Channel Assignment Algorithms Satisfying Cochannel and Adjacent Channel Reuse Constraints in Cellular Mobile Networks

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Abstract

Recently improved channel assignment algorithms for cellular networks were designed by modelling the interference constraints in terms of a hypergraph [1]. However these algorithms only considered cochannel reuse constraints. Receiver filter responses impose restrictions on simultaneous adjacent channel usage in the same cell or in neighbouring cells. We first present some heuristics for designing fixed channel assignment algorithms with a minimum number of channels satisfying both cochannel and adjacent channel reuse constraints. An asymptotically tight upper bound for the traffic carried by the system in the presence of arbitrary cochannel and adjacent channel reuse constraints was developed in [2]. However this bound is computationally intractable even for small systems like a regular hexagonal cellular system of 19 cells. We have obtained approximations to this bound using

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the optimal solutions for cochannel reuse constraints only, and a further graph theoretic approach. Our approximations are computationally much more efficient and have turned out to track very closely the exact performance bounds in most cases of interest.

I. Introduction

In a cellular system, the coverage area is logically divided into cells. Each cell has a cell site or a base station. The communication from the mobile user is directed to a central switching office by the base station. The central switching office directs this communication to the destination. Depending on the mode of multiple access used by the mobile customers, cellular systems can be broadly classified into *channelized* and *nonchannelized* systems. In a channelized cellular system the multiple access is TDMA or FDMA or a combination of both. The term *channel* refers to a time slot in TDMA, a frequency slot in FDMA and a combination of both in TDMA/FDMA systems like GSM. The traffic in a cellular systems is usually too high to allow the use of a channel for one call at a time; radio channels must be used simultaneously for more than one call. This is known as *channel reuse*. *Channel reuse* causes interference, which in turn degrades the transmission quality. Transmission quality requirements impose certain cochannel reuse constraints. Cochannel reuse constraints in cellular systems have been modelled by regular hexagonal channel reuse patterns [3] for a long time. Cochannel reuse constraints in a cellular system were first modelled by a hypergraph in [4]. This has been found to be the most efficient model for cochannel reuse constraints [5]. We describe below certain performance measures of interest in cellular systems. These have been formulated using hypergraph modelling of cellular systems.

We first describe the hypergraph model briefly. A hypergraph H is formally defined as $H = (V, E)$, where V is the set of vertices and E is the set of edges, where each edge e is a nonempty subset of V such that $\cup_{e \in E} e = V$ [6]. A hypergraph is a generalization of a graph in that an edge can have no more than two vertices in a graph but this restriction does not hold for a hypergraph. Hypergraph modelling of cellular systems is as follows:

- Each cell corresponds to a vertex.
- A *forbidden set* is a group of cells *all* of which cannot use a channel simultaneously. If no proper subset of a forbidden set is forbidden, then it is a *minimal forbidden set*. An edge is a *minimal forbidden set*.
- A set of vertices which does not contain an edge is an *independent* set. Any group of cells which may use the same channel simultaneously forms an *independent set* of the underlying hypergraph. If an independent set is not a proper subset of another independent set, then it is a *maximal independent set*.

We make the following assumptions about the system. The system consists of N cells and the underlying offered traffic model is independent from cell to cell; in particular, we ignore the effects of call handovers and intercell calls. However it is likely that we can extend our results to the case in which this independence assumption is dropped and handovers and intercell calls are included. Our optimism is derived from the fact that the results of [4], to which we shall refer extensively, have been extended to include handovers in [7]. The underlying model of offered traffic satisfies the "asymptotic traffic property" (ATP) [4] which states that

$$\lim_{n\to\infty} C(k,n)/n = \min(r,\,1), \text{ where } \lim_{n\to\infty} k/n = r.$$

$C(k,n)$ is the carried traffic in a one-cell system, when the offered traffic is k and the number of available channels is n. Many offered traffic models including the common Poisson arrivals and exponential holding times satisfy the ATP. The N cells share a common set of n channels. Let the hypergraph H modelling the cochannel reuse constraints have M maximal independent sets and let N_j denote the size of the jth maximal independent set. We define

$$a_{ij} = \begin{cases} 1, & \text{if the } i\text{th cell is in the } j\text{th maximal} \\ & \text{independent set of the hypergraph modelling} \\ & \text{the cochannel reuse constraints, and} \\ 0, & \text{otherwise.} \end{cases}$$

The expected number of calls that would be in progress in cell i if all call requests could be honored is known as the *offered traffic* in cell i. If A_i denotes the offered traffic in cell i then A_i/n is the *offered traffic intensity* in cell i. The offered traffic intensity in the system, r, is the sum of the offered traffic intensities in the cells; thus $r = \sum A_i/n$. We assume that r is rational. The ratio $p_i = A_i / \sum_{i=1}^{N} A_i$ represents the fraction of the total offered traffic in cell i and the vector $\tilde{p} = (p_1, p_2, \ldots, p_N)$ is the *traffic pattern*. We assume that the p_is are rational. The carried traffic intensity in cell i, x_i, is the carried traffic (expected number of calls in progress) in cell i per available channel in the system. The optimal value of the objective function of the following linear program, denoted by $T(r)$, is an asymptotically[1] tight upper bound on the total carried traffic intensity in the cellular system with cochannel reuse constraints only [4].

$$\text{Maximize } \sum_{i=1}^{N} x_i, \text{ subject to}$$

$$\begin{aligned} x_i &\leq \sum_{j=1}^{M} X_j a_{ij}, \qquad i = 1,2,\ldots,N \\ x_i &\leq p_i r, \qquad i = 1,2,\ldots,N \\ X_j &\geq 0, \qquad j = 1,2,\ldots,M \\ \sum_{j=1}^{M} X_j &= 1. \end{aligned}$$

We denote this linear program by LP1. This linear program has $M+N$ nonnegative variables and $2N+1$ constraints. It was also proved in [4] under the same assumptions as those in this paper but in the presence of cochannel reuse constraints only, that if the offered traffic

[1]The number of channels and the offered traffic are made arbitrarily large while keeping the ratio finite.

intensity, r, is less than or equal to a certain quantity r_0, which depends on the cellular system and the traffic pattern, there exists a channel assignment algorithm which achieves arbitrarily low blocking probabilities, if the number of available channels is sufficiently large. For $r > r_0$, no channel assignment algorithm can produce zero blocking for any number of channels. r_0 has been termed the *capacity* of the system. r_0 is given by the optimal value of the objective function of the following linear program:

$$\max \left\{ r : \sum_{j=1}^{M} X_j = 1, X_j \geq 0, \right.$$
$$\left. \sum_{j=1}^{M} X_j a_{ij} \geq p_i r, i = 1, \ldots, N. \right\}$$

We denote this linear program by LP2. This linear program has $M+1$ nonnegative variables and $N+1$ constraints. The capacity of a system is a measure of the reuse offered by the system as informally speaking each channel can carry r_0 calls simultaneously in the system on an average. It can also serve as a good operating load.

Imperfect receiver filter responses impose restrictions on simultaneous use of adjacent channels in nearby cells. By "adjacent" channels we mean consecutive carrier frequencies in a FDMA system. In the case of FDMA/TDMA systems, we assume all the time slots in a frequency channel are allocated to the same cell so that we can treat each carrier frequency as a channel. In the rest of this paper, we assume that a channel is a carrier frequency. The n channels are numbered $1, 2, \ldots, n$ with adjacent channels given consecutive numbers. The adjacent channel use constraints can be modelled by an $N \times N$ matrix B, such that

$$b_{ij} = \begin{cases} 1, & \text{if the } i\text{th and the } j\text{th cells can use} \\ & \text{adjacent channels simultaneously, and} \\ 2, & \text{otherwise.} \end{cases}$$

For example, $b_{ij} = 1$ for all i, j means that there are no adjacent channel use constraints. Typically $b_{ij} = 1$ if cells i and j are separated by a distance greater than some d and $b_{ij} = 2$ otherwise. Typical values of d are 0 and $\sqrt{3}R$ in regular hexagonal cellular systems, where R is the cell radius. If $d = 0$, adjacent channels cannot be used simultaneously in the same cell and if $d = \sqrt{3}R$ adjacent channels cannot be used simultaneously in the same cell and in adjacent cells. [2] extended these notions to include the effect of arbitrary adjacent channel[2] use constraints. We briefly describe the relevant portion of the theory developed in [2].

The state of a channel is an N-tuple whose elements are either 0 or 1. The ith element is 1 iff the channel is carrying a call in the ith cell. The state of a channel (other than the one represented by the all zero N-tuple) represents an independent set of the underlying

[2]Imperfect receiver filter responses may produce interference between nonadjacent channels also. The theory developed in [2] takes care of the constraints on simultaneous use of c contiguous channels, where c is any positive integer. However we shall concentrate on adjacent channel use constraints.

hypergraph modelling the cochannel reuse constraints. Let τ denote the set of states of a channel. (If the cochannel reuse constraints are modelled by a hypergraph, the elements of τ, with the exception of the all zero N-tuple, have a one-to-one correspondence with the set of independent sets of the hypergraph.) The hyperstate of a channel i, $1 \leq i < n$, is (p, q), if the channel i is in state p and the channel $i+1$ is in state q. $\Omega \subseteq \tau \times \tau$ is the set of hyperstates of a channel.[3] $d_{i\omega} = 1$ iff a channel carries a call in cell i in the hyperstate $\omega \in \Omega$. $t_\omega = \sum_{i=1}^{N} d_{i\omega}$. Let $|\Omega| = R$. $\Omega_U(\eta) = \{\omega : \omega = (\eta, \eta') \in \Omega, \eta' \in \tau\}$, and $\Omega_L(\eta) = \{\omega : \omega = (\eta', \eta) \in \Omega, \eta' \in \tau\}$, $\eta \in \tau$. $\Omega_U(\eta), \Omega_L(\eta) \subseteq \Omega$. Let

$$S = \left\{ \begin{array}{lll} (s_1, \ldots, s_R) : & & \\ \sum_{\omega \in \Omega} d_{i\omega} s_\omega & \leq p_i r, & i = 1, 2, \ldots N, \\ s_\omega & \geq 0, & \omega \in \Omega, \\ \sum_{\omega \in \Omega_U(\eta)} s_\omega & = \sum_{\omega \in \Omega_L(\eta)} s_\omega, & \eta \in \tau, \\ \sum_{\omega \in \Omega} s_\omega & = 1, & \end{array} \right\}$$

and

$$T_A(r) = \max \left\{ \sum_{\omega \in \Omega} t_\omega s_\omega : s = (s_1, s_2, \ldots, s_R) \in S \right\}.$$

$T_A(r)$ can be determined by computing a linear program with R nonnegative variables and $N + 1 + |\tau|$ constraints. We denote this linear program as LP3.

