}}, j \in \mathcal{M}_{\mathcal{S}}$.

Constraints (1) ensure that each service unit k communicates less than 1 unit of time[1]. The fraction of time customer j is served is below 1 by (2). Any allocation $\{\alpha_{kj}, k \in \mathcal{B}_{\mathcal{S}}, j \in \mathcal{M}_S\}$ that satisfies $1-3$ is called a feasible joint action of coalition $\mathcal{S}$. Note that for any feasible allocation $\{\alpha_{kj}, k \in \mathcal{B}_{\mathcal{S}}, j \in \mathcal{M}_{\mathcal{S}}\}$, there is a schedule that allocates service units to customers, ensuring that for all $k \in \mathcal{B}_{\mathcal{S}}, j \in \mathcal{M}_{\mathcal{S}}$, service unit k serves customer j an amount of α_{kj} unit of time, by [6]. Let $A(\mathcal{S})$ denote the joint action space of providers in coalition $\mathcal{S}$.

Consider a joint action $a \in A(\mathcal{S})$ and a vector of payoffs $\mathbf{x} \in \mathbb{R}^{|\mathcal{S}|}$. Now define $\mathcal{F}^{\mathcal{S}}(a)$ to be the payoff vector generated by joint action a. That is $\mathbf{x} = \mathcal{F}^{\mathcal{S}}(a)$ if a) $\mathbf{x}_i = \sum_{\omega \in \Omega} \mathbb{P}(\omega) f_i(\mathbf{y}_i^{\mathcal{S}}(\omega)),\ \forall i \in \mathcal{S}$, and b) $(\mathbf{y}_i^{\mathcal{S}}, i \in \mathcal{S})$ is the vector of service rates resulting from the joint action a.

Associated with each coalition $\mathcal{S}$, there is a set of feasible payoff vectors $v(\mathcal{S})$ defined as:

$$v(\mathcal{S}) = \{\mathbf{x} \in \mathbb{R}^{|\mathcal{S}|} : \exists \mathbf{z} \in \mathbb{R}^{|\mathcal{S}|}, \mathbf{z} \geq \mathbf{x},\ \mathbf{z} = \mathcal{F}^{\mathcal{S}}(a) \text{ for some } a \in A(\mathcal{S})\}. \tag{1}$$

In other words, $v(\mathcal{S})$ contains all payoff vectors which are less than or equal to at least one payoff vector generated by a feasible joint action. Any $\mathbf{x} \in v(\mathcal{S})$ is called a feasible payoff profile. Now the stage is set for the following definition.

Definition 3.2: A nontransferable payoff cooperative (NTU) game consists of a pair $(\mathcal{N}, v)$, where $\mathcal{N}$ is the set of players, and $v(\mathcal{S})\ \forall \mathcal{S} \subseteq \mathcal{N}$ is the set of feasible payoff profiles satisfying

1) For each $\mathcal{S}$, $v(\mathcal{S})$ is a closed set.
2) If $\mathbf{z} \in v(\mathcal{S})$ and $\mathbf{x} \in \mathbb{R}^{|\mathcal{S}|}$ with $\mathbf{x} \leq \mathbf{z}$, then $\mathbf{x} \in v(\mathcal{S})$.
3) The set of vectors in $v(\mathcal{S})$ in which each player in $\mathcal{S}$ receives no less than the maximum that he can obtain by himself is a nonempty, bounded set.

If providers agree to cooperate and form the grand coalition, they can take any feasible joint action, and consequently, achieve any payoff profile in $v(\mathcal{N})$. However, there is a need for a criterion that determines which payoff profile in $v(\mathcal{N})$ will be acceptable to the providers.

Definition 3.3: A payoff profile $\mathbf{x} \in v(\mathcal{N})$ is said to be *blocked* by coalition $\mathcal{S}$, if there is a payoff profile $\mathbf{z} \in v(\mathcal{S})$ such that $\mathbf{z}_i > \mathbf{x}_i$ for all $i \in \mathcal{S}$, i.e., $\mathbf{z}$ makes every provider in $\mathcal{S}$ better off.

Note that providers in $\mathcal{S}$ can object to a payoff profile $\mathbf{x} \in v(\mathcal{N})$ that is blocked by $\mathcal{S}$.

We use the well known solution concept in coalitional game theory, namely the *core*. The idea behind the core in a cooperative game is analogous to that behind a Nash equilibrium of a noncooperative game: an outcome is stable if no deviation is profitable. Roughly speaking, a payoff profile is in the core, if no sub-group of providers have any incentive to split from the grand coalition. In other words, the core $\mathcal{C}$ of the game is the set of all feasible payoff profiles which cannot be blocked by any coalition. That is,

$$\mathcal{C} = \{\mathbf{x} \in v(\mathcal{N}) : \forall \mathcal{S}, \nexists \mathbf{z} \in v(\mathcal{S}) \text{ such that } \mathbf{z}_i > \mathbf{x}_i\ \forall i \in \mathcal{S}\} \tag{2}$$

We now discuss the importance of the concept of core in coalitional games. Suppose $\mathcal{C} \neq \emptyset$. Let providers form the grand coalition and select a joint action corresponding to a payoff profile $\mathbf{x}$ in the core. Now suppose a group of providers $\mathcal{S}$ leave the grand coalition and choose their own joint action and a corresponding payoff profile $\mathbf{z} \in v(\mathcal{S})$. However, they will do so only if they all receive a higher payoff than what they could in the grand coalition, i.e., $\mathbf{z}_i > \mathbf{x}_i, i \in \mathcal{S}$. But this is in contradiction with the fact that $\mathbf{x} \in \mathcal{C}$. In other words, no group of provider has any incentive to split from the grand coalition. Thus the grand coalition is stable, which is a desirable outcome since the grand coalition has the potential to achieve higher efficiency.

Example 3.1: Consider a network setup with 3 providers $\mathcal{N} = \{1,2,3\}$, and 1 state $|\Omega| = 1$. Suppose each provider has 1 service unit $\mathcal{B}_i = \{i\}, i = 1,2,3$, and 1 customer $\mathcal{M}_i = \{i\}, i = 1,2,3$. Now let the service rates be as follows. $r_{jj} = 1 \forall j$, $r_{kj} = 3$ when $j = (k+1 \mod 3)\ \forall k$, and $r_{kj} = 2$ when $j = (k-1 \mod 3)\ \forall k$. Suppose that the payoff of each provider equals the service rate of its customers. When a provider does not cooperate, it is clear that its maximum feasible payoff equals 1. In other words, $v(\{i\}) = [0,1]\ \forall i$. For the coalition $\{1,2\}$, we can similarly specify the feasible payoff profile as $v(\{1,2\}) = \{(x_1, x_2) : x_1 \leq 2, x_2 \leq 3\}$. Finally, for the grand coalition we have $v(\{1,2,3\}) = \{(x_1,x_2,x_3) : x_i \leq 3\ \forall i\}$. It is easy to verify

[1]This condition can be modified to capture the scenario when a service unit has access to multiple channels with only 1 radio, as follows. The modified constraint (1) for a service unit, bounds the sum of α_{kj} over customers j, and channels k accessed by that service unit, by 1. It can be shown that all the results obtained later extends to this scenario as well.

that the feasible payoff profile $(3,3,3)$ it in the core, since is is not blocked by any coalition.

The core in several coalitional games is empty, i.e., the grand coalition cannot be stabilized (Example 260.3 p. 260 [7]), and in general it is NP-hard to determine whether the core of a coalitional game is nonempty([8]). In the following sections we show that the core of the game we consider is nonempty, and obtain a payoff profile in the core of the game.

IV. The Core

We now proceed to show that the core of the game $(\mathcal{N}, v)$ is nonempty.

Definition 4.1: A collection of coalitions $\mathcal{I} \subset 2^{\mathcal{N}} \backslash \emptyset$ is called *balanced* if there exist nonnegative weights $(\lambda_{\mathcal{S}}, \mathcal{S} \in \mathcal{I})$ such that

$$\sum_{\mathcal{S} \in \mathcal{I}:\ i \in \mathcal{S}} \lambda_{\mathcal{S}} = 1,\ \forall i \in \mathcal{N}.$$

Accordingly, a game is balanced if for every balanced collection $\mathcal{I}$, if $u \in \mathbb{R}^n$ and $u^{\mathcal{S}} \in v(\mathcal{S})$ for all $\mathcal{S} \in \mathcal{I}$, then $u \in v(\mathcal{N})$.[2]

Example 4.1: Let $\mathcal{N} = \{1,2,3\}$. Then $\mathcal{I}_1 = \{\{1,2\},\{2,3\},\{1,3\}\}$ is balanced since every player is exactly in two of the coalitions. So $\lambda = \frac{1}{2}$ is the balancing weight for all coalitions in $\mathcal{I}_1$. On the other hand $\mathcal{I}_2 = \{\{1,2\},\{2,3\}\}$ is not balanced, since there does not exist nonnegative λ_1 and λ_2 such that $\lambda_1 = 1$, $\lambda_1 + \lambda_2 = 1$, and $\lambda_2 = 1$.

