Channel Assignment Algorithms Satisfying Cochannel and Adjacent Channel Reuse Constraints in Cellular Mobile Networks

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Abstract—**Improved channel assignment algorithms for cellular networks were designed by modeling the interference constraints in terms of a hypergraph [1]. However these algorithms only considered co-channel reuse constraints. Receiver filter responses impose restrictions on simultaneous adjacent channel usage in the same cell or in neighboring cells. We first present some heuristics for designing fixed channel assignment algorithms with a minimum number of channels satisfying both co-channel and adjacent channel reuse constraints. An asymptotically tight upper bound for the traffic carried by the system in the presence of arbitrary co-channel and adjacent channel use constraints was developed in [2]. However this bound is computationally in-tractable even for small systems like a regular hexagonal cellular system of 19 cells. We have obtained approximations to this bound using the optimal solutions for co-channel reuse constraints only, and a further graph theoretic approach. Our approximations are computationally much more efficient and have turned out to track very closely the exact performance bounds in most cases of interest.**

Index Terms: Cellular systems, Channel assignment algorithms, Cochannel interference, Adjacent channel interference, Graph models, Hypergraph models

I. Introduction

In a cellular system, the coverage area is logically divided into cells. Each cell has a cell site or a base station. The communication from the mobile user is directed to a central switching office by the base station. The central switching office directs this communication to the destination. Depending on the mode of multiple access used by the mobile customers, cellular systems can be broadly classified into *channelized* and *non-channelized* systems. In a channelized cellular system the multiple access is TDMA or FDMA or a combination of both. The term *channel* refers to a time slot in TDMA, a frequency slot in FDMA and a combination of both in TDMA/FDMA systems like GSM. The traffic in a cellular systems is usually too high to allow the use of a channel for one call at a time; radio channels must be used simultaneously for more than one call. This is known as *channel reuse*. *Channel reuse* causes interference, which in turn degrades the transmission quality. Transmission quality requirements impose certain co-channel reuse constraints, i.e., the same channels may not be used simultaneously within certain distance.

Interference is also caused by simultaneous use of adjacent channels in close proximity. This happens because of imperfections in filters. Use of guard band between adjacent frequencies somewhat mitigates this interference. However, guard bands need to be significantly large in order to reduce this interference below an acceptable threshold. This leads to poor utilization of the limited radio spectrum. A better strategy is to eliminate the adjacent channel interference by not using adjacent channels in neighboring cells at the same time. This results in better use of radio spectrum as all available channels can be used in accommodating calls in the system. However, in this case, channel allocation strategies need to satisfy co-channel and adjacent channel use constraints. While frequency allocation in presence of co-channel reuse constraint has received significant attention, efficient channel allocation strategies which satisfy both co-channel and adjacent channel use constraints do not exist. This paper is directed towards addressing both these constraints.

We first describe our system model and the existing theoretical results for cellular systems with different types of channel allocation constraints, e.g., co-channel reuse constraints only[4], and both co-channel and adjacent channel use constraints[2] (Section II). Next, we present efficient fixed channel allocation strategies which attains low blocking probabilities in presence of limited spectrum availability in systems with co-channel and adjacent channel constraints (Section III). The channel allocation strategy we present closely approximates the minimum number of channels required to attain certain desired blocking probabilities in cells. Performance bounds are known for cellular systems with different channel use constraints. The efficiency of actual channel allocation strategies can be determined by comparing their performances with these bounds. However, these bounds are computationally complex when both co-channel and adjacent channel constraints are considered. We present computationally simple approximation for these bounds, which track the exact bounds closely (Sections IV and V). Channel use constraints need not be limited to adjacent channels, but may extend to simultaneous use of arbitrary channels. This is because transmissions in non-adjacent channels may also cause interference. However, the interference produced by simultaneous use of non-adjacent channels in vicinity is negligible in general. Nevertheless, this interference may be significant in some cases, and as such we address this generalization in Section VI.

II. System Model and Analytical performance Bounds

We describe the system model in this section. We introduce some notations for this purpose. Table I summarizes all notations used extensively in the paper. The system consists of N cells which share a common set of n channels. The underlying offered

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Symbol	Meaning
N	Number of cells
M	Number of maximal independent sets of the hypergraph modeling co-channel reuse constraints
a_{ij}	Indicates whether cell i is in jth maximal independent set
b_{ij}	Indicates whether cells i and j can use an adjacent channel simultaneously
r	Offered traffic
p_i	Fraction of offered traffic for cell i
τ	Number of possible states of a channel
R	Number of hyperstates of a channel
r_0	System capacity without adjacent channel use constraints
$T(r)$	Traffic carried by system in absence of adjacent channel use constraints
r_0^A	System capacity with adjacent channel use constraints
$T_A(r)$	Traffic carried by system in presence of adjacent channel use constraints
r_0^{AL}	Lower bound to r_0^A
$T_{AL}(r)$	Lower bound to $T_A(r)$