$T_A(r)$ is an asymptotically tight upper bound[4] on the total carried traffic intensity in the system in the presence of cochannel reuse constraints modelled by A (elements a_{ij}) and the adjacent channel use constraints modelled by B. Let

$$S' = \left\{ \begin{array}{lll} (s_1, \ldots, s_R) : & & \\ \sum_{\omega \in \Omega} d_{i\omega} s_\omega & \geq p_i r & i = 1, 2, \ldots N, \\ s_\omega & \geq 0 & \omega \in \Omega, \\ \sum_{\omega \in \Omega_U(\eta)} s_\omega & = \sum_{\omega \in \Omega_L(\eta)} s_\omega & \eta \in \tau, \\ \sum_{\omega \in \Omega} s_\omega & = 1. & \end{array} \right\}$$

$$r_0^A = \max \{ r : s = (s_1, s_2, \ldots, s_R) \in S' \}$$

where r_0^A is the capacity in the presence of cochannel reuse and adjacent channel use constraints. r_0^A can be determined by computing a linear program with $R + 1$ nonnegative variables and $N + |\tau| + 1$ constraints. We denote this linear program as LP4. r_0^A has the same properties for systems with cochannel reuse and adjacent channel use constraints that r_0 has for systems with cochannel reuse constraints only. In particular, for $r \leq r_0^A$ there exists a channel assignment algorithm which achieves arbitrarily low blocking probabilities, if the number of available channels is sufficiently large, and for $r > r_0^A$, no channel assignment algorithm can produce zero blocking for any number of channels.

[3] Ω is determined by τ and the matrix B.

[4] Unlike $T(r)$ which is the upper bound for all n, $T_A(r)$ is the upper bound only in the asymptotic case.

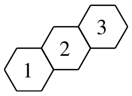

Fig. 1. A 3-cell system

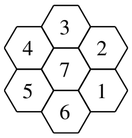

Fig. 2. A 7-cell system

In general both R and $|\tau|$ are very large. When cochannel reuse constraints are modelled by a hypergraph, and adjacent channels not used in the same cell ($d = 0$), $R \sim 10$, $|\tau| \sim 10$ for a 3-cell system (refer figure 1), $R \sim 100$, $|\tau| \sim 10$ for a 7-cell system (refer figure 2), $R \sim 10^6$, $|\tau| \sim 10^3$ for a 19-cell system (refer figure 3) and $R \sim 10^9$, $|\tau| \sim 10^6$ for a 37-cell system (refer figure 4). Thus both these linear programs are computationally intractable for systems of reasonable size.

We first present some heuristics for designing fixed channel assignment algorithms with a minimum number of channels satisfying both cochannel and adjacent channel reuse constraints in Section II. In Sections III and IV we derive lower bounds on r_0^A and $T_A(r)$ which we denote by r_0^{AL} and $T_{AL}(r)$ respectively. These lower bounds can be computed using the solutions to the linear programs for computing $T(r)$ and r_0 and a further graph theoretic approach. These lower bounds are computationally much simpler as $M \ll R$ and $N \ll |\tau| \ll R$. We have also observed that in many cases $T_{AL}(r)$ and r_0^{AL} are quite close to $T_A(r)$ and r_0^A respectively. Note that upper bounds on $T_A(r)$ and r_0^A are already known in the form of $T(r)$ and r_0 respectively. Finally in Section V we discuss possible extensions to the above techniqes to cover the cases where there are restrictions on the simultaneous use of nonadjacent channels also.

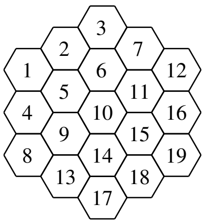

Fig. 3. A 19-cell system

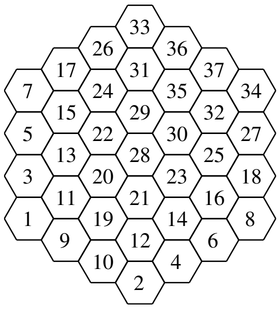

Fig. 4. A 37-cell system

II. A Heuristic Approach for the Determination of the Number of Channels Required for a Given Fixed Channel Allocation

A channel allocation algorithm in which groups of channels are allocated to cells *a priori* and a cell accepts a requested call only if it has a free channel amongst those allocated to it is known as a *fixed channel allocation* algorithm. Though much more sophisticated algorithms, e.g., *dynamic channel allocation* schemes, have been devised which sometimes outperform the fixed channel allocation schemes, fixed channel allocation algorithms are still the ones in actual use because they are easy to implement. Given a fixed channel allocation scheme which allocates n_i channels to the ith cell, for $1 \leq i \leq N$, we may be interested in $n_{\min}$, the minimum number of channels necessary to make the allocation, satisfying the cochannel reuse and the adjacent channel use constraints imposed by the transmission quality requirements. The fixed channel allocation scheme may be determined from the offered traffic estimates and the blocking probability requirements. We have not found any efficient algorithm for this purpose in the literature. We suggest a heuristic approach here.

We first describe the concept of a graphs $F(\tilde{X})$, $F'(\tilde{X})$ induced by a M-tuple $\tilde{X} = (X_1, X_2, \ldots, X_M)$. $\tilde{X}$ be any M-tuple such that X_j is a nonnegative rational number and M is the number of maximal independent sets of the hypergraph modelling the cochannel reuse constraints of the cellular system. Let $X_{k_1}, X_{k_2}, \ldots, X_{k_V}$ be the ones with nonzero values. Let $X_{k_1} : X_{k_2} : \ldots : X_{k_V} = m_{k_1} : m_{k_2} : \ldots : m_{k_V}$, where $m_{k_1}, m_{k_2}, \ldots, m_{k_V}$ are relatively prime positive integers. If $X_j = 0$, $m_j = 0$. It is convenient to think of m_{k_l} as being (proportional to) the number of channels that are to be assigned to maximal independent set X_{k_l}. Form a graph $F(\tilde{X}) = (V_F, E_F)$[5] as follows. $V_F = \{v_{k_1}, v_{k_2}, \ldots, v_{k_V}\}$. Vertices v_{k_l} and v_{k_t} are joined by an edge iff $b_{ij} = 1$ for each cell i in the k_lth maximal independent set and each cell j in the k_tth maximal independent set of the hypergraph modelling the cochannel reuse constraints for the cellular system, i.e., if any two cells such that one is in the k_lth maximal independent set and the other in the k_tth maximal independent set can use adjacent channels simultaneously. Next form a graph $F'(\tilde{X}) = (V_{F'}, E_{F'})$. $V_{F'} = \{v_{k_1 1}, \ldots, v_{k_1 m_{k_1}}, v_{k_2 1}, \ldots, v_{k_2 m_{k_2}}, \ldots, v_{k_V 1}, \ldots, v_{k_V m_{k_V}}\}$.

[5] In our notation in any graph $G = (V, E)$, V is the vertex set and E is the edge set.

Vertices $v_{k_l a}$ and $v_{k_t b}$ have an edge between them iff the vertices v_{k_l} and v_{k_t} have an edge between them in $F(\tilde{X})$. In effect, each vertex in $F(\tilde{X})$ represents a maximal independent set (which is to be assigned at least one channel) and to obtain $F'(\tilde{X})$ each vertex in $F(\tilde{X})$ is replaced by m vertices where m is (proportional to) the number of channels to be assigned to the maximal independent set represented by that vertex. We say that $F(\tilde{X})$ and $F'(\tilde{X})$ have been induced by the M-tuple $\tilde{X} = (X_1, X_2, \ldots, X_M)$.

Theorem 1: Let $F(\tilde{X})$ and $F'(\tilde{X})$ be the graphs induced by some M-tuple $\tilde{X} = (X_1, X_2, \ldots, X_M)$. Let $F'(\tilde{X})$ become hamiltonian[6] upon the addition of some p edges. Let $q(p + \sum_{j=1}^{M} m_j) - 1_{\{pq>0\}}$[7] channels be available. Then there exists a fixed channel allocation algorithm which allocates $q \sum_{j=1}^{M} m_j a_{ij}$ channels to the ith cell, for $1 \leq i \leq N$, for any nonnegative integer q.

Note that the proof exhibits a channel allocation algorithm which attains the purpose.

Proof: Let $q(p + \sum_{j=1}^{M} m_j) - 1_{\{pq>0\}}$ channels be available. If $q = 0$ the lemma is trivially true. Let $q > 0$. We say that we allocate a channel to a maximal independent set when we allocate the channel to each cell in the maximal independent set. First we show that it is possible to allocate qm_j channels to the jth maximal independent set without violating the cochannel reuse and adjacent channel use constraints. Let the hamilton cycle of vertices of F' formed upon the addition of p edges to F' be $v_{i_0,j_0}, v_{i_1,j_1}, \ldots, v_{i_{U-1},j_{U-1}}, v_{i_0,j_0}$. Note that $U = \sum_{j=1}^{M} m_j$. Some of the consecutive vertices in this cycle are linked by an edge in F' and some by an added edge if $p > 0$. If $p > 0$, w.l.o.g. $v_{i_{U-1},j_{U-1}}$ and $v_{i_0 j_0}$ are joined by an added edge. Allocate channels to the maximal independent sets as per the following algorithm:

1. $a = 0$, $b = 0$, $c = 1$.
2. Give channel numbered b to the i_ath maximal independent set.
3. If $a = U - 1$, $c \rightarrow c + 1$. If $c > q$ stop.
4. If v_{i_a,j_a} and $v_{i_{(a+1) \bmod U} \, j_{(a+1) \bmod U}}$ are linked by an edge in F', then $b \rightarrow b + 1$; else $b \rightarrow b + 2$. $a \rightarrow (a + 1) \bmod U$. Go to step 2.