We will make use of the following theorem which holds for any coalitional game [9].

Theorem 4.1: A balanced game always has a nonempty core.

Here is the main result.

Theorem 4.2: The coalitional game among providers, $(\mathcal{N}, v)$, is balanced and hence has a nonempty core.

Proof: Consider a balanced collection of coalitions $\mathcal{I}$. Let $(\lambda_{\mathcal{S}}, \mathcal{S} \in \mathcal{I})$ be the corresponding nonnegative weights. Also, let $u \in \mathbb{R}^{|\mathcal{N}|}$ be such that $u^{\mathcal{S}} \in v(\mathcal{S})$ for all $\mathcal{S} \in \mathcal{I}$, i.e, there exists joint action $\{\alpha^{\mathcal{S}}_{kj},\ k \in \mathcal{B}_{\mathcal{S}}, j \in \mathcal{M}_{\mathcal{S}}\}$ for all $\mathcal{S} \in \mathcal{I}$ such that

1) $\{\alpha^{\mathcal{S}}_{kj},\ k \in \mathcal{B}_{\mathcal{S}}, j \in \mathcal{M}_{\mathcal{S}}\}$ satisfies feasibility constraints $1-3$ in section III, for all $\mathcal{S} \in \mathcal{I}$.
2) $u_i \le \sum_{\omega \in \Omega} \mathbb{P}(\omega) f_i(\mathbf{y}^{\mathcal{S}}_i(\omega)), \forall i \in \mathcal{S}$, where $\mathbf{y}^{\mathcal{S}}_i$ denotes the rate vector corresponding to joint action $\{\alpha^{\mathcal{S}}_{kj},\ k \in \mathcal{B}_{\mathcal{S}}, j \in \mathcal{M}_{\mathcal{S}}\}$.

We next show that $u \in v(\mathcal{N})$. Thus, the game is balanced.

Now define a joint action set $\{\alpha_{kj},\ k \in \mathcal{B}_{\mathcal{N}}, j \in \mathcal{M}_{\mathcal{N}}\}$ as follows

$$\alpha_{kj}(\omega) = \sum_{\mathcal{S} \in \mathcal{I}:\ \substack{k \in \mathcal{B}_{\mathcal{S}} \\ j \in \mathcal{M}_{\mathcal{S}}}} \lambda_{\mathcal{S}} \alpha^{\mathcal{S}}_{kj}(\omega). \tag{3}$$

The rest of the proof consists of two steps.

Step 1: We show that $\{\alpha_{kj},\ k \in \mathcal{B}_{\mathcal{N}}, j \in \mathcal{M}_{\mathcal{N}}\}$ satisfy feasibility constraints $1-3$ in section III.

$$\begin{aligned} \sum_{j \in \mathcal{M}_{\mathcal{N}}} \alpha_{kj}(\omega) &= \sum_{j \in \mathcal{M}_{\mathcal{N}}} \sum_{\mathcal{S} \in \mathcal{I}:\ \substack{k \in \mathcal{B}_{\mathcal{S}} \\ j \in \mathcal{M}_{\mathcal{S}}}} \lambda_{\mathcal{S}} \alpha^{\mathcal{S}}_{kj}(\omega) \\ &= \sum_{\mathcal{S} \in \mathcal{I}:\ k \in \mathcal{B}_{\mathcal{S}}} \lambda_{\mathcal{S}} \sum_{j \in \mathcal{M}_{\mathcal{S}}} \alpha^{\mathcal{S}}_{kj}(\omega) \\ &\le \sum_{\mathcal{S} \in \mathcal{I}:\ k \in \mathcal{B}_{\mathcal{S}}} \lambda_{\mathcal{S}} \\ &= \sum_{\mathcal{S} \in \mathcal{I}:\ i \in \mathcal{S}} \lambda_{\mathcal{S}} \ (\text{where } k \in \mathcal{B}_i) \\ &= 1. \end{aligned}$$

The first equality follows from (3). The inequality follows by feasibility of $\{\alpha^{\mathcal{S}}_{kj}\}$ and constraint (1) in section III.

Similarly, one can show that constraint 2 is also satisfied. Constraint 3 is trivial. Thus, the joint action $\{\alpha_{kj},\ k \in \mathcal{B}_{\mathcal{N}}, j \in \mathcal{M}_{\mathcal{N}}\}$ satisfies feasibility.

Step 2: We show that $u_i \le \sum_{\omega \in \Omega} \mathbb{P}(\omega) f_i(\mathbf{y}_i(\omega)), \forall i \in \mathcal{N}$, where $\mathbf{y}_i$ is the rate vector given by joint action $\{\alpha_{kj},\ k \in \mathcal{B}_{\mathcal{N}}, j \in \mathcal{M}_{\mathcal{N}}\}$.

Using (3), it is easy to verify that $\mathbf{y}_i(\omega)$ satisfies

$$\mathbf{y}_i(\omega) = \sum_{\mathcal{S} \in \mathcal{I}: i \in \mathcal{S}} \lambda_{\mathcal{S}} \mathbf{y}^{\mathcal{S}}_i(\omega) \tag{4}$$

That is, $\mathbf{y}_i$ is the convex combination of $\{\mathbf{y}^{\mathcal{S}}_i, \mathcal{S} \in \mathcal{I} : i \in \mathcal{S}\}$. Since $f_i(.)$s are concave, for each provider i we have:

$$\begin{aligned} \sum_{\omega \in \Omega} \mathbb{P}(\omega) f_i(\mathbf{y}_i(\omega)) &\ge \sum_{\omega \in \Omega} \mathbb{P}(\omega) \sum_{\mathcal{S} \in \mathcal{I}: i \in \mathcal{S}} \lambda_{\mathcal{S}} f_i(\mathbf{y}^{\mathcal{S}}_i(\omega)) \\ &= \sum_{\mathcal{S} \in \mathcal{I}: i \in \mathcal{S}} \lambda_{\mathcal{S}} \sum_{\omega \in \Omega} \mathbb{P}(\omega) f_i(\mathbf{y}^{\mathcal{S}}_i(\omega)) \\ &\ge \sum_{\mathcal{S} \in \mathcal{I}: i \in \mathcal{S}} \lambda_{\mathcal{S}} u_i \\ &= u_i \end{aligned}$$

It now follows from theorem 4.1 that the core of the game is nonempty. ■

V. Computation of a Payoff Profile in the Core

In the previous section, it was shown that the NTU game $(\mathcal{N}, v)$ has a nonempty core. Another interesting issue now is computing a payoff profile in the core $\mathcal{C}$. Towards that end, we construct an "exchange market" setting, a concept borrowed from microeconomics (section V-A). Next we show that the *market equilibrium* in this setting, if existent, corresponds to a payoff profile in the core of our NTU game (section V-B). Note that the fact that two different concepts are equivalent in this context, can be of independent interest. Finally, we show that the equilibrium exists, which can be computed using several available algorithms (section V-C).

A. Exchange Market Preliminaries

We now proceed to introduce the "exchange market" concept from microeconomic theory. Consider an exchange market with $\mathcal{N} = \{1, \ldots, N\}$ as the set of agents. Let $\mathcal{L} = \{1, \ldots, L\}$ denote the set of goods in the markets. Each

[2] For any $u \in \mathbb{R}^n$, $u^{\mathcal{S}} \in \mathbb{R}^{|\mathcal{S}|}$ is defined by $u^{\mathcal{S}}_i = u_i, \forall i \in \mathcal{S}$.

agent has a positive initial endowment of the goods given by the vector $\mathbf{e}_i = (e_i^1, \ldots, e_i^L)$. Associated with each agent i is a utility function $u_i(.) : \mathbb{R}_+^L \rightarrow \mathbb{R}$, where $u_i(\mathbf{x}_i)$ represents the satisfaction level of agent i from the allocation of goods $\mathbf{x}_i = (x_i^1, \ldots, x_i^L)$. Now let vector $\mathbf{p} = (p_1, \ldots, p_L)$ denote the price of goods in the market. The agents will then try to maximize their utilities through trading of goods according to prices given by $\mathbf{p}$. We can now present the definition of the market equilibrium (p.579 [10]).

Definition 5.1: An allocation $\mathbf{x}^*$ and a price vector $\mathbf{p} = (p_1, \ldots, p_L)$ constitute a market equilibrium if

i) $\mathbf{x}_i^* \in \arg\max_{\mathbf{x}_i \in \mathbb{R}_+^L} u_i(\mathbf{x}_i)$ subject to $\mathbf{p}.\mathbf{x}_i \leq \mathbf{p}.\mathbf{e}_i$; $\forall i \in \mathcal{N}$. Note that $\mathbf{p}.\mathbf{e}_i$ is the value of agent i's endowment, which clearly cannot be larger that the value of his allocation after trading (budget constraint).

ii) $\sum_i (\mathbf{x}_i^* - \mathbf{e}_i) = 0$, that is, it is possible to provide the agents' desired allocation, just by using the total endowments present in the market (market clearing).