TABLE I
SYMBOL TABLE

traffic model is independent from cell to cell; in particular, we ignore the effects of call handovers and inter-cell calls. However it is likely that we can extend our results to the case in which this independence assumption is dropped and handovers and inter-cell calls are included. Our optimism is derived from the fact that the results of [4], to which we shall refer extensively, have been extended to include handovers in [7]. The underlying model of offered traffic satisfies the "asymptotic traffic property" (ATP) [4] which states that

$$\lim_{n\to\infty} C(k,n)/n = \min(r,1), \text{ where } \lim_{n\to\infty} k/n = r.$$

$C(k,n)$ is the carried traffic in a system with 1 cell, when the offered traffic is k and the number of available channels is n. Many offered traffic models including the common Poisson arrivals and exponential holding times satisfy the ATP.

The expected number of calls that would be in progress in cell i if all call requests could be honored is known as the *offered traffic* in cell i. If A_i denotes the offered traffic in cell i then A_i/n is the *offered traffic intensity* in cell i. The offered traffic intensity in the system, r, is the sum of the offered traffic intensities in the cells; thus $r = \sum A_i/n$. We assume that r is rational. The ratio $p_i = A_i/\sum_{i=1}^{N} A_i$ represents the fraction of the total offered traffic in cell i and the vector $\tilde{p} = (p_1, p_2, \ldots, p_N)$ is the *traffic pattern*. We assume that the p_is are rational. The carried traffic intensity in cell i, x_i, is the carried traffic (expected number of calls in progress) in cell i per available channel in the system. An objective is to maximize the total traffic carried by a system, $\sum_{i=1}^{N} x_i$.

We will first describe a mathematical model and performance bounds for systems with co-channel reuse constraints only. These have been presented in [4]. Subsequently, we present a generalization for including the adjacent channel use constraints. This generalization was developed in [2].

Mathematical Model for systems with Co-channel Reuse Constraints: Co-channel reuse constraints in cellular systems have been modeled by regular hexagonal channel reuse patterns [3] for a long time. A hypergraph model was first used in [4]. This has been found to be the most efficient model for co-channel reuse constraints [5]. We first describe the hypergraph model briefly. A hypergraph H is formally defined as $H = (V, E)$, where V is the set of vertices and E is the set of edges, where each edge e is a nonempty subset of V such that $\cup_{e\in E} e = V$ [6]. A hypergraph is a generalization of a graph in that an edge can have no more than two vertices in a graph but this restriction does not hold for a hypergraph. Hypergraph modeling of cellular systems is as follows:

- Each cell corresponds to a vertex.
- A *forbidden set* is a group of cells *all* of which cannot use a channel simultaneously. If no proper subset of a forbidden set is forbidden, then it is a *minimal forbidden set*. An edge is a *minimal forbidden set*.
- A set of vertices which does not contain an edge is an *independent* set. Any group of cells which may use the same channel simultaneously forms an *independent set* of the underlying hypergraph. If an independent set is not a proper subset of another independent set, then it is a *maximal independent set*.

Let the hypergraph H modeling the co-channel reuse constraints have M maximal independent sets and let N_j denote the size of the jth maximal independent set. We define

$$a_{ij} = \begin{cases} 1, & \text{if the } i\text{th cell is in the } j\text{th maximal} \\ & \text{independent set of the hypergraph modeling} \\ & \text{the co-channel reuse constraints, and} \\ 0, & \text{otherwise.} \end{cases}$$

Now, [4] presents an asymptotically[1] tight upper bound on the total carried traffic intensity in the cellular system with co-channel reuse constraints. The upper bound $T(r)$ is a function of the offered traffic r and is given by the optimal value of the objective function of the following linear program.

$$\text{Maximize } \sum_{i=1}^{N} x_i, \text{ subject to}$$

$$\left.\begin{array}{rcl} x_i & \le & \sum_{j=1}^{M} X_j a_{ij}, \quad i = 1,2,\ldots,N \\ x_i & \le & p_i r, \quad i = 1,2,\ldots,N \\ X_j & \ge & 0, \quad j = 1,2,\ldots,M \\ \sum_{j=1}^{M} X_j & = & 1 \end{array}\right\} \text{LP1}$$

We denote this linear program by LP1. The intuition behind LP1 can be described as follows. The variable x_i denotes the traffic carried in cell i. The objective is to maximize the sum of the traffic carried in all cells. Note that