Observe that we are moving along the hamilton cycle q times and each time the vertex $v_{i_l j_l}$ is encountered the i_lth maximal independent set is given a channel. Since we are allocating different channels to different maximal independent sets, the cochannel reuse constraints are satisfied. Two cells i and j can get adjacent channels only if i is in a maximal independent set k_1 and j in k_2, such that the vertices corresponding to the two maximal independent sets k_1 and k_2 are linked by an edge in F' and hence in F (v_{de} and v_{fg} have an edge between them in F' iff v_d and v_f have an edge between them in F.). This means that any cell i in the maximal independent set k_1 and j in k_2 are permitted to use adjacent channels simultaneously. Thus the adjacent channel use constraints are satisfied.

[6] A cycle in a graph is a sequence of vertices such that consecutive vertices in the cycle have an edge between them and no vertex in the sequence occurs more than once except the first (or the last) which occurs twice, once in the first position and again in the last position. A graph is hamiltonian iff it has a cycle consisting of all vertices. Such a cycle is known as a hamilton cycle.

[7] $1_{\{pq>0\}} = 1$ iff $pq > 0$ and $1_{\{pq>0\}} = 0$ otherwise.

The vertex $v_{i_l j_l}$ is encountered q times while traversing the hamilton cycle q times giving q different channels to the i_lth maximal independent set, for each i_l such that $X_{i_l} > 0$ and $j_l \in \{1, 2, \ldots, m_{i_l}\}$. Thus each of the following vertices $v_{i_l 1}, v_{i_l 2}, \ldots, v_{i_l, m_{i_l}}$ give q channels to the i_lth maximal independent set, giving it qm_{i_l} channels, if $X_{i_l} > 0$. If $X_j = 0$, the jth maximal independent set gets $qm_j = 0$ channels. If $p = 0$ no channel has been skipped.[8] If $p > 0$, $p-1$ channels are skipped the last time the hamilton cycle is traversed and p channels are skipped each of the other $q-1$ times the hamilton cycle is traversed, because of the p added edges. Thus $q(p + \sum_{j=1}^{M} m_j) - 1_{\{pq>0\}}$ channels have been used to allocate qm_j channels to the jth maximal independent set satisfying the cochannel reuse and adjacent channel use constraints. This fixed channel allocation algorithm allocates $q \sum_{j=1}^{M} m_j a_{ij}$ channels to the ith cell. ■

Consider the following integer linear program:

$$\left.\begin{array}{l} \text{Minimize} \sum_{j=1}^{M} W_j \\ \sum_{j=1}^{M} W_j a_{ij} \geq n_i, \quad i = 1, 2, \ldots N \\ W_j \geq 0, \quad j = 1, 2, \ldots, M, \quad W_j \text{ integer.} \end{array}\right\} \text{(ILP)}$$

Let n be the optimal value of the objective function. Consider graphs $G(\tilde{X})$ and $G(\tilde{X})'$ induced by the optimal solution $(W_1^O, \ldots, W_M^O)$. As usual $W_{i_1}^O : W_{i_2}^O : \ldots : W_{i_V}^O = m_{i_1} : m_{i_2} : \ldots : m_{i_V}$, where $m_{i_1}, m_{i_2}, \ldots, m_{i_V}$ are relatively prime positive integers and $W_{i_k} > 0$, $1 \leq k \leq V$. $V_G = \{v_{i_1}, v_{i_2}, \ldots, v_{i_V}\}$ and $m_j = 0$ if $W_j = 0$. E_G is determined from the adjacent channels use constraint given by B. G' is constructed from G and $m_{i_1}, m_{i_2}, \ldots, m_{i_V}$ as before.

The following Corollary follows from Theorem 1.

Corollary 1: Let G' become hamiltonian on the addition of p edges. Then $n + (qp-1)^+$ channels are sufficient for the fixed channel allocation algorithm which allocates n_i channels to the ith cell, for $1 \leq i \leq N$, where $q =$ g.c.d. of nonzero W_j^Os, if there exists at least one nonzero W_j^O and $q = 0$ otherwise.

We summarize the heuristic approach as follows:

1. Compute the ILP. Let n be the optimal value of the objective function. Let $(W_1^O, W_2^O, \ldots, W_M^O)$ be the optimal solution. $q =$ g.c.d. of nonzero W_i^O, if there exists at least one nonzero W_j^O, and $q = 0$ otherwise.
2. Form G and G' from the optimal solution as discussed above.
3. Compute the minimum number of edges $p_{\min}$ required to make G' hamiltonian, or an approximation to it, say p. We discuss how to compute of $p_{\min}$ in Section III.
4. Use $n + (pq-1)^+$ channels to make the fixed channel allocation. The closer p is to $p_{\min}$, fewer the number of additional edges required to make G' hamiltonian, and lesser the number of channels used.
5. The algorithm for making the required fixed channel allocation using $n + (pq-1)^+$ channels follows from the proof of Theorem 1. Find the channels allocated to the jth maximal independent set by the algorithm described in the proof to allocate $W_j^O = qm_j$

[8] A channel is said to be "skipped" if it is not allocated to any maximal independent set.

channels to the jth maximal independent set. All the channels allocated to the jth maximal independent set are allocated to each of the cells in the jth maximal independent set. This gives the actual channel allocation to the cells.

If pq is small compared to n, then the number of channels obtained from our heuristic will be close to the minimum since n channels are necessary for making the fixed channel allocation even while satisfying only the cochannel reuse constraints. We illustrate the actual channel allocation to the cells in the following examples:

Example II.1: Consider the 7-cell system of figure 2. Let the cochannel reuse constraint be that the maximum interference should not exceed 0.15. The hypergraph modelling this cochannel reuse constraint has 10 maximal independent sets: $\{1,3\}$, $\{1,4\}$, $\{1,5\}$, $\{2,4\}$, $\{2,5\}$, $\{2,6\}$, $\{3,5\}$, $\{3,6\}$, $\{4,6\}$, $\{7\}$. We need to allocate 4 channels to each cell. Thus $n_i = 4$, for $1 \leq i \leq 7$. The ILP gives nonzero values to only the following maximal independent sets: $a \rightarrow \{1,3\}, b \rightarrow \{2,4\}, c \rightarrow \{3,5\}, d \rightarrow \{4,6\}, e \rightarrow \{1,5\}, f \rightarrow \{2,6\}, g \rightarrow \{7\}$. $a = b = c = d = e = f = 2$, $g = 4$, $n = 16$, $q = 2$, $m_a = m_b = m_c = m_d = m_e = m_f = 1$ and $m_g = 2$. We shall consider two different adjacent channel use constraints.

Case 1: Adjacent channels cannot be used simultaneously only in the same cell. The graphs G and G' have been shown in figure 5. G' is hamiltonian. Thus $p_{\min} = 0$ and 16 channels are sufficient. $a\ g1\ b\ c\ d\ g2\ e\ f\ a$ is a hamilton cycle in G'. Proceeding as per the algorithm in Theorem 1 with $q = 2$, a gets channels $1, 9$, $g1$ gets $2, 10$, b gets $3, 11$, c gets $4, 12$, d gets $5, 13$, $g2$ gets $6, 14$, e gets $7, 15$ and f gets $8, 16$. Thus cell 1 gets channels $\{1,7,9,15\}$, cell 2 gets $\{3,8,11,16\}$, cell 3 gets $\{1,4,9,12\}$, cell 4 gets $\{3,5,11,13\}$, cell 5 gets $\{4,7,12,15\}$, cell 6 gets $\{5,8,13,16\}$, and cell 7 gets $\{2,6,10,14\}$.

Case 2: Again assume that adjacent channels cannot be used simultaneously in the s ame cell and also that cell 7 cannot use a channel if its adjacent channel is being used in either cell 2, 4 or 6. Refer to figures 6(a) and (b) for G and G' respectively. G' is not hamiltonian. However it becomes hamiltonian when edges are added between the pairs of vertices $(g1, b)$ and $(d, g2)$. Thus $p_{\min} \leq 2$ and 19 channels are sufficient. $b\ c\ d\ g2\ e\ f\ a\ g1\ b$ is a hamilton cycle in G' with the added edges. Proceeding as per the algorithm in Theorem 1 with $q = 2$, b gets channels $1, 11$, c gets $2, 12$, d gets $3, 13$, $g2$ gets $5, 15$, e gets $6, 16$, f gets $7, 17$, a gets $8, 18$ $g1$ gets $9, 19$. Thus cell 1 gets channels $\{6,8,16,18\}$, cell 2 gets $\{1,7,11,17\}$, cell 3 gets $\{2,8,12,18\}$, cell 4 gets $\{1,3,11,13\}$, cell 5 gets $\{2,6,12,16\}$, cell 6 gets $\{3,7,13,17\}$, and cell 7 gets $\{5,9,15,19\}$.

For large systems the ILP is difficult to evaluate. The integer constraints in the ILP may be relaxed to get an approximate solution. $n_{\text{approx}} = \sum_{j=1}^{M} \lceil W_j^O \rceil$ can then be taken as an approximation to n and $(\lceil W_1^O \rceil, \lceil W_2^O \rceil, \ldots, \lceil W_M^O \rceil)$ can be used in place of the optimal solution to the ILP. Clearly $n_{\text{approx}} + (pq-1)^+$ channels, with p obtained from the corresponding G and G' and q from the $(\lceil W_1^O \rceil, \lceil W_2^O \rceil, \ldots, \lceil W_M^O \rceil)$, is sufficient for the required fixed channel allocation. In general n_{approx} is a good approximation to n, the optimal value of the objective function of the ILP.

III. Approximation to r_0^A

Let G and G' be the graphs induced by a rational M-tuple $(X_1^O, X_2^O, \ldots, X_M^O)$ which forms an optimal solution of LP2. (Clearly any extreme point of the feasible region corresponds to rational values of $X_1^O, X_2^O, \ldots, X_M^O$ because the constraints involve rational numbers only. We know at least one of the extreme points gives the optimal solution, if there exists an optimal solution [8]. LP2 always has an optimal solution [4]. Thus there always exists a rational M-tuple $(X_1^O, X_2^O, \ldots, X_M^O)$ which forms an optimal solution of LP2.) γ_G (the number of vertices in G) $= V \leq N$ and $\gamma_{G'} = \sum_{j=1}^{M} m_j$.