The following well known theorem provides a sufficient condition for a market equilibrium to exist (p.585 [10]).

Theorem 5.1: Suppose that for every consumer i, $u_i(.)$ is continuous, strictly concave, and strictly increasing. Suppose also that $\sum_i \mathbf{e}_i \in \mathbb{R}_{++}^L$. Then a market equilibrium exists, with the property that the price vector is strictly positive, i.e., $\mathbf{p} \in \mathbb{R}_{++}^L$.

Now suppose instead of trading, agents pool their goods and reallocate them among each other. The amount of goods allocated to each agent in such manner has to be commensurate with agents initial endowments, or some agents would not agree to it. Consequently, one can use the *core of the market* concept as a policy to determine the allocation of goods among agents as follows. An allocation $\mathbf{x}^* = (x_1^*, \ldots, x_N^*) \in \mathbb{R}_+^{LN}$ is in the core of the market, $\mathfrak{C}$, if is cannot be blocked by any coalition of agents $\mathcal{S} \subseteq \mathcal{N}$, i.e., for all $\mathcal{S} \subset \mathcal{N}$, there does not exist an allocation $\mathbf{x}_i$ with the properties:

i) $u_i(\mathbf{x}_i) > u_i(\mathbf{x}_i^*)$ for every $i \in \mathcal{S}$.

ii) $\sum_{i \in \mathcal{S}} \mathbf{x}_i \leq \sum_{i \in \mathcal{S}} \mathbf{e}_i$.

The following well known theorem states the relation between the market equilibrium and the core of the market $\mathfrak{C}$ (p.654 [10]).

Theorem 5.2: Any market equilibrium allocation is in the core of the market $\mathfrak{C}$.

B. Relating the NTU game to An Exchange Market

Consider the NTU game defined in section III. Now think of the set of providers $\mathcal{N}$ as the agents in the market. The goods in the market will then be the right to access each of the service units in $\mathcal{B}_{\mathcal{N}}$ when the outcome is ω, given in unit of time. Subsequently, the initial endowment of the providers will be the full access to the set of service units they own. In other words, for a provider i, $e_i^{k\omega} = 1$ if service unit k belongs to i and $e_i^{k\omega} = 0$, otherwise.

Now consider an allocation of goods $\mathbf{x}$ in this setup. We define the providers' corresponding utility functions to be the maximum payoff they can obtain by serving their own customers using their access level to the service units specified by the allocation $\mathbf{x}$. In other words,
$u_i(\mathbf{x}_i) = \max \sum_{\omega \in \Omega} \mathbb{P}(\omega) f_i(\mathbf{y}_i(\omega))$
subject to:

1) $\mathbf{y}_i(\omega) = (y_{kj}(\omega), k \in \mathcal{B}_{\mathcal{N}}, j \in \mathcal{M}_i)$.
2) $y_{kj}(\omega) = \alpha_{kj}(\omega) r_{kj}(\omega), k \in \mathcal{B}_{\mathcal{N}}, j \in \mathcal{M}_i, \omega \in \Omega$.
3) $\sum_{k \in B_{\mathcal{N}}} \alpha_{kj}(\omega) \leq 1,\ j \in \mathcal{M}_i, \omega \in \Omega$.
4) $\sum_{j \in \mathcal{M}_i} \alpha_{kj}(\omega) \leq x_i^{k\omega}, k \in \mathcal{B}_{\mathcal{N}}, \omega \in \Omega$.
5) $\alpha_{kj}(\omega) \geq 0,\ k \in \mathcal{B}_{\mathcal{N}}, j \in \mathcal{M}_i$.

We next show how an allocation in the core of the market can be used to obtain a payoff profile in the core of the NTU game.

Theorem 5.3: Consider any allocation $\mathbf{x}_i, i \in \mathcal{N}$ belonging to $\mathfrak{C}$. Let $\{\alpha_{kj}^*, k \in \mathcal{B}_{\mathcal{N}}, j \in \mathcal{M}_i\}$ be an optimal solution of the optimization defining $u_i(\mathbf{x}_i)$. Now let $\mathbf{z}$ be the payoff profile corresponding to joint action $\{\alpha_{kj}^*, k \in \mathcal{B}_{\mathcal{N}}, j \in \mathcal{M}_{\mathcal{N}}\}$. Then, $\mathbf{z} \in \mathcal{C}$.

Proof: First notice that $\{\alpha_{kj}^*, k \in \mathcal{B}_{\mathcal{N}}, j \in \mathcal{M}_{\mathcal{N}}\}$ constitute a feasible joint action of providers in $\mathcal{N}$. Also note that $\mathbf{z}_i = u_i(\mathbf{x}_i)\ \forall i \in \mathcal{N}$.

Next, consider any payoff profile $\hat{\mathbf{z}} \in v(\mathcal{S})$. We argue that there exists an allocation $\hat{\mathbf{x}}_i, i \in \mathcal{S}$ such that (i) $\sum_{i \in \mathcal{S}} \hat{\mathbf{x}}_i \leq \sum_{i \in \mathcal{S}} \mathbf{e}_i$ and (ii) $u_i(\hat{\mathbf{x}}_i) \geq \hat{\mathbf{z}}_i\ \forall i \in \mathcal{S}$, as follows. Consider the joint action $\{\alpha_{kj}, k \in \mathcal{B}_{\mathcal{S}}, j \in \mathcal{M}_{\mathcal{S}}\}$, corresponding to payoff profile $\hat{\mathbf{z}}$. Now define $\hat{\mathbf{x}}_i = (\sum_{j \in \mathcal{M}_i} \alpha_{kj}(\omega), k \in \mathcal{B}_{\mathcal{S}}, \omega \in \Omega)$. Since $\{\alpha_{kj}, k \in \mathcal{B}_{\mathcal{S}}, j \in \mathcal{M}_{\mathcal{S}}\}$ is a feasible joint action of providers in $\mathcal{S}$, it satisfies constraint (1) in section III. Also, from the definition of $\hat{\mathbf{x}}_i$, it follows that $\hat{x}_i^{k\omega} = \sum_{j \in \mathcal{M}_{\mathcal{S}}} \alpha_{kj}(\omega) \leq 1, k \in \mathcal{B}_{\mathcal{S}}, \omega \in \Omega$. Now using the definition of $\mathbf{e}_i$, it is clear that (i) holds.

Next, notice that for the given $\hat{\mathbf{x}}_i$s, $\{\alpha_{kj}, k \in \mathcal{B}_{\mathcal{S}}, j \in \mathcal{M}_i\}$ is a feasible solution of the optimization defining $u_i(\hat{\mathbf{x}}_i)$, while $\hat{\mathbf{z}}_i$ is the value of its objective function. (ii) immediately follows.

We now prove the claim by contradiction. Suppose $\mathbf{z} \notin \mathcal{C}$. Then there exist a coalition $\mathcal{S}$ and a payoff profile $\hat{\mathbf{z}} \in v(\mathcal{S})$ such that $\hat{\mathbf{z}}_i > \mathbf{z}_i$ for all $i \in \mathcal{S}$. Since $\hat{\mathbf{z}} \in v(\mathcal{S})$, by the above argument, there exists an allocation $\hat{\mathbf{x}}_i, i \in \mathcal{S}$ such that (i) $\sum_{i \in \mathcal{S}} \hat{\mathbf{x}}_i \leq \sum_{i \in \mathcal{S}} \mathbf{e}_i$ and (ii) $u_i(\hat{\mathbf{x}}_i) \geq \hat{\mathbf{z}}_i\ \forall i \in \mathcal{S}$. Consequently, $u_i(\hat{\mathbf{x}}_i) > \mathbf{z}_i = u_i(\mathbf{x}_i)\ \forall i \in \mathcal{S}$. This is in contradiction with $\mathbf{x} \in \mathfrak{C}$. ■

Theorem 5.4: If the market equilibrium $\mathbf{x}^*$ in the exchange market exists, the corresponding payoff profile $(u_i(\mathbf{x}_i^*), i \in \mathcal{N})$ is in the core of the NTU game.

Proof: Using theorems 5.2, 5.3, the claim immediately follows. ■

C. Existence and Computation of The Market Equilibrium

In this section, we proceed to establish the existence of the market equilibrium in our model. We make the following technical assumptions.

1) f_is are strictly concave, strictly increasing, and smooth functions (i.e., the first two derivatives exist and are continuous).
2) For any arbitrary feasible allocation $\mathbf{x}_i$, constraint (3) in the optimization defining $u_i(.)$ is never binding.

We originally considered f_is to be concave functions. Assumption (1) imposes stronger conditions on f_is. Assumption (2), on the other hand, can be motivated by considering the number of customers high enough so that it is always sub-optimal to serve any customer the whole time.