Theorem 2: Let G' become hamiltonian upon the addition of some p edges. Then $r_0^A \geq r_0/(1+p/\sum_{j=1}^{M} m_j)$.

Proof: Let there be n channels. Then $n = q(p + \sum_{j=1}^{M} m_j) + t$ where $0 \leq t < (p+\sum_{j=1}^{M} m_j)$ and q is a nonnegative integer (Euclidean division theorem.). By Theorem 1 we have a fixed channel allocation algorithm that allocates $q\sum_{j=1}^{M} m_j a_{ij}$ channels to the ith cell, for $1 \leq i \leq N$. Using the ATP and the independence of offered traffic from cell to cell it can be shown that this fixed channel allocation algorithm carries a traffic intensity of $\min\left(p_i r, \frac{\sum_{j=1}^{M} m_j a_{ij}}{p+\sum_{j=1}^{M} m_j}\right)$ in the asymptotic limit ($n \to \infty$) in cell i (refer to the Appendix for the proof). Let $r \leq r_0/(1+p/\sum_{j=1}^{M} m_j)$. Then

$$
\begin{aligned}
p_i r &\leq p_i r_0/(1+p/\textstyle\sum_{j=1}^{M} m_j) \\
&\leq \frac{\sum_{j=1}^{M} X_j^O a_{ij}}{(1+p/\sum_{j=1}^{M} m_j)} && \text{from the third set of inequalities in LP2} \\
&= \frac{\frac{\sum_{j=1}^{M} m_j a_{ij}}{\sum_{k=1}^{M} m_k}}{(1+p/\sum_{j=1}^{M} m_j)} && \text{since } X_j^O = \frac{m_j}{\sum_{k=1}^{M} m_k} \\
&= \frac{\sum_{j=1}^{M} m_j a_{ij}}{(p+\sum_{j=1}^{M} m_j)}.
\end{aligned}
$$

Thus in the asymptotic limit the carried traffic intensity in the ith cell is $\lim_{n\to\infty} C(p_i r n, q\sum_{j=1}^{M} m_j a_{ij})/n = \min(p_i r, \sum_{j=1}^{M} m_j a_{ij}/(p+\sum_{j=1}^{M} m_j)) = p_i r$. Thus the blocking probability is 0 in the asymptotic limit for this fixed channel allocation algorithm if $r \leq r_0/(1+p/\sum_{j=1}^{M} m_j)$. Thus $r_0^A \geq r_0/(1+p/\sum_{j=1}^{M} m_j)$, since r_0^A is the largest value of r for which a channel assignment algorithm that has asymptotically zero blocking exists. ■

Let $r_0^{AL} = r_0/(1+p/\sum_{j=1}^{M} m_j)$. Clearly the lower bound is the tightest if $p = p_{\min}$, where $p_{\min}$ is the minimum number of edges which must be added to G' to make it hamiltonian. In general, the problem of finding $p_{\min}$ is NP-complete [9] but $p_{\min}$ can be found easily in the following special cases.

1. G is a complete[9] graph. Let G' have $\gamma_{G'}$ vertices. Then

$$p_{\min} = \begin{cases} (\max_j m_j - \sum_{\substack{j=1 \\ j \neq \arg\max_k m_k}}^{M} m_j)^+, & \text{if } \gamma_{G'} \geq 3, \\ 0, & \text{if } \gamma_{G'} = 2 \\ 1, & \text{if } \gamma_{G'} = 1. \end{cases}$$

The proof goes as follows. Let $\gamma_{G'} \geq 3$, and let $\max_j m_j \leq \sum_{\substack{j=1 \\ j \neq \arg\max_k m_k}}^{M} m_j$. Let v_{jk} be a vertex in G'. $\gamma_{G'} = \sum_{i=1}^{M} m_i$. The degree of a vertex v in a graph G (denoted by $d_G(v)$) is the number of edges incident on v. $d_{G'}(v_{jk}) = \sum_{\substack{i=1 \\ i \neq j}}^{M} m_i = \gamma_{G'} - m_j$.

$$\begin{aligned} \max_i m_i &\leq \sum_{\substack{i=1 \\ i \neq \arg\max_l m_l}}^{M} m_i \\ 2 \max_i m_i &\leq \sum_{\substack{i=1 \\ i \neq \arg\max_l m_l}}^{M} m_i + \max_i m_i \\ 2 \max_i m_i &\leq \sum_{i=1}^{M} m_i \\ \max_i m_i &\leq \frac{\sum_{i=1}^{M} m_i}{2} \\ m_j &\leq \frac{\gamma_{G'}}{2} \quad \text{for all } j \\ d_{G'}(v_{jk}) &\geq \frac{\gamma_{G'}}{2}. \end{aligned}$$

Thus G' is hamiltonian ([10], page 54, Theorem 4.3). Thus $p_{\min} = 0$ and the relation holds in this case.

Let $\max_j m_j > \sum_{\substack{j=1 \\ j \neq \arg\max_k m_k}}^{M} m_j$. Let $i = \arg\max_k m_k$. $v_{i1}, v_{i2}, \ldots, v_{im_i}$ must be part of the hamilton cycle. Clearly v_{ia}, v_{ib} do not have an edge between them in G', since G is simple. Thus an extra edge must be added between v_{ia} and v_{ib} if they are consecutive vertices in the cycle. There must be at least $m_i - \sum_{\substack{j=1 \\ j \neq i}}^{M} m_j$ pairs v_{ia} and v_{ib} such that the vertices of the pair occupy consecutive positions in the cycle. Thus $\max_j m_j - \sum_{\substack{j=1 \\ j \neq \arg\max_k m_k}}^{M} m_j$ edges must be added to make G' hamiltonian. By placing one v_{jk} for some k and some $j \neq i$ between v_{il} and v_{il+1} till the v_{jk}s are exhausted, exactly $m_i - \sum_{\substack{j=1 \\ j \neq i}}^{M} m_j$ pairs v_{ia} and v_{ib} will be there such that the vertices of the pair occupy consecutive positions in the cycle.

[9] A graph is simple if no edge joins a vertex to itself.A graph is complete if it is simple and any two distinct vertices in the graph has an edge between them [10].

Thus only $\max_j m_j - \sum_{\substack{j=1 \\ j\neq \arg\max_k m_k}}^{M} m_j$ edges need be added to make G' hamiltonian. Thus $p_{\min} = \max_j m_j - \sum_{\substack{j=1 \\ j\neq \arg\max_k m_k}}^{M} m_j$ in this case.
Let $\gamma_{G'} = 1$. Since G is simple, G' must be simple and thus 1 edge must be added to make G' hamiltonian. Let $\gamma_{G'} = 2$. Note that $\gamma_G \le \gamma_{G'}$. If $\gamma_G = 1$, and $V_G = v_l$, $m_l = 1$. ($X_l^O > 0$, $X_i^O = 0$, if $i \neq l$, $\sum_{i=1}^{M} X_i^O = 1 \Rightarrow X_l^O = 1$, $m_l = 1$.) This means $\gamma_{G'} = 1$ which is a contradiction. Thus $\gamma_G = 2$. Let $V_{G'} = \{v_{i1}, v_{j1}\}$. Since G is complete, so is G'. Thus there is an edge between v_{i1} and v_{j1}. As per our definition, $v_{i1}v_{j1}v_{i1}$ is a hamilton cycle. Thus no edge need be added to G' to make it hamiltonian. Thus $p_{\min} = 0$ if $\gamma_{G'} = 2$.
G is complete for many systems. For example, G is complete for a linear array of cells, where any two cells can use the same channel simultaneously iff they are separated by a distance $D \ge uR$, where R is the center to center distance between adjacent cells, and the adjacent channel use constraint is that the same cell cannot use adjacent channels simultaneously, and $p_i = p_{i+u}$ for all i. Thus $p_{\min}$ can be computed easily for these systems.
2. G is completely disconnected, i.e., G has no edge. Thus G' is also completely disconnected. Thus $p_{\min} = \gamma_{G'} = \sum_{j=1}^{M} m_j$. We have found that G is in general disconnected if adjacent channels cannot be used simultaneously in the same or adjacent cells.
3. There are certain sufficient conditions involving the degree sequence and the number of edges in G' ([10], [11]) for G' to be hamiltonian. If these are satisfied, then $p_{\min} = 0$. Often a good approximation to $p_{\min}$ can be found from these sufficiency conditions.

If none of the above cases apply then we need to resort to other means to find $p_{\min}$. The problem of finding $p_{\min}$ can be reduced to a travelling salesman problem [10]. The travelling salesman problem is an NP-complete problem but there exist techniques which yield good approximations to the required result [10], [9].

We have used the following seemingly crude method which surprisingly gives small values of p in a very short time for all the cases we have studied and whose results we shall present in tables I and II. We first briefly describe our method. Finding a hamilton cycle in G' is equivalent to finding a closure-possible walk[10] in G ($\gamma_G \le N$) traversing vertex v_j m_j times. The length of the walk must be $\gamma_{G'}$. We have observed that a simple *branch and backtrack*[11] technique gives a walk of length, say T, very fast, where T is fairly close to $\gamma_{G'}$. Moreover this walk traverses no vertex v_j more than m_j times but after the length of the walk increases to T any further increase takes place very slowly. We stop the branch and backtrack process after a walk which is T vertices long is obtained. If this walk traverses the vertex v_j n_j times, we try to insert $m_j - n_j$ v_js in the walk that has already been obtained. If we cannot do so, we add extra edges which increase the value of p we can get. The number T is decided upon after a few observations. We illustrate the branch and backtrack

[10] A walk in a graph is a sequence of vertices such that the consecutive vertices in the sequence have an edge between them. A walk is closure-possible if the first and the last vertices have an edge between them.

[11] This procedure traverses along the graph without visiting any vertex v_j more than m_j times as long as it can, and when it cannot it backtracks along the traced path, till it can proceed along some branch which has not yet been visited.

procedure by the following example.

Example III.1: Let the graph G consist of W vertices numbered $1, 2, \ldots, W$ where W is quite large. The edges in G are as follows: i and $i+1$ are joined by an edge for all $i < W-1$. W has an edge with 1 and 2. $W-1$ has an edge with 1. There is no other edge in G. $m_i = 1$ for all i. The only closure-possible walk covering each vertex once is $1\ W\ 2\ 3 \ldots W-1$. (We consider cyclic shifts such as $1\ W\ 2\ 3 \ldots W-1$ and $W\ 2\ 3 \ldots W-1\ 1$ as being equivalent.) However the branch and backtrack procedure may move along the branch $1\ 2\ 3 \ldots W-1$. Thus a walk of length $W-1$ is obtained very fast but then on, the process backtracks in order to insert W, and this may take a long time. If the procedure is terminated when the walk of length $W-1$ is obtained, then W can be very easily inserted between 1 and 2 and thus a closure-possible walk is obtained and $p=0$. If W could not be inserted i.e., if it did not have edges joining it to 1 and 2, then 2 edges could have been added, one joining W to 1 and the other to 2, and a closure-possible walk could have been obtained, giving $p=2$. This value of p is small, particularly if W is large and thus we get a good approximation to $p_{\min}$ in a very short time.

If the graph G' does not give a small value of p fast with the above method with a particular optimal solution, the graph G' obtained from some other rational optimal solution may be tried. This often helps when there are multiple optimal solutions (degeneracy).

We now present our results for the regular hexagonal 3-cell, 7-cell, 19-cell and 37-cell systems (refer figures 1 to 4). We have assumed that the cochannel reuse in the system is constrained by the maximum tolerable interference. (Any two calls in the system using the same channel simultaneously interfere with each other. The interference diminishes with increase in distance between the callers.) The assumed model of interference is as follows:

- Interference produced in cell u due to the usage of the same channel in cell v equals $d(u,v)^{-4}$, where $d(u,v)$ is the center-to-center distance between cells u and v.
- Total interference produced in cell u = interference produced by all other cells using the same channel $= \sum_{v \in C(u)} d(u,v)^{-4}$, where $C(u)$ is the set of cells using the same channel as u, barring u. An *additive* model of interference is thus assumed.
- The cell radius is assumed to be $1/\sqrt{3}$ or equivalently the distance between adjacent cells is taken to be 1.
- Let the requisite transmission quality be that the maximum interference must be less than or equal to some given threshold. This limits cochannel reuse.

This model for interference is the same as that used in [4]. The cochannel reuse constraints have been modelled by a hypergraph. We illustrate the computation of the approximation to r_0^A by the following example.

Example III.2: Consider the 7-cell system of figure 2 described in Example II.1. The same model for interference is assumed as in Section II. The cochannel reuse constraints, and hence the maximal independent sets of the hypergraph modelling the cochannel reuse constraints, remain the same as in Example II.1. LP2 gives nonzero values to only the following maximal independent sets: $a \to \{1,3\}, b \to \{2,4\}, c \to \{3,5\}, d \to \{4,6\}, e \to \{1,5\}, f \to \{2,6\}, g \to \{7\}$. $a = b = c = d = e = f = 1/8$, $g = 1/4$, $m_a = m_b = m_c = m_d = m_e = m_f = 1$, and $m_g = 2$. We shall consider two different adjacent channel use constraints

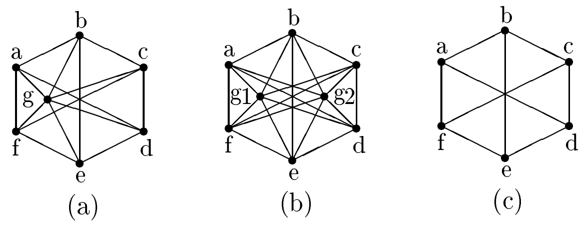

Fig. 5. The graphs for the 7-cell system when adjacent channel use is prohibited in the same cell.

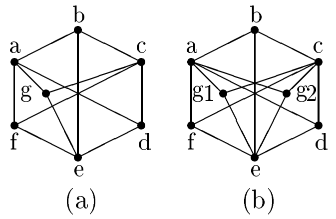

Fig. 6. The graphs for the 7-cell system when adjacent channel use is prohibited in the same cell, and also cell 7 cannot use a channel if its adjacent channel is being used in cell 2, 4 or 6.

as in Example II.1:
Case 1: Adjacent channels cannot be used simultaneously only in the same cell. Thus $b_{ii} = 2$ for all i and $b_{ij} = 1$, if $i \neq j$. The graphs G and G' have been shown in figure 5(a) and (b) respectively. G' is hamiltonian. Thus $p_{\min} = 0$ and $r_0^{AL} = r_0 = 1.75$. Clearly $r_0^{AL} = r_0^A$ here. $a\ g1\ b\ c\ d\ g2\ e\ f\ a$ is a hamilton cycle in G'.
Case 2: Assume adjacent channels cannot be used simultaneously in the same cell and also that cell 7 cannot use a channel if its adjacent channel is being used in either cell 2, 4 or 6. Thus

$$b_{ij} = \begin{cases} 2, & \text{if } i = j, \text{ or } (i,j) \in \{(2,7), \\ & \qquad (4,7),(6,7),(7,2),(7,4),(7,6)\}, \\ 1, & \text{otherwise.} \end{cases}$$

Refer to figure 6(a) and (b) for G and G' respectively. G' is not hamiltonian. However it becomes hamiltonian when edges are added between $g1, b$ nd $d, g2$. Thus $p_{\min} \leq 2$ and $r_0^{AL} = 0.8r_0 = 1.4$. Using $r_0^{AL} \leq r_0^A \leq r_0$, $1.4 \leq r_0^A \leq 1.75$. $b\ c\ d\ g2\ e\ f\ a\ g1\ b$ is a hamilton cycle in G' with the added edges.

We will study two cases of adjacent channel use constraints. The first prevents the use of adjacent channels in the same cell simultaneously but allows any other form of adjacent channel use. All nondiagonal elements of the corresponding B matrix are 1 and the diagonal elements are 2. The second adjacent channel use constraint prevents the simultaneous use of adjacent channels in cells separated by a distance ≤ 1, i.e., $b_{ij} = 1$ iff $d(i,j) > 1$ and $b_{ij} = 2$ otherwise.

We have also studied two different traffic patterns: Uniform traffic pattern (UT) and the Nonuniform traffic pattern (NUT). In the former the same amount of traffic is offered in each cell, i.e., $p_i = 1/N$, for all i. In the latter the maximum traffic is offered in the central cell, one half of that in the next ring of cells, one third of that in the next ring, and so on. Nonuniform traffic pattern has been studied for the 19-cell system only. For the 19-cell system,

$$p_i = \begin{cases} 1/24, & i \in \{1,\ldots,4,\ 7,\ 8,\ 12,\ 13,\ 16,\ldots,19\}, \\ 1/16, & i \in \{5,6,9,11,14,15\}, \text{ and} \\ 1/8, & i = 10. \end{cases}$$

This model of nonuniform traffic pattern may be representative of cities where more traffic is offered in the center and less in the outskirts.

The results for the first case have been tabulated in Table I and Table II and for the second case in Table III and Table IV. The number of vertices in G and G' have not been listed for the second case because the graph G is completely disconnected in all these cases and we get $p_{\min}$ and hence r_0^{AL} from the special case (2). The number of vertices in G and G' are anyway the same as the corresponding ones in Table I.

In Table I we have not listed the number of vertices of G and G' for the 19-cell system with nonuniform traffic pattern for interference threshold ≥ 0.4 because an optimal solution of LP2 in each of these cases is same as that for an interference threshold of 0.375. Thus the graphs G and G' are also the same as that for an interference threshold of 0.375. We have listed the number of variables, constraints, and so on, in the linear program used in [2]

TABLE I

A COMPARISON OF THE COMPLEXITIES OF DEORA'S METHOD [2] AND OUR METHOD IN THE CASE WHERE ADJACENT CHANNEL USE IN THE SAME CELL ALONE IS PROHIBITED.

		Deora's method[2]		Our method					
No. of cells	Cochannel Interf. Threshold	Number of vars.	Number of constrs.	Number of vars	Number of constrs	Number of vertices in G UT	Number of vertices in G NUT	Number of vertices in G' UT	Number of vertices in G' NUT
3	(0, 1)	17	9	3	4	2		2	
7	0.15	201	25	11	8	7		8	
19	0.1	53515	311	85	20	17	18	98	74
19	0.2	344539	833	249	20	8	14	18	624998
19	0.3	681543	1238	242	20	18	17	147	50
19	0.375	802297	1373	188	20	17	11	49	500006
19	0.4	824137	1397	194	20	18		82	
19	0.5	856699	1432	167	20	6		10	
19	0.6	867139	1444	146	20	9		19	
19	2/3	867971	1445	140	20	3		3	
37	0.1	235796085	21310	4987	38	34		1111	

in one of the cases for the 37 cell system. The numbers are even larger for the other cases for the 37 cell system when the interference threshold is higher.

The following observations may be made from the tabulated data:

1. We know that $r_0^{AL} \leq r_0^A \leq r_0$. Thus r_0^{AL} is a good approximation to r_0^A if $r_0^{AL}/r_0 = (1+p/\sum_{j=1}^M m_j)^{-1}$ is close to 1, and is the same as r_0^A if $r_0^{AL}/r_0 = 1$. Thus Table II indicates that r_0^{AL} is a very good approximation to r_0^A and often gives the exact r_0^A in the first case, i.e., when adjacent channels cannot be used simultaneously in the same cell only. Also the proximity of r_0^{AL}/r_0 to 1 indicates that $r_0^A \approx r_0$ in this case. Table I indicates that the computation of the exact value of r_0^A as per [2] may be impossible in this case even for the 19-cell system. We could compute the approximations for the 19-cell system using no more than 0.4 seconds of system time on an IBM SP2 machine. Our computations took less than 0.5 minutes of system time on an IBM SP2 for the 37-cell system.