Using assumption (2) we can rewrite the agents i's utility function $u_i(.)$ as
$u_i(\mathbf{x}_i) = \max \sum_{\omega\in\Omega} \mathbb{P}(\omega) f_i(\mathbf{y}_i(\omega))$
subject to:

1) $\mathbf{y}_i(\omega) = (y_{kj}(\omega), k \in \mathcal{B}_\mathcal{N}, j \in \mathcal{M}_i)$.
2) $y_{kj}(\omega) = \alpha_{kj}(\omega) r_{kj}(\omega), k \in \mathcal{B}_\mathcal{N}, j \in \mathcal{M}_i, \omega \in \Omega$.
3) $\sum_{j\in\mathcal{M}_i} \alpha_{kj}(\omega) \le x_i^{k\omega}, k \in \mathcal{B}_\mathcal{N}, \omega \in \Omega$.
4) $\alpha_{kj}(\omega) \ge 0$, $k \in \mathcal{B}_\mathcal{N}, j \in \mathcal{M}_i$

It follows by assumptions (1) and (2) that $u_i(.)$s are continuous, strictly increasing, strictly concave, and smooth functions. Then it immediately follows from theorem 5.1 that market equilibrium exists. Then by theorem 5.4, a payoff profile in the core of the NTU game can be obtained by computing the market equilibrium.

We next discuss how to compute a market equilibrium. For a price vector $\mathbf{p}$, define the demand vector of agent i as $\mathbf{d}_i(\mathbf{p}) = \arg\max_{\mathbf{x}_i\in\mathbb{R}_+^L} u_i(\mathbf{x}_i)$ subject to $\mathbf{p}.\mathbf{x}_i \le \mathbf{p}.\mathbf{e}_i$, that is, an allocation of goods to agent i, that maximizes his utility, subject to his budget constraint. Then the aggregate excess demand in the market is the function ξ given by $\xi(\mathbf{p}) = \sum_i (\mathbf{d}_i(\mathbf{p}) - \mathbf{e}_i)$, i.e., the aggregate demand minus the total endowment. From definition 5.1, $\mathbf{p}$ is an equilibrium price vector, if $\xi(\mathbf{p}) = 0$. This equation can be solved by using the *global Newton method* proposed in [11]. Having the equilibrium prices, the equilibrium allocation $\mathbf{x}^*$ can then be computed as the maximizer of agents utilities $u_i(\mathbf{x_i}$. The payoff vector in the NTU game corresponding to this equilibrium is then $(u_i(\mathbf{x}_i^*), i \in \mathcal{N})$, which is in the core, by theorem 5.4.

VI. Customers As Decision Makers

In this section, we consider another type of cooperative game in which customers are also players in the game and choose their providers. We formulate this problem as a nontransferable payoff coalitional game, and show that it has a nonempty core (section VI-A). We then investigate a method to compute a payoff profile in the core of this game (section VI-B).

A. An NTU Game Formulation

So far, we investigated how to incentivize providers to cooperate with each other by adopting a cooperation strategy that makes it sub-optimal for any group of providers to split from the grand coalition. We showed that such a strategy indeed exists, and as a result, the grand coalition is stabilizable. In this model, we did not consider customers' actions, and assumed that they will stick to the same provider irrespective of whether providers cooperate. However, this may not be the case if customers can decide their providers. To elucidate this further, consider a simple setting with 2 providers. Let the link rates be functions over a space with two states, i.e., $\Omega = \{\omega_1, \omega_2\}$. There is a customer who intends to subscribe to one of the 2 providers. Suppose this customer requires a minimum service rate in both states (note that this can be implemented by appropriately selecting the utility function of the customer), and that only provider 2 has the resources to satisfy it in both states. Consequently, under noncooperative regime, the customer subscribes to provider 2. Now if providers cooperate by sharing parts of their resources with each other, provider 1 may be able to satisfy the customer's constraints in both states, and offer a high enough rate in 1 state to win the customer over. In this scenario, if the customer were aware of the cooperation between the 2 providers, he would subscribe to 1, which is a different decision than in noncooperative regime. Therefore, by merely deciding to cooperate, some providers may improve their customer base, while some incur losses. At the end of the day, this very reason can weaken the cooperation scheme. To summarize this, we say that cooperation may turn out to be sub-optimal for some providers *ex ante*, although we proved in section IV that it is always *ex post* efficient.

In this section, we propose a revised cooperation strategy, in which customers are also decision makers and can subscribe to the network of their choice. Towards that end, consider the network model as in section III. We now redefine a coalition as follows.

Definition 6.1: A coalition $(\mathcal{S}, \mathcal{T}), \mathcal{S} \subseteq \mathcal{N}, \mathcal{T} \subseteq \mathcal{M}$, is a group of providers and customers who cooperate, that is each customer in $\mathcal{T}$ agrees to subscribe to one of the providers in $\mathcal{S}$, and providers in $\mathcal{S}$ will jointly serve customers in $\mathcal{T}$. The *grand coalition* now refers to the coalition $(\mathcal{N}, \mathcal{M})$.

Let $y_{kj}^{\mathcal{S},\mathcal{T}}$ denote the rate customer $j \in \mathcal{T}$ receives from service unit k. $y_{kj}^{\mathcal{S},\mathcal{T}}(\omega) = \alpha_{kj}(\omega) r_{kj}(\omega)$ if $k \in \mathcal{B}_\mathcal{S}$, and 0, otherwise. Now define $\mathbf{y}_j^{\mathcal{S},\mathcal{T}}(\omega) = (y_{kj}^{\mathcal{S},\mathcal{T}}(\omega), k \in \mathcal{B}_\mathcal{N})$. We assume that for serving customer j, provider i receives a payoff equal to $f_{ij}(\mathbf{y}_j^{\mathcal{S},\mathcal{T}})$, while customer j attains a payoff (or, utility) $g_j(\mathbf{y}_j^{\mathcal{S},\mathcal{T}})$, which can be a function of j's received rate, power consumption, etc. $f_{ij}(.)$s and $g_j(.)$s are considered to be concave functions. Consequently, the expected payoff of provider $i \in \mathcal{S}$ will be $\sum_{\substack{j\in\mathcal{T}\\ \omega\in\Omega}} \mathbb{P}(\omega) f_{ij}(\mathbf{y}_j^{\mathcal{S},\mathcal{T}}(\omega))$. Likewise, the expected payoff of customer $j \in \mathcal{T}$ is $\sum_{\omega\in\Omega} \mathbb{P}(\omega) g_j(\mathbf{y}_j^{\mathcal{S},\mathcal{T}}(\omega))$.

Similar to that in section III, we can define a feasible joint action of coalition $(\mathcal{S}, \mathcal{T})$ as an allocation $\{\alpha_{kj}, k \in \mathcal{B}_\mathcal{S}, j \in \mathcal{T}\}$ that satisfies the following conditions.

1) $\sum_{j\in\mathcal{T}} \alpha_{kj}(\omega) \le 1, k \in \mathcal{B}_\mathcal{S}, \omega \in \Omega$
2) $\sum_{k\in\mathcal{B}_\mathcal{S}} \alpha_{kj}(\omega) \le 1, j \in \mathcal{T}, \omega \in \Omega$
3) $\alpha_{kj} \ge 0, k \in \mathcal{B}_\mathcal{S}, j \in \mathcal{T}$.

Note that for any feasible allocation $\{\alpha_{kj}, k \in \mathcal{B}_\mathcal{S}, j \in \mathcal{T}\}$, there is a schedule that allocates service units to customers, ensuring that for all $k \in \mathcal{B}_\mathcal{S}, j \in \mathcal{T}$, service unit k serves customer j an amount of α_{kj} unit of time, by [6]. Let $A(\mathcal{S}, \mathcal{T})$ denote the joint action space of coalition $(\mathcal{S}, \mathcal{T})$. For a joint action $a \in A(\mathcal{S}, \mathcal{T})$, let $\mathcal{F}^{\mathcal{S},\mathcal{T}}(a)$ be the payoff profile resulting from a. We now define the set of feasible payoff profiles $v(\mathcal{S}, \mathcal{T})$ as follows.

$$v(\mathcal{S},\mathcal{T}) = \{\mathbf{x} \in \mathbb{R}^{|\mathcal{S}|+|\mathcal{T}|} : \exists \mathbf{z} \in \mathbb{R}^{|\mathcal{S}|+|\mathcal{T}|}, \mathbf{z} \geq \mathbf{x}, \\ \mathbf{z} = \mathcal{F}^{\mathcal{S},\mathcal{T}}(a) \text{ for some } a \in A(\mathcal{S},\mathcal{T})\}. \quad (5)$$

That is, $v(\mathcal{S},\mathcal{T})$ is the set of all payoff profiles which are achievable through some joint action of coalition $(\mathcal{S},\mathcal{T})$, and all the payoffs lower than those. Now according to definition 3.2, $((\mathcal{N},\mathcal{M}),v)$ is a well defined NTU game. The core of the game is then defined as follows.

$$\mathcal{C} = \{\mathbf{x} \in v(\mathcal{N},\mathcal{M}) : \forall(\mathcal{S},\mathcal{T}), \nexists \mathbf{z} \in v(\mathcal{S},\mathcal{T}) \text{ such that} \\ \mathbf{z}_i > \mathbf{x}_i, \mathbf{z}_j > \mathbf{x}_j \ \forall i \in \mathcal{S}, j \in \mathcal{T}\} \quad (6)$$

We now demonstrate the significance of the core. Suppose $\mathcal{C} \neq \emptyset$. Now let customers subscribe to the network of their choice, and together with their providers then form the grand coalition and produce a payoff profile in the core. Now suppose a group of providers $\mathcal{S}$ and customers $\mathcal{T}$ are not satisfied with this outcome. Thus they split from the grand coalition. Then each customer in $\mathcal{T}$ subscribes to one of the providers in $\mathcal{S}$ according to his own preferences, and they altogether, form the coalition $(\mathcal{S},\mathcal{T})$. But, by the definition of the core, there exists no joint action of coalition $(\mathcal{S},\mathcal{T})$ that makes every one better off. Therefore, no subset of providers and customers has any incentive to leave the grand coalition. Thus, if the core is nonempty, the grand coalition is stabilizable. Also note that the condition for a payoff profile to be in the core, as given in (6), does not depend on which provider an arbitrary customer j subscribes to. Therefore, a customer cannot improve his payoff by changing subscription, provided that the grand coalition has been formed.

We now proceed to prove that the core of this game is nonempty. Recall that by theorem 4.1, any balanced coalitional game has a nonempty core. Then is suffices to show that $((\mathcal{N},\mathcal{M}),v)$ is balanced.

Theorem 6.1: The nontransferable payoff game $((\mathcal{N},\mathcal{M}),v)$ is balanced, and therefore, has a nonempty core.

Proof: Consider a balanced collection of coalitions $\mathcal{I} = 2^{\mathcal{N}\cup\mathcal{M}}\backslash\emptyset$ and the corresponding nonnegative weights $(\lambda_{\mathcal{S},\mathcal{T}}, (\mathcal{S},\mathcal{T}) \in \mathcal{I})$. Now consider the payoff profile $u \in \mathbb{R}^{|\mathcal{N}|+|\mathcal{T}|}$ be such that $u^{\mathcal{S},\mathcal{T}} \in v(\mathcal{S},\mathcal{T})$ for all $(\mathcal{S},\mathcal{T}) \in \mathcal{I}$. In other words, there exists joint action $\{\alpha_{kj}^{\mathcal{S},\mathcal{T}},\ k \in \mathcal{B}_{\mathcal{S}}, j \in \mathcal{T}\}$ for all $(\mathcal{S},\mathcal{T}) \in \mathcal{I}$ such that

a) $\{\alpha_{kj}^{\mathcal{S},\mathcal{T}},\ k \in \mathcal{B}_{\mathcal{S}}, j \in \mathcal{T}\}$ satisfies feasibility constraints $1-3$ introduced in this section, for all $(\mathcal{S},\mathcal{T}) \in \mathcal{I}$.

b) $u_i \leq \sum_{\substack{j\in\mathcal{T}\\ \omega\in\Omega}} \mathbb{P}(\omega) f_{ij}(\mathbf{y}_j^{\mathcal{S},\mathcal{T}}(\omega)), \forall i \in \mathcal{S}$, where $\mathbf{y}_j^{\mathcal{S},\mathcal{T}}$s denote the rate vectors corresponding to joint action $\{\alpha_{kj}^{\mathcal{S},\mathcal{T}},\ k \in \mathcal{B}_{\mathcal{S}}, j \in \mathcal{T}\}$.

c) $u_j \leq \sum_{\omega\in\Omega} \mathbb{P}(\omega) g_j(\mathbf{y}_j^{\mathcal{S},\mathcal{T}}(\omega)), \forall j \in \mathcal{T}$

By definition, if we show that $u \in v(\mathcal{N},\mathcal{M})$, the game is balanced. Then by theorem 4.1, it has a nonempty core. The procedure is similar to that in the proof of theorem 4.2. First define a joint action set $\{\alpha_{kj},\ k \in \mathcal{B}_{\mathcal{N}}, j \in \mathcal{M}\}$ as follows

$$\alpha_{kj}(\omega) = \sum_{(\mathcal{S},\mathcal{T})\in\mathcal{I}:\substack{k\in\mathcal{B}_{\mathcal{S}}\\ j\in\mathcal{T}}} \lambda_{\mathcal{S},\mathcal{T}} \alpha_{kj}^{\mathcal{S},\mathcal{T}}(\omega) \quad (7)$$

The following two steps, concludes the proof.

Step 1: We need to show that $\{\alpha_{kj},\ k \in \mathcal{B}_{\mathcal{N}}, j \in \mathcal{M}\}$ satisfy feasibility constraints $1-3$. The argument is similar to that in step 1 of the proof of theorem 4.2, and is eliminated for brevity.

Step 2: (i) $u_i \leq \sum_{\substack{j\in\mathcal{M}\\ \omega\in\Omega}} \mathbb{P}(\omega) f_{ij}(\mathbf{y}_j^{\mathcal{N},\mathcal{M}}(\omega)), \forall i \in \mathcal{N}$ and (ii) $u_j \leq \sum_{\omega\in\Omega} \mathbb{P}(\omega) g_j(\mathbf{y}_j^{\mathcal{N},\mathcal{M}}(\omega)), \forall j \in \mathcal{M}$, where $\mathbf{y}_j^{\mathcal{N},\mathcal{M}}$s are the rate vectors given by joint action $\{\alpha_{kj},\ k \in \mathcal{B}_{\mathcal{N}}, j \in \mathcal{M}\}$. Using the concavity of $f_{ij}(.)$s and $g_j(.)$s, it is straightforward to show (i) and (ii) (Refer to step 2 of the proof of theorem 4.2 for a similar argument).

It follows from Steps 1 and 2 that $u \in v(\mathcal{N})$, and the claim follows ■

B. Computation of A Payoff Profile In The Core

In section V, we showed how the concept of market equilibrium can be used to obtain a payoff profile in the core. However, the coalitional game $((\mathcal{N},\mathcal{M}),v)$ cannot be similarly related to the exchange market setting we constructed in section V. Consequently, we turn to a more general method to obtain an element of the core, proposed in [9].

Consider a coalitional game $(\mathcal{Q},V)$, where $\mathcal{Q}$ is the set of players and $V^{\mathcal{S}}$ is the set of feasible payoff profiles for all coalitions $\mathcal{S} \subseteq \mathcal{Q}$. For all proper coalitions $\mathcal{S} \subset \mathcal{N}$, consider an arbitrary finite list of payoffs $u^{1,\mathcal{S}}, u^{2,\mathcal{S}}, \ldots, u^{k_{\mathcal{S}},\mathcal{S}}$ belonging to $v(\mathcal{S})$. Using this finite list as input, the algorithm in [9] will then calculate a pay off vector $\hat{\mathbf{x}} \in \mathbb{R}^{|\mathcal{Q}|}$ such that $\hat{\mathbf{x}}$ cannot be blocked by any proper coalition, using a payoff vector from this finite list to block. If, in addition, the game is balanced, we have $\hat{\mathbf{x}} \in V(\mathcal{Q})$. Although the computed payoff vector $\hat{\mathbf{x}}$ need not be in the core of $(\mathcal{Q},V)$, it can be made arbitrarily close to the core by selecting the above finite list of payoffs appropriately. A notion of proximity to the core will be discussed shortly.

We now proceed to discuss the above procedure in details, in the following 3 steps. 1) Introducing the notion of approximate core. 2) Construction of an appropriate finite list of payoffs to be used by the algorithm to generate a payoff profile in the approximate core with arbitrary precision. 3) Apply the algorithm to the finite list.

Step 1: Consider the coalitional game $((\mathcal{N},\mathcal{M}),v)$ defined earlier in this section. We defined the core of this game to be the set of all feasible payoff profiles that cannot be blocked by any coalition. We can generalize this definition as follows. A feasible payoff profile is said to be in the *approximate core* of the game $((\mathcal{N},\mathcal{M}),v)$, $\mathcal{C}_\epsilon$, if it cannot be blocked by at least a margin of ϵ, by any coalition. Formally,

$$\mathcal{C}_\epsilon = \{\mathbf{x} \in v(\mathcal{N},\mathcal{M}) : \forall(\mathcal{S},\mathcal{T}), \nexists \mathbf{z} \in v(\mathcal{S},\mathcal{T}) \text{ such that } \mathbf{z}_i > \mathbf{x}_i + \epsilon, \mathbf{z}_j > \mathbf{x}_j + \epsilon \ \forall i \in \mathcal{S}, j \in \mathcal{T}\} \tag{8}$$

It is straightforward that for $\epsilon < 0$, $\epsilon = 0$, and $\epsilon > 0$, $\mathcal{C}_\epsilon$ is a subset of, equal to, and superset of the core, respectively. Here we are naturally interested in the approximate core for strictly positive values of ϵ. It is now evident from the definition of the approximate core (8), that by letting ϵ go to 0, payoff profiles in the approximate core get closer to those in the core of the game, hence the term *approximate core*.