2. Consider Table IV where we list the results for the case where adjacent channels cannot be used simultaneously in the same cell and in adjacent cells. The graph G is completely disconnected in this case and $p_{\min} = \gamma_{G'}$. Thus $r_0^{AL}/r_0 = (1+p/\sum_{j=1}^M m_j)^{-1} = 0.5$. Thus the value of r_0^{AL}/r_0 does not guarantee that our approximation is good. However in this case the number of variables and constraints in the linear program used in [2] are not that large

[11] From the hamilton cycle, $r_0^{AL} = (1+p/\sum_{j=1}^M m_j)^{-1} r_0 = 4.5$. Using the fact that r_0^A increases with increase in allowable interference threshold and $r_0^A = 5.04$ for a lower value of the interference threshold, namely 0.375, we get a better lower bound. We have similarly improved the lower bound for the 37-cell system with an interference threshold of 0.5.

TABLE II

THE LOWER BOUNDS ON CAPACITIES OBTAINED USING OUR METHOD IN THE CASE WHERE ADJACENT CHANNEL USE IN THE SAME CELL ALONE IS PROHIBITED.

No. of cells	Cochannel Interf. Threshold	r_0^{AL}/r_0 UT	NUT	r_0^{AL} UT	NUT	r_0 UT	NUT
3	(0,1)	1.0		1.50		1.50	
7	0.15	1.0		1.75		1.75	
19	0.02	1.0	1.0	2.11	1.85	2.11	1.85
19	0.1	1.0	1.0	3.10	2.56	3.10	2.56
19	0.2	1.0	1.0	4.22	3.20	4.22	3.2
19	0.3	1.0	1.0	4.65	3.84	4.65	3.84
19	0.375	1.0	1.0	5.04	4	5.04	4
19	0.4	0.99	1.0	5.04	4	5.10	4
19	0.5	1.0	1.0	5.73	4	5.73	4
19	0.6	1.0	1.0	6.00	4	6.00	4
19	2/3	1.0	1.0	6.33	4	6.33	4
37	0.1	1.0		5.13		5.13	
37	0.12	1.0		5.69		5.69	
37	0.39	1.0		9.25		9.25	
37	0.5	1.0		9.25		9.26	
37	0.6	1.0		10.74		10.74	
37	0.7	1.0		10.96		10.98	
37	0.74	1.0		12.33		12.33	

TABLE III

A COMPARISON OF THE COMPLEXITIES OF DEORA'S METHOD [2] AND OUR METHOD IN THE CASE WHERE ADJACENT CHANNEL USE IN THE SAME CELL AND IN ADJACENT CELLS IS PROHIBITED.

		Deora's method[2]		Our method	
No. of cells	Cochannel Interference Threshold	Number of vars.	constrs.	Number of vars	constrs
3	(0 1)	11	8	3	4
7	0.15	63	24	11	8
19	0.1	7368	311	86	20
19	0.2	19097	832	249	20
19	0.3	24131	1237	242	20
19	0.375	25125	1322	188	20
19	0.4	25244	1396	194	20
19	0.5	25429	1431	167	20
19	0.6	25465	1443	146	20
19	2/3	25467	1444	140	20

TABLE IV

THE LOWER BOUNDS ON CAPACITIES OBTAINED USING OUR METHOD IN THE CASE WHERE ADJACENT CHANNEL USE IN THE SAME CELL AND IN ADJACENT CELLS IS PROHIBITED.

No. of cells	Cochannel Interference Threshold	r_0^{Al}/r_0^A		r_0^{AL}		r_0^A	
		UT	NUT	UT	NUT	UT	NUT
3	(0 1)	1.0		0.75		0.75	
7	0.15	0.75		0.875		1.17	
19	0.1	0.65	0.80	1.55	1.28	2.37	1.6
19	0.2	0.76	0.80	2.11	1.6	2.78	2.0
19	0.3	0.73	0.96	2.33	1.92	3.17	2.0
19	0.375	0.80	1.0	2.52	2.0	3.17	2.0
19	0.4	0.81	1.0	2.55	2.0	3.17	2.0
19	0.5	0.90	1.0	2.85	2.0	3.17	2.0
19	0.6	0.95	1.0	3.0	2.0	3.17	2.0
19	2/3	1.0	1.0	3.17	2.0	3.17	2.0

and we could compute r_0^A as per [2] for the 3-cell, 7-cell and 19-cell systems. Comparison of r_0^{AL} with r_0^A indicates that r_0^{AL} is reasonably close to r_0^A in most of the cases but there is a significant difference in some of the cases. Better heuristics than those we have proposed here may be needed in such cases. The results indicate that r_0 (in this case $r_0 = 2r_0^{AL}$) is significantly higher than r_0^A. Our approximation is much easier to compute in this case even for the 19-cell system. For the 37-cell system again the computation of the exact value of r_0^A as per [2] may be impossible.

IV. Approximation to $T_A(r)$

Let LP1 have a rational optimal solution, $(X_1^O, \ldots, X_M^O)$. Now let G and G' be the graphs induced by the M-tuple $(X_1^O, X_2^O, \ldots, X_M^O)$. Clearly $X_k^O = m_k / \sum_{j=1}^M m_j$, for $1 \le k \le M$.

Theorem 3: Let G' become hamiltonian upon the addition of p edges. Then $T_A(r) \ge T(r)/(1 + p/\sum_{j=1}^M m_j)$.

Proof: This proof is similar to that of Theorem 2. Let there be n channels. Then $n = q(p + \sum_{j=1}^M m_j) + t$ for some t and q where $0 \le t < (p + \sum_{j=1}^M m_j)$ and q is a nonnegative integer (Euclidean division theorem). By Theorem 1 we have a fixed channel allocation algorithm allocating $q\sum_{j=1}^M m_j a_{ij}$ channels to the ith cell for $1 \le i \le N$. Using the ATP and the independence of offered traffic it can be shown that this fixed channel allocation algorithm carries a traffic intensity of min $\left(p_i r, \frac{\sum_{j=1}^M m_j a_{ij}}{p+\sum_{j=1}^M m_j}\right)$ in the asymptotic limit ($n \to \infty$) in cell i (refer to the Appendix for the proof). Let the total traffic intensity carried by this algorithm in the system be $T_{FCA}(r)$. Then

$$T_{FCA}(r) = \sum_{i=1}^{N} \min(p_i r, \frac{\sum_{j=1}^M m_j a_{ij}}{p + \sum_{j=1}^M m_j})$$

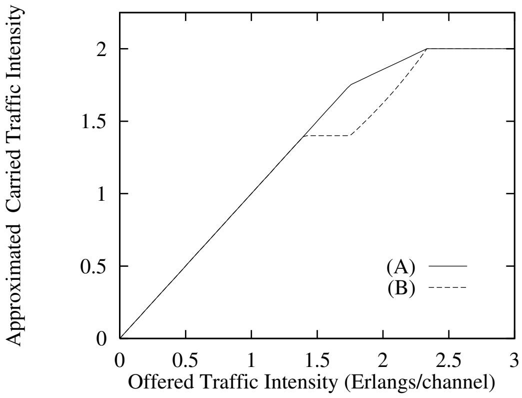

Fig. 7. The curve (A) gives $T_{AL}(r)$, $T(r)$ and $T_A(r)$ for Example II.3.1. The curve (B) gives $T_{AL}(r)$ for Example II.3.2. $T(r)$ is given by curve (A). $T_A(r)$ is between curves (A) and (B) in this case.

$$\begin{aligned}
&\geq \sum_{i=1}^{N} \frac{\sum_{j=1}^{M} m_j}{p+\sum_{j=1}^{M} m_j} \min(p_i r, \frac{\sum_{j=1}^{M} m_j a_{ij}}{\sum_{j=1}^{M} m_j}) \\
&= \frac{\sum_{j=1}^{M} m_j}{p+\sum_{j=1}^{M} m_j} \sum_{i=1}^{N} \min(p_i r, \sum_{j=1}^{M} X_j^O a_{ij}) \\
&\qquad \text{since } X_j^O = m_j / \sum_{k=1}^{M} m_k \\
&= \frac{\sum_{j=1}^{M} m_j}{p+\sum_{j=1}^{M} m_j} T(r).
\end{aligned}$$

Thus $T_A(r) \geq T_{FCA}(r) \geq \frac{\sum_{j=1}^{M} m_j}{p+\sum_{j=1}^{M} m_j} T(r) = T_{AL}(r)$ where the last equality defines $T_{AL}(r)$ ■

Again the lower bound is the tightest if $p = p_{\min}$. The techniques for finding $p_{\min}$ in the special cases and the approximation to $p_{\min}$ in the general case discussed in Section III apply here. Certain other observations simplify the computation of $T_A(r)$, e.g., $T_A(r) = r$ for $r \leq r_0^{AL}$. (This follows from the definition of r_0^A and the fact that $r_0^{AL} \leq r_0^A$.) Since $T_{AL}(r) = T(r)/(1+p/\sum_{j=1}^{M} m_j) \leq T_A(r) \leq T(r)$, one can be assured that the approximation is doing well if p is small compared to $\sum_{j=1}^{M} m_j$. In general it is much easier to compute $T_{AL}(r)$ than $T_A(r)$ as per [2].

We would also like to point out that since $T_A(r) = r$, for $r \leq r_0^{AL} \leq r_0^A$, if r_0^{AL} is known then we can set $T_{AL}(r) = r$, for $r \leq r_0^{AL}$.