Step 2: We next discuss how to construct a finite list of payoff profiles such that, the payoff profile $\hat{\mathbf{x}}$ computed by the algorithm using this finite list is in $\mathcal{C}_\epsilon$, for any arbitrary $\epsilon > 0$.

Suppose that for all coalitions $(\mathcal{S},\mathcal{T}) \subsetneqq (\mathcal{N},\mathcal{M})$, we can find a finite list of payoff profiles in $v(\mathcal{S},\mathcal{T})$, $V^{(\mathcal{S},\mathcal{T})} = \{u^{1,(\mathcal{S},\mathcal{T})}, u^{2,(\mathcal{S},\mathcal{T})}, \ldots, u^{k_{\mathcal{S},\mathcal{T}},(\mathcal{S},\mathcal{T})}\}$, such that every other payoff profile in $v(\mathcal{S},\mathcal{T})$ is within ϵ distance of at least one vector in $V^{(\mathcal{S},\mathcal{T})}$. In other words

$$\forall \mathbf{x} \in v(\mathcal{S},\mathcal{T}), \exists u^{m,(\mathcal{S},\mathcal{T})} \in V^{(\mathcal{S},\mathcal{T})} \text{ s.t. } \mathbf{x} - u^{m,(\mathcal{S},\mathcal{T})} \leq \epsilon.\mathbb{1}_{1\times(|\mathcal{S}|+|\mathcal{T}|)}. \tag{9}$$

Now consider any arbitrary payoff profile $\mathbf{x} \in v(\mathcal{N},\mathcal{M})$ that is not blocked by any payoff vector in $V^{(\mathcal{S},\mathcal{T})}$ for all proper coalitions $(\mathcal{S},\mathcal{T})$. That is, $\forall(\mathcal{S},\mathcal{T}) \subset (\mathcal{N},\mathcal{M})$, there does not exist $\hat{\mathbf{z}} \in V^{(\mathcal{S},\mathcal{T})}$ such that $\hat{\mathbf{z}}_i > \mathbf{x}_i \ \forall i \in \mathcal{S}$ and $\hat{\mathbf{z}}_j > \mathbf{x}_j \ \forall j \in \mathcal{T}$. Then it follows from (9) that there does not exist $\mathbf{z} \in v(\mathcal{S},\mathcal{T})$ such that $\mathbf{z}_i > \mathbf{x}_i + \epsilon \ \forall i \in \mathcal{S}$ and $\mathbf{z}_j > \mathbf{x}_j + \epsilon \ \forall j \in \mathcal{T}$. Thus $\mathbf{x} \in \mathcal{C}_\epsilon$. Since $\mathbf{x} \in v(\mathcal{N},\mathcal{M})$ was arbitrarily selected, it might as well be the payoff profile $\hat{\mathbf{x}}$ computed by the algorithm (Note that $\hat{\mathbf{x}}$ is guaranteed to be in $v(\mathcal{N},\mathcal{M})$). Consequently, $\hat{\mathbf{x}} \in \mathcal{C}_\epsilon$. Therefore, $(V^{(\mathcal{S},\mathcal{T})}, (\mathcal{S},\mathcal{T}) \subset (\mathcal{N},\mathcal{M}))$ is indeed the finite list of payoffs we seek.

It remains to show how to construct $V^{(\mathcal{S},\mathcal{T})}$, for all $(\mathcal{S},\mathcal{T}) \subset (\mathcal{N},\mathcal{M})$. First notice that we can restrict our search to the payoff profiles in $v(\mathcal{S},\mathcal{T})$ that are Pareto-optimal, that is, no other payoff profile in $v(\mathcal{S},\mathcal{T})$ can be found that gives every one in $(\mathcal{S},\mathcal{T})$ a strictly higher payoff. Also, every Pareto-optimal payoff profile $\mathbf{x}$ in $v(\mathcal{S},\mathcal{T})$, can be obtained as a solution of the following optimization OPT $(\lambda^{\mathcal{S},\mathcal{T}})$, for different values of $\lambda^{\mathcal{S},\mathcal{T}}$, where $\lambda^{\mathcal{S},\mathcal{T}} \neq 0$ is a set of nonnegative weights.

OPT$(\lambda^{\mathcal{S},\mathcal{T}})$: Maximize $\sum_{i\in\mathcal{S}} \lambda_i^{\mathcal{S},\mathcal{T}} \mathbf{x}_i + \sum_{j\in\mathcal{T}} \lambda_j^{\mathcal{S},\mathcal{T}} \mathbf{x}_j$

1) $\mathbf{x}_i = \sum_{\substack{j\in\mathcal{T}\\ \omega\in\Omega}} \mathbb{P}(\omega) f_{ij}(\mathbf{y}_j^{\mathcal{S},\mathcal{T}}(\omega))$
2) $\mathbf{x}_j = \sum_{\omega\in\Omega} \mathbb{P}(\omega) g_j(\mathbf{y}_j^{\mathcal{S},\mathcal{T}}(\omega))$
3) $y_{kj}^{\mathcal{S},\mathcal{T}}(\omega) = \alpha_{kj}(\omega) r_{kj}(\omega), k \in \mathcal{B}_\mathcal{S}, j \in \mathcal{T}, \omega \in \Omega$
4) $\sum_{j\in\mathcal{T}} \alpha_{kj}(\omega) \leq 1, k \in \mathcal{B}_\mathcal{S}, \omega \in \Omega$
5) $\sum_{k\in\mathcal{B}_\mathcal{S}} \alpha_{kj}(\omega) \leq 1, j \in \mathcal{T}, \omega \in \Omega$
6) $\alpha_{kj} \geq 0, k \in \mathcal{B}_\mathcal{S}, j \in \mathcal{T}$.

Let $\mathbf{x}(\lambda^{\mathcal{S},\mathcal{T}})$ be the solution of OPT $(\lambda^{\mathcal{S},\mathcal{T}})$. Notice that the function $\mathbf{x}(\lambda^{\mathcal{S},\mathcal{T}})$ is continuous in $\lambda^{\mathcal{S},\mathcal{T}}$. Also note that since scaling $\lambda^{\mathcal{S},\mathcal{T}}$ does not change the solution of the above optimization, we can set $\sum_{i\in\mathcal{S}} \lambda_i^{\mathcal{S},\mathcal{T}} + \sum_{j\in\mathcal{T}} \lambda_j^{\mathcal{S},\mathcal{T}} = 1$. As a result, we have a continuous function over a bounded set $\{\lambda^{\mathcal{S},\mathcal{T}} : \sum_{i\in\mathcal{S}} \lambda_i^{\mathcal{S},\mathcal{T}} + \sum_{j\in\mathcal{T}} \lambda_j^{\mathcal{S},\mathcal{T}} = 1\}$, whose range covers the set of Pareto-optimal feasible payoff profiles. It then follows that by selecting a collection of weights $\{\lambda^{1,\mathcal{S},\mathcal{T}}, \ldots, \lambda^{k_{(\mathcal{S},\mathcal{T})},(\mathcal{S},\mathcal{T})}\}$ appropriately, the set of feasible payoff profiles obtained by solving the above optimization, will be the desired set $V^{(\mathcal{S},\mathcal{T})}$.

Step 3: Now, we can apply the algorithm presented in [9] to $V^{(\mathcal{S},\mathcal{T})}, \forall(\mathcal{S},\mathcal{T}) \subset (\mathcal{N},\mathcal{M})$ to obtain a payoff profile $\hat{\mathbf{x}} \in v(\mathcal{N},\mathcal{M})$. This algorithm is similar to the simplex method, and performs several consecutive row and column operations on a matrix containing the payoffs in $V^{(\mathcal{S},\mathcal{T})}$ to reach the answer. We have the following.

Theorem 6.2: The payoff profile $\hat{\mathbf{x}}$ computed by the algorithm is in $\mathcal{C}_\epsilon$.

Proof: Using the result in [9] and the above steps, the claim follows ■

Discussion: Note that to compute a payoff profile in the core by this method, we first have to construct a finite list of payoff profiles, whose size is exponential in parameters of the network and ϵ. As a result, regardless of the efficiency of the algorithm proposed in [9], the computational time will be exponential. The computational time of the algorithm discussed in section V-C, on the other hand, is dominated by the *Global Newton Method*. This method, despite not being guaranteed to run in polynomial time, it has demonstrated the capability of handling real world applications (see [12], p. 670).

VII. GENERALIZATION

The cooperative games studied in previous sections have the following two properties. (i) Each player (providers and customers) benefits exclusively from the payoff he earned by himself. In other words, the players do not share their payoffs. (ii) All players have scaler payoff functions. In this section, we first relax (i) and generalize this framework to accommodate more general payoff sharing rules, such as when there are groups of providers, and providers in each group are willing to share their payoffs. We formulate this problem as a coalitional game and show that it has a nonempty core (section VII-A). Subsequently, we relax (ii) and let players have vector payoff functions. Each component of this payoff vector can then follow different sharing rules. For instance, one component could be nontransferable, while another could be shared by the players in specific groups. We formulate a coalitional game based on such payoff functions, and investigate it core VII-B.

A. Payoff Sharing

In this section, we reexamine the game $((\mathcal{N},\mathcal{M}), v)$ defined in section VI in presence of a more general payoff sharing rule. Payoff sharing has its potential advantages, as well as some practical difficulties. When a group of providers, for instance, agree to share their payoffs, instead of each one trying to maximize its own payoff, they attempt to maximize their aggregate payoff, which can be generally higher than the sum of the maximized individual payoffs. As a result, providers can enjoy higher payoffs, when they share. On the other hand, trust relations of certain individuals can prevent them from payoff sharing. For instance, two providers are unlikely to share their revenues with each other, unless both parties believe that the other is honest in reporting its actual revenue. In addition, not all types of payoffs can be shared, since they may not have monetary equivalence. For example, fairness in the offered service may contribute to a provider's total payoff, but it would be difficult to translate this into monetary units.

We consider a general sharing model, in which there are several groups of providers. If providers in the same group decide to cooperate, they will share their payoffs. Providers from different groups, on the other hand, cannot share in any case. Let $\mathcal{P} = \{\mathcal{N}_1, \ldots, \mathcal{N}_L\}$ be a partition of the set of providers. We assume that for all $l \in \{1, \ldots, L\}$, the providers in $\mathcal{N}_l$ that are in coalition, share their payoffs. For example, if $\mathcal{P} = \{\mathcal{N}\}$, then all providers that are in coalition will share payoffs, and if $\mathcal{P} = \{\{1\}, \{2\}, \ldots, \{N\}\}$, then no one shares payoffs. Now consider a coalition $(\mathcal{S}, \mathcal{T})$. Define $\mathcal{S}_l = \mathcal{S} \cap \mathcal{N}_l$. Then it follows from the above construction that for all $l \in \{1, \ldots, L\}$, providers in $\mathcal{S}_l$ engage in payoff sharing. Note that for simplicity, we consider sharing only among providers. All the formulations and results extend to the scenario where there are groups of customers, and those from the same group can share payoffs.

Consider the network setup in section VI. The feasibility constraints, and consequently the set of feasible joint actions do not vary due to payoff sharing. However, we need to redefine the set of feasible payoff profiles. Previously, this set included every payoff vector that is less than or equal to some payoff profile obtained through a feasible joint action. But since now some providers share their payoffs, for a payoff vector to be in the set of feasible payoff profiles, we only need the *aggregate* payoffs of each sharing group to be less than or equal to that in some payoff profile given by a feasible joint action. Formally, we define the set of feasible payoff profiles $v(\mathcal{S}, \mathcal{T})$ as follows

$$\begin{aligned} v(\mathcal{S}, \mathcal{T}) = \{\mathbf{x} \in \mathbb{R}^{|\mathcal{S}|+|\mathcal{T}|} : \exists \mathbf{z} \in \mathbb{R}^{|\mathcal{S}|+|\mathcal{T}|}, \sum_{i \in \mathcal{S}_l} \mathbf{z}_i \geq \sum_{i \in \mathcal{S}_l} \mathbf{x}_i \\ \forall i \in \{1, \ldots, L\}, \mathbf{z}_j \geq \mathbf{x}_j \ \forall j \in \mathcal{T}, \\ \mathbf{z} = \mathcal{F}^{\mathcal{S},\mathcal{T}}(a) \text{ for some } a \in A(\mathcal{S}, \mathcal{T})\}. \end{aligned} \tag{10}$$

With this definition, the coalitional game $((\mathcal{N}, \mathcal{M}), v)$ is now well defined. The definition of the core of this game then will be the same as (6). Using a similar technique as that used in the proof of theorems 4.2 and 6.1, one can show that this game is balanced. It then follows from theorem 4.1 that the core of the game is nonempty, and thus the grand coalition is stabilizable. Also, a payoff profile in the core of this game can be computed by an algorithm similar to one discussed in section VI-B.

B. Modeling Individuals' Profits By Vector Functions

We have so far focused on coalitional games with scaler payoff functions. In this section, we examine the scenario where players have vector payoff functions. Such payoff functions can have several utility components of different types. For instance, a provider's payoff can be a vector of its total revenue, its competitive power in the market, fairness in the network, reputational issues, social welfare, among others. A customer's payoff, on the other hand, may consist of his service rate and cost, power consumption, the size of the network, and so on. Note that it is possible to have payoff vectors with mixed transferable and nontransferable utility components. Then payoff sharing can be feasible, which not only depend on the players in a coalition, but also on the type of utilities. Specifically, there could be groups of players, and players in each group would share the transferable types of utility, provided that they are in a coalition.

In this section, we investigate cooperation among providers in presence of a vector payoff function of two components; a transferable utility and a nontransferable one. We consider the scenario where all providers is a coalition would share the transferable utility. The formulations and results can extend to more general cases, where payoff functions have several components, and payoff sharing of individual components occurs only among providers from the same group, for given groups of providers.

Consider the scenario that when a provider i is in coalition $\mathcal{S}$ and the state is ω, i enjoys utilities $f_i^t(\mathbf{y}_i^{\mathcal{S}}(\omega))$ and $f_i^n(\mathbf{y}_i^{\mathcal{S}}(\omega))$, where the superscripts t, n stand for transferable and nontransferable, respectively. $f_i^t(.)$s and $f_i^n(.)$s are considered to be concave functions. The definitions of a rate vector $\mathbf{y}_i^{\mathcal{S}}(\omega)$, a feasible payoff profile $\{\alpha_{kj}, k \in \mathcal{B}_{\mathcal{S}}, j \in \mathcal{M}_S\}$, and the joint action space $A(\mathcal{S})$ are exactly the same as those in section III. Now consider a joint action $a \in A(\mathcal{S})$. We define $\mathcal{F}^{\mathcal{S}}(a)$ to be the payoff profile corresponding to joint action a. In other words, $\mathcal{F}^{\mathcal{S}}(a) = (\mathbf{x}^t, \mathbf{x}^n)$ if a) $\mathbf{x}_i^t = \sum_{\omega \in \Omega} \mathbb{P}(\omega) f_i^t(\mathbf{y}_i^{\mathcal{S}}(\omega))$ and $\mathbf{x}_i^n = \sum_{\omega \in \Omega} \mathbb{P}(\omega) f_i^n(\mathbf{y}_i^{\mathcal{S}}(\omega)) \ \forall i \in \mathcal{S}$, and b) $(\mathbf{y}_i^{\mathcal{S}}, i \in \mathcal{S})$ is the vector of service rates generated by joint action a.

We now define the set of feasible payoff profiles $v(\mathcal{S})$ as follows.

$$\begin{aligned} v(\mathcal{S}) = \{(\mathbf{x}^t, \mathbf{x}^n) \in \mathbb{R}^{|\mathcal{S}|} \times \mathbb{R}^{|\mathcal{S}|} : \exists (\mathbf{z}^t, \mathbf{z}^n) \in \mathbb{R}^{|\mathcal{S}|} \times \mathbb{R}^{|\mathcal{S}|} \text{ s.t.} \\ \mathbf{z}^t . \mathbb{1}_{1 \times |\mathcal{S}|} \geq \mathbf{x}^t . \mathbb{1}_{1 \times |\mathcal{S}|}, \mathbf{z}^n \geq \mathbf{x}^n, \ \mathbf{z} = \mathcal{F}^{\mathcal{S}}(a) \\ \text{for some } a \in A(\mathcal{S})\}. \end{aligned} \tag{11}$$

In words, $v(\mathcal{S})$ is the set of all payoff profiles for which the nontransferable utilities of providers in $\mathcal{S}$, as well as the sum of their transferable utilities, are either equal or less than that of a payoff profile generated by a feasible joint action.

We now proceed to define the core of this game. Note that to define the core of the game, an ordering relation between two different payoff vectors of a provider is necessary, i.e., we have to know which of the two payoffs $(\mathbf{x}_i^t, \mathbf{x}_i^n)$ and $(\mathbf{z}_i^t, \mathbf{z}_i^n)$ provider i prefers. We consider a lexicographic ordering relation, in which providers prefer the payoff profile that offers the highest of nontransferable utility. In case there are several payoff profiles with this property, the one among them that offers more of the transferable utility is preferred.