Example IV.1: Consider the 7-cell system of figure 2 described in Example II.1. The same model for interference is assumed as in Section II The cochannel reuse constraint and hence the maximal independent sets of the hypergraph modelling the cochannel reuse constraint remains the same as in Example II.1. LP1 gives the following solution:

$$T(r) = \begin{cases} r, \text{ if } r \leq 7/4, \\ 1 + 3r/7, \text{ if } 7/4 < r \leq 7/3, \\ 2, \text{ if } r \geq 7/3. \end{cases}$$

$$\text{For } r \leq 7/4, \quad X_i(r) = \begin{cases} r/14, \text{ if } i \in \{a, b, \ldots, f\}, \\ r/7, \text{ if } i = g, \\ 0, \text{ otherwise.} \end{cases}$$

$$\text{For } 7/4 < r \leq 7/3, \quad X_i(r) = \begin{cases} r/14, \text{ if } i \in \{a, b, \ldots, f\}, \\ 1 - 3r/7, \text{ if } i = g, \\ 0, \text{ otherwise.} \end{cases}$$

$$\text{For } r \geq 7/3, \quad X_i(r) = \begin{cases} 1/6, \text{ if } i \in \{a, b, \ldots, f\}, \\ 0, \text{ otherwise.} \end{cases}$$

Throughut in this example we shall assume that $r \in Q$. We shall consider two different adjacent channel use constraints as in Example II.1:

Case 1: Adjacent channels cannot be used simultaneously only in the same cell. We have found in Example III.2 that $r_0^A = r_0^{AL} = 7/4$. So $T_{AL}(r) = r$, for $r \leq 7/4$. Let $h = \text{LCM}(r, 14 - 6r)$. For $7/4 < r < 7/3$,

$$m_i = \begin{cases} h\frac{r}{\min(r, 14-6r)}, & \text{if } i \in \{a, b, \ldots, f\} \\ h\frac{14-6r}{\min(r, 14-6r)}, & \text{if } i = g \\ 0, & \text{otherwise.} \end{cases} \qquad \text{where } h = \text{LCM}(r, 14 - 6r)$$

Since $m_i \neq 0$, $i \in \{a, b, \ldots, g\}$, G is given by figure 5(a). G' is always hamiltonian in this range. Let $m_i = x, i \in \{a, b, \ldots, f\}$ and $m_g = y$. $V'_G = \{a_1, a_2, \ldots, a_x, b_1, b_2, \ldots, b_x, \ldots f_1, f_2, \ldots, f_x, g_1, g_2, \ldots, g_y\}$. $a_1 b_1 \ldots f_1 a_2 b_2 \ldots f_2 \ldots a_x b_x \ldots f_x a_1$ is a cycle in G'. Now insert g_1 between a_1 and b_1, g_2 between b_1 and c_1, and so on in the above cycle till the g_is are exhausted. The g_is will be exhausted before one reaches $b_{\lceil x/3 \rceil}$ since $0 < y < 2x$ for $7/4 < r < 7/3$. This gives a hamilton cycle in G' since g_i and p_j, $p \in \{a, b, \ldots, f\}$ $1 \leq i \leq y$ and $1 \leq j \leq x$ are connected by an edge in G'. Thus $p_{\min} = 0$. $T_{AL}(r) = T(r)$ for $7/4 < r < 7/3$. For $r \geq 7/3$, $m_i = 1$, $i \in \{a, b, \ldots, f\}$ and $m_i = 0$ otherwise. G and G' are given by figure 5(c). G' is clearly hamiltonian. $abcdefa$ is the hamilton cycle. Thus $T_{AL}(r) = T(r)$, $r > 7/4$. Thus $T_{AL}(r) = T(r)$ for all r. Hence $T_A(r) = T(r)$ for all r. Refer to figure 7 for $T_{AL}(r)$, $T_A(r)$ and $T(r)$. Recall that we had observed earlier that r_0^{AL} is very close to r_0 and hence to r_0^A for the same adjacent channel use constraint as in this case and we find that $T_{AL}(r) = T_A(r) = T(r)$ for the 7-cell system. Hence we expect that $T_{AL}(r)$ will track $T(r)$ and hence $T_A(r)$ very closely even for larger systems with the same adjacent channel use constraint as in this example.

Case 2: Again assume that adjacent channels cannot be used simultaneously in the same cell and also that cell 7 cannot use a channel if its adjacent channel is being used in either cell 2, 4 or 6. We know from Example III.2 that $r_0^{AL} = 7/5$. Thus $T_{AL}(r) = r$ for $r \leq 7/5$. For $7/5 < r \leq 7/4$,

$$m_i = \begin{cases} 1, & \text{if } i \in \{a, b, \ldots, f\}, \\ 2, & \text{if } i = g, \\ 0, & \text{otherwise.} \end{cases}$$

Refer to figures 6(a) and (b) for G and G' respectively. G' is not hamiltonian. However it becomes hamiltonian when edges are added between the pairs of vertices $(g1, b)$ and $(d, g2)$. Thus $p_{\min} \leq 2$ and $T_{AL}(r) = 0.8T(r)$, for $7/5 < r \leq 7/4$. $bcdg2efag1b$ is a hamilton cycle in G' with the added edges. However we can get a better upper bound by observing that $T_{AL}(7/5) = 7/5$ and $T_{AL}(r) \geq T_{AL}(r')$, if $r \geq r'$. Using this $T_{AL}(r) = 7/5$, for $7/5 < r \leq 7/4$. For $7/4 < r < 7/3$, m_is are given by those in case 1 for the same range of r. G is given by figure 6(a). V'_G is also the same as that given in case 1 for the same range of r. G' is not, in general, hamiltonian. However G' can be made hamiltonian by adding edges between the pairs of vertices (g_1, b_1), (g_2, d_1), (g_3, b_2), (g_4, d_2) and so on till the g_is are exhausted. (All g_is will be exhausted because $0 < y < 2x$ here, where x, y are as defined in case 1 for the same range of r.) $b_1c_1d_1g_2e_1f_1a_1g_1b_2c_2d_2g_4e_2f_2a_2g_3 \ldots b_lc_ld_lg_{2l}e_lf_la_l\ g_{2l-1} \ldots b_xc_xd_xe_xf_xa_xb_1$ is a hamilton cycle in G' with the added edges. Thus for $7/4 < r < 7/3$, $p_{\min} \leq y$ and

$$\begin{aligned} T_{AL}(r) &= \frac{T(r)}{1 + \frac{y}{\sum_{i \in \{a,\ldots,g\}} m_i}}, \\ &= \frac{T(r)}{2 - 3r/7}, \quad \text{(using the values of } y \text{ and } m_i\text{s)} \\ &= \frac{7 + 3r}{14 - 3r}. \quad \text{(using the value of } T(r)) \end{aligned}$$

For $r \geq 7/3$, m_is are given by those in case 1 for the same range of r. G and G' are given by figure 5(c). G' is clearly hamiltonian. $abcdefa$ is the hamilton cycle. Thus $T_{AL}(r) = T(r)$. Summarizing,

$$T_{AL}(r) = \begin{cases} r, & \text{if } r \leq 7/5, \\ 7/5, & \text{if } 7/5 < r \leq 7/4, \\ \frac{7+3r}{14-3r} & \text{if } 7/4 < r < 7/3, \\ 2 & \text{if } r \geq 7/3. \end{cases}$$

V. Approximation in the Presence of Arbitrary Restrictions on Simultaneous Use of any Two Different Channels

Simultaneous use of any two channels can produce interference even if they are not the same or adjacent. In most cases the filter responses are good enough so that the interference produced by nonadjacent channels is negligible. Nevertheless sometimes there may be restrictions on the simultaneous use of nonadjacent channels also. As mentioned before [2] formulates linear programs for the computation of $T_A(r)$ and r_0^A in the presence of any such restriction but these linear programs are incomputable because of the large number of

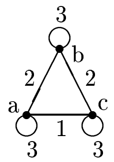

Fig. 8. The weighted D graph for a 3-cell system. The numbers against the edges indicate their weights.

variables and constraints even for systems of moderate size, e.g., a 19-cell system.[12] We can extend our approximations to this general case under certain special circumstances.

The cochannel reuse constraints are modelled by a hypergraph as before. The channels are numbered 1,2,…,n. The distance between channels numbered i, j is defined to be $|i-j|$. The numbering should be such that this distance is proportional to the actual separation between them in the radio spectrum. The adjacent channel use constraints are modelled by a matrix B, which is defined slightly differently from the corresponding definition in Section I. B is a $N \times N$ matrix, such that if cells i, j can use adjacent channels simultaneously, then $b_{ij} = 1$. Otherwise b_{ij} is the minimum distance between the channels that can be used simultaneously in cells i and j. Note that if the restriction is only on adjacent channel use, as assumed in Sections III and II, then the elements of B are 1 and 2. (This agrees with the definition of B in Section III.)

A graph D is induced by an M-tuple $(X_1, X_2, \ldots, X_M)$ as follows: Let $(X_1, X_2, \ldots, X_M)$ be any M-tuple, such that X_j is a nonnegative rational number and M is the number of maximal independent sets of the hypergraph modelling the cochannel reuse constraints of the cellular system. Let $X_{k_1}, X_{k_2}, \ldots, X_{k_V}$ be the ones with nonzero values. Let $X_{k_1} : X_{k_2} : \ldots : X_{k_V} = m_{k_1} : m_{k_2} : \ldots : m_{k_V}$, where $m_{k_1}, m_{k_2}, \ldots, m_{k_V}$ are relatively prime positive integers. If $X_j = 0$, $m_j = 0$. Form a weighted graph $D = (V_D, E_D)$ as follows: $V_D = \{v_{k_1 1}, \ldots, v_{k_1 m_{k_1}}, v_{k_2 1}, \ldots, v_{k_2 m_{k_2}}, \ldots, v_{k_V 1}, \ldots, v_{k_V m_{k_V}}\}$.
There is an edge between any two vertices v_i and v_j (even if $v_i = v_j$). The weight of an edge between the vertices $v_{k_l a}$ and $v_{k_t b}$ is

$$\begin{aligned} w(v_{k_l a}, v_{k_t b}) &= \max_{1 \le i,j \le N} b_{ij} a_{ik_l} a_{jk_t} \\ &= \max_{\substack{1 \le i \le N, a_{ik_l}=1 \\ 1 \le j \le N, a_{jk_t}=1}} b_{ij}. \end{aligned}$$

Example V.1: Consider the 3-cell example of figure 1. Let the minimum separation between channels used simultaneously in the same cell be 3, i.e., cell 1 can not use channels $1, 2$ or $1, 3$ simultaneously, and so on. Adjacent cells cannot use adjacent channels simultaneously. Any other cells can use adjacent channels simultaneously. Let there be no cochannel reuse in the system, i.e., there are 3 maximal independent sets a, b, c each consisting of a single

[12] If there are restrictions on the simultaneous use of c channels, $\Omega \subseteq \tau \times \tau \times \ldots \times \tau$, c times. Refer to the relevant discussion in Section I of the work in [2]. $|\tau|$ is large and $|\Omega|$ is very large making the number of variables extremely large.

cell. $a = \{1\}$, $b = \{2\}$, $c = \{3\}$.

$$B = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

The graph D induced by $(2,2,2)$ is shown in figure 8.
We present the following results:

Theorem 4: Let D be the weighted graph induced by some M-tuple $(X_1, X_2, \ldots, X_M)$. Let D satisfy the triangle inequality.[13] Let there be a hamilton cycle in D of weight[14] L. Let $w_{\max}$ be the weight of an edge which has the maximum weight amongst all edges in the hamilton cycle. Let $(qL - w_{\max} + 1)^+$ channels be available. Then there exists a fixed channel allocation algorithm that allocates $q\sum_{j=1}^{M} m_j a_{ij}$ channels to the ith cell, for $1 \leq i \leq N$, for any nonnegative integer q.