With lexicographic ordering relation, the core of the game is defined as follows: the set of payoff profiles that cannot be blocked lexicographically, by any coalition, i.e.,

$$\mathcal{C} = \{(\mathbf{x}^t, \mathbf{x}^n) \in v(\mathcal{N}) : \forall \mathcal{S}, \nexists (\mathbf{y}^t, \mathbf{y}^n) \in v(\mathcal{S}) \text{ such that } \mathbf{y}_i^n > \mathbf{x}_i^n \ \forall i \in \mathcal{S} \text{ or } \mathbf{y}^n \geq \mathbf{x}^n \text{ and } \mathbf{y}_i^t > \mathbf{x}_i^t \ \forall i \in \mathcal{S}.\} \tag{12}$$

We seek to show that $\mathcal{C}$ is nonempty. First note that as $f_i^t(.)$s and $f_i^n(.)$s are concave, using a similar technique used in the proof of theorem 4.2, it is straightforward to verify that this game is balanced. But since the coalitional games considered in [9] have scaler payoff functions, it is not evident whether balancedness lead to nonemptiness of the core. However, with a slight twist in the definition of the core, we can use this theorem and derive interesting results. Towards that end, we first define the approximate core for this game as follows.

$$\mathcal{C}_\epsilon = \{(\mathbf{x}^t, \mathbf{x}^n) \in v(\mathcal{N}) : \forall \mathcal{S}, \nexists (\mathbf{y}^t, \mathbf{y}^n) \in v(\mathcal{S}) \text{ such that } \mathbf{y}_i^n > \mathbf{x}_i^n + \epsilon \ \forall i \in \mathcal{S} \text{ or } \mathbf{y}^n \geq \mathbf{x}^n + \epsilon \text{ and } \mathbf{y}_i^t > \mathbf{x}_i^t \ \forall i \in \mathcal{S}\} \tag{13}$$

In words, $\mathcal{C}_\epsilon$ is the set of all payoff profiles that cannot be lexicographically blocked by a margin of ϵ, by any coalition $\mathcal{S} \subset \mathcal{N}$. Here is the main result.

Theorem 7.1: For any $\epsilon > 0$, $\mathcal{C}_\epsilon$ is nonempty.

Remark: Note that theorem 7.1 does not imply that $\mathcal{C}$ is nonempty, however, in practice it is as desirable.

Proof: Suppose that instead of lexicographic ordering relation, providers use a linear ordering to compare different payoff profiles. That is $(\mathbf{x}_i^t, \mathbf{x}_i^n)$ is preferred over $(\mathbf{z}_i^t, \mathbf{z}_i^n)$ if $\lambda \mathbf{x}_i^t + \mathbf{x}_i^n \geq \lambda \mathbf{z}_i^t + \mathbf{z}_i^n$, for some given $\lambda > 0$. We can then define the core, $\hat{\mathcal{C}}(\lambda)$, based on this ordering relation as the following.

$$\hat{\mathcal{C}}(\lambda) = \{(\mathbf{x}^t, \mathbf{x}^n) \in v(\mathcal{N}) : \forall \mathcal{S}, \nexists (\mathbf{y}^t, \mathbf{y}^n) \in v(\mathcal{S}) \text{ such that } \lambda \mathbf{y}_i^t + \mathbf{y}_i^n > \lambda \mathbf{x}_i^t + \mathbf{x}_i^n \ \forall i \in \mathcal{S}.\} \tag{14}$$

We first argue that $\hat{\mathcal{C}}(\lambda) \neq \emptyset \ \forall \lambda > 0$. The reason is that now, we can assume that providers have a scaler payoff function given by $\lambda f_i^t(.) + f_i^n(.)$ and redefine the set of feasible payoff profile $\hat{v}(\mathcal{S})$ according to the new payoff function. The definition of the core will be the same as $\hat{\mathcal{C}}(\lambda)$. Now theorem 4.1 applies. It is then straightforward to verify that the coalitional game $(\mathcal{N}, \hat{v})$ is balanced. Then by theorem 4.1, we conclude that $\hat{\mathcal{C}}(\lambda)$ is nonempty.

We now claim that $\hat{\mathcal{C}}(\lambda) \subset \mathcal{C}_\epsilon$, if $\lambda = \epsilon / (\max_{i \in \mathcal{N}, (\mathbf{z}^t, \mathbf{z}^n) \in v(\mathcal{N})} \mathbf{z}_i^t)$. We prove this claim by contradiction. Consider a payoff profile $(\mathbf{x}^t, \mathbf{x}^n) \in \hat{\mathcal{C}}(\lambda)$. Suppose that $(\mathbf{x}^t, \mathbf{x}^n) \notin \mathcal{C}_\epsilon$. Then by (13), $\exists (\mathbf{y}^t, \mathbf{y}^n) \in v(\mathcal{S})$ such that either of the following holds

i) $\mathbf{y}_i^n > \mathbf{x}_i^n + \epsilon \ \forall i \in \mathcal{S}$

ii) $\mathbf{y}^n \geq \mathbf{x}^n + \epsilon$ and $\mathbf{y}_i^t > \mathbf{x}_i^t \ \forall i \in \mathcal{S}$.

If (ii) holds, then $\lambda \mathbf{y}^t + \mathbf{y}^n > \lambda \mathbf{x}^t + \mathbf{x}^n$, and thus $(\mathbf{x}^t, \mathbf{x}^n) \notin \hat{\mathcal{C}}(\lambda)$, which is a contradiction. Therefore, only (i) holds. Thus, we have

$$\mathbf{y}_i^n > \mathbf{x}_i^n + \epsilon \ \forall i \in \mathcal{S} \tag{15}$$

Also, since $(\mathbf{x}^t, \mathbf{x}^n) \in \hat{\mathcal{C}}(\lambda)$, it cannot be blocked in linear ordering sense, by any coalition, i.e.,

$$\lambda \mathbf{y}_i^t + \mathbf{y}_i^n \leq \lambda \mathbf{x}_i^t + \mathbf{x}_I^n \Rightarrow \mathbf{y}_i^n - \mathbf{x}_i^n \leq \lambda (\mathbf{x}_i^t - \mathbf{y}_i^t) \leq \epsilon \ \forall i \in \mathcal{S} \tag{16}$$

The last inequality follows by $\lambda = \epsilon / (\max_{i \in \mathcal{N}, (\mathbf{z}^t, \mathbf{z}^n) \in v(\mathcal{N})} \mathbf{z}_i^t)$. It is clear that (16) is in contradiction with (15). Thus, the claim and subsequently, the theorem follows. ■

Discussion: We now discuss computing a payoff profile in the approximate core. As $\hat{\mathcal{C}}(\lambda) \subset \mathcal{C}_\epsilon$, we can obtain a payoff profile in $\mathcal{C}_\epsilon$ by finding one in $\hat{\mathcal{C}}(\lambda)$. And since $\hat{\mathcal{C}}(\lambda)$ can be considered as the core of a corresponding coalitional game with scaler payoff functions $\lambda f_i^t(.) + f_i^n(.) \ i \in \mathcal{N}$, a payoff profile in $\hat{\mathcal{C}}(\lambda)$ can be computed by the algorithm discussed in section VI-B.

REFERENCES

[1] Z. Han and H. V. Poor, "Coalitional games with cooperative transmission: A cure for the curse of boundary nodes in selfish packet-forwarding wireless networks," in *5th International Symposium on Modeling and Optimization in Mobile, Ad Hoc, and Wireless Networks*, 2007.

[2] S. Mathur, L. Sankaranarayanan, and N. Mandayam, "Coalitional games in cooperative radio networks," *Fortieth Asilomar Conference on Signals, Systems and Computers*, pp. 1927–1931, Oct.-Nov. 2006.

[3] R. La and V. Anantharam, "A game theoretic look at the gaussian multiaccess channel," in *Proc. of the DIMACS Workshop on Network Information Theory*, (New Jersey, NY, USA), Mar. 2003.

[4] S. Mathur, L. Sankaranarayanan, and N. Mandayam, "Coalitions in cooperative wireless network," *IEEE J. Select. Areas Commun.*, vol. vol. 26, p. 11041115, Sep. 2008.

[5] A. Aram, C. Singh, S. Sarkar, and A. Kumar, "Cooperative profit sharing in coalition based resource allocation in wireless networks," in *Proc. of IEEE INFOCOM*, (Rio de Janeiro, Brazil), Apr. 2009.

[6] B. Hajek and G. Sasaki, "Link scheduling in polynomial time," *IEEE Transactions on Information Theory*, vol. 34, no. 5, 1988.

[7] M. Osborne and A. Rubinstein, *A Course in Game Theory*. The MIT press, 1999.

[8] V. Conitzer and T. Sandholm, "Complexity of determining nonemptiness of the core," Technical Reprt CS-02-137, CMU, 2002.

[9] H. E. Scarf, "The core of an n person game," *Econometrica*, vol. 35, pp. 50–69, January 1967.

[10] A. Mas-colell, M. D. Whinston, and J. R. Green, *Microeconomic Theory*. Oxford University Press.

[11] S. Smale, "A convergent process of price adjustment," *Journal of Mathematical Economics*, vol. 3, pp. 107–120, 1976.

[12] M. Ferris and J. Pang, "Engineering and economic applications of complementarity problems," *SIAM Review*, vol. 39, pp. 669–713, 1997.