Proof: This proof is along the same lines as that of Theorem 1. Again if $q = 0$ the lemma is trivially true ($w_{\max} \geq 1$). Let $q > 0$. Let the hamilton cycle of weight L be $v_{i_0 j_0}, v_{i_1, j_1}, \ldots, v_{i_{U-1}, j_{U-1}}, v_{i_0 j_0}$. $U = \sum_{j=1}^{M} m_j$. Without loss of generality $w(v_{i_{U-1}, j_{U-1}}, v_{i_0 j_0}) = w_{\max}$. If the algorithm given in the proof of Theorem 1 is slightly modified, it allocates qm_j channels to the jth maximal independent set satisfying the cochannel reuse and the adjacent channel use constraints, using $(qL - w_{\max} + 1)^+$ channels. The following modification is required in step 4:
$b \rightarrow b + w(v_{i_a, j_a}, v_{i_{(a+1) \bmod U} j_{(a+1) \bmod U}})$. $a \rightarrow (a+1) \bmod U$. Go to step 2.
Clearly this allocation satisfies cochannel reuse constraints. Let cell i get channel k_1 and cell j get channel k_2. Let channel k_1 be allocated to the sth maximal independent set and channel k_2 be allocated to the tth maximal independent set. Thus cell i is in the sth maximal independent set and cell j in the tth maximal independent set. $|k_1 - k_2|$ is equal to the total weight of at least one of the paths between v_{k_1} and v_{k_2} on the hamilton cycle. Because of the triangle inequality, the weight of both the paths on the hamilton cycle is greater than or equal to $w(v_{k_1}, v_{k_2})$ and this weight is greater than or equal to b_{ij}, the minimum separation between the channels which can be simultaneously used in these cells. Thus this allocation satisfies the adjacent channel use constraints. This is the advertised fixed channel allocation. ■

The following Corollary follows from Theorem 4.

Corollary 2: Let a fixed channel allocation that allocates n_i channels to the ith cell be desired. Let D be the graph induced by an optimal solution of the ILP, $(X_1^O, X_2^O, \ldots, X_M^O)$. Let q = g.c.d. of nonzero X_j^Os if there exists at least one nonzero W_j^O and let $q = 0$ otherwise. Let there be a hamilton cycle in D of weight L. $w_{\max}$ is the weight of an edge which has the maximum weight amongst all edges in the hamilton cycle. If D satisfies the triangle inequality, $(qL - w_{\max} + 1)^+$ channels are sufficient to achieve the required fixed channel allocation.

[13] $w(v_1, v_2) \leq w(v_1, v_3) + w(v_3, v_2)$
[14] The weight of a hamilton cycle in a weighted graph is the sum of the weights of the edges in the cycle.

Theorem 5: Let D be the graph induced by a rational optimal solution of LP2, $(X_1^O, X_2^O, \ldots, X_M^O)$. Let there be a hamilton cycle in D of weight L. If D satisfies the triangle inequality, $r_0^A \geq r_0 \sum_{j=1}^{M} m_j/L$.

Proof: The proof follows from Theorem 4, Lemmas 1 and 2 along the same lines as the proof of Theorem 2 follows from Theorem 1, and Lemmas 1 and 2. ■

Remarks: $r_0 \sum_{j=1}^{M} m_j/L \leq r_0^A \leq r_0$. The lower bound is tightest if $L = L_{\min}$, where $L_{\min}$ is the weight of the minimum weight hamilton cycle in the graph D induced by $(X_1^O, X_2^O, \ldots, X_M^O)$, a rational optimal solution of LP2.

Theorem 6: Let D be the graph induced by a rational optimal solution of LP1, $(X_1^O, X_2^O, \ldots, X_M^O)$. Let there be a hamilton cycle in D of weight L. If D satisfies the triangle inequality, $T_A(r) \geq T(r) \sum_{j=1}^{M} m_j/L$.

The proof follows the same lines as that of Theorem 3.

Remarks: $T(r) \sum_{j=1}^{M} m_j/L \leq T_A(r) \leq T(r)$. Again the lower bound is the tightest if $L = L_{\min}$, where $L_{\min}$ is the weight of the minimum weight hamilton cycle in the graph D induced by $(X_1^O, X_2^O, \ldots, X_M^O)$, a rational optimal solution of LP1. Note that if the restriction is only on adjacent channel use, as assumed in Sections III and II, then the elements of B are 1 and 2. (This agrees with the definition of B in Section III.) This makes D always satisfy the triangle inequality in this case. Also the lower bounds with $L_{\min}$ given by Theorems 5 and 6 turn out to be the same as those given by Theorems 2 and 3 respectively with $p_{\min}$ in these cases. The same observation applies for the results of Corollaries 1 and 2.

The problem of finding the minimum weight hamilton cycle in D is the same as the travelling salesman problem which is NP-complete. However if D satisfies the triangle inequality, there are some polynomial time algorithms which will produce, within known bounds, an approximation to the travelling salesman problem [9]. Thus this technique is useful, whenever D satisfies the triangle inequality, to get an estimate of r_A^0 and $T_A(r)$, more so because the exact values of these quantities are incomputable even for systems of reasonably small size.

VI. Conclusion and Summary

We now summarize the contributions of this paper. There are various performance bounds for a cellular system which are extremely useful from the theoretical as well as from the network operators' point of view. The computation of the exact values of these performance bounds in the presence of cochannel reuse and adjacent channel use constraints is extremely difficult or impossible even for systems of reasonably small size. We have developed approximations to these performance bounds in the presence of cochannel reuse and adjacent channel use constraints which are computationally much simpler. These approximations track very closely the actual performance bounds in most cases. We have also presented good heuristics for the problem of finding the minimum number of channels necessary for achieving any given fixed channel allocation algorithm in the presence of cochannel reuse and adjacent channel use constraints.

We make one observation before conclusion: we have made all computations in two steps. The LPs or the ILP takes care of the cochannel reuse constraints and the graph theoretic approach takes care of the adjacent channel use constraint. The advantage of this modularization is that if for some reason the adjacent channel use constraints change but the cochannel reuse constraints remain the same, then only the graph theoretic computations need be repeated. This is quite helpful because for actual systems both the ILP and the LPs may be computationally quite intensive.

Appendix

I. Appendix

Lemma 1: Let the carried traffic in a one cell system be denoted by $C(r_n, h_n)$ when h_n channels are allocated to the cell and r_n is the offered traffic in the cell. $\lim_{n\to\infty} C(r_n, h_n)/n = \min(r, h)$ if $\lim_{n\to\infty} r_n/n = r$, $\lim_{n\to\infty} h_n/n = h$ and $h > 0$.

Proof: This proof follows easily from the ATP introduced in [4]. We give the proof here for the sake of completeness.

$$\begin{aligned}
\lim_{n\to\infty} \frac{C(r_n,h_n)}{n} &= \lim_{n\to\infty} \frac{h_n}{n} \lim_{n\to\infty} \frac{C(r_n,h_n)}{h_n} \\
&= h \lim_{h_n\to\infty} \frac{C(r_n,h_n)}{h_n} \\
&\quad (h_n \to \infty \text{ as } n \to \infty \text{ since } h > 0). \\
&= h \min(\lim_{h_n\to\infty} \frac{r_n}{h_n}, 1) \quad \text{by the ATP} \\
&= h \min(\frac{r}{h}, 1) \\
&= \min(r, h).
\end{aligned}$$

■

Lemma 2: Let $n = q(p+\sum_{j=1}^M m_j)+t$ channels be available where $0 \le t < (p+\sum_{j=1}^M m_j)$ and q is a nonnegative integer. The fixed channel allocation algorithm which allocates $q\sum_{j=1}^M m_j a_{ij}$ channels to the ith cell carries a traffic intensity of $\min\left(p_i r, \frac{\sum_{j=1}^M m_j a_{ij}}{p+\sum_{j=1}^M m_j}\right)$ in the asymptotic limit $(n \to \infty)$ in the ith cell.

Proof: As argued in [4], because of the independence of the offered traffic and the fixed channel allocation algorithm, each cell behaves as a one cell system with offered load $p_i rn$ and $q\sum_{j=1}^M m_j a_{ij}$ channels. Thus the carried traffic intensity in the ith cell is $C(p_i rn, q\sum_{j=1}^M m_j a_{ij})/n$. If $\sum_{j=1}^M m_j a_{ij} = 0$, $C(p_i rn, q\sum_{j=1}^M m_j a_{ij})/n = 0$, for each n. Thus $\lim_{n\to\infty} C(p_i rn, q\sum_{j=1}^M m_j a_{ij})/n = 0 = \min(p_i r, \sum_{j=1}^M m_j a_{ij}/(p + \sum_{j=1}^M m_j)$. Let $\sum_{j=1}^M m_j a_{ij} > 0$. Then $p_i rn/n \overset{n\to\infty}{\to} p_i r$ and $\lim_{n\to\infty} \frac{q}{n} = \frac{1}{(p+\sum_{j=1}^M m_j)}$ since $0 \le t < (p + \sum_{j=1}^M m_j)$ and p, m_j, a_{ij}, M are fixed constants independent of n. Thus $\lim_{n\to\infty} \frac{q\sum_{j=1}^M m_j a_{ij}}{n} = \frac{\sum_{j=1}^M m_j a_{ij}}{p+\sum_{j=1}^M m_j} > 0$. Thus $\lim_{n\to\infty} C(p_i rn, q\sum_{j=1}^M m_j a_{ij})/n =$

$\min(p_i r, \frac{\sum_{j=1}^{M} m_j a_{ij}}{p+\sum_{j=1}^{M} m_j})$ by Lemma 1. ∎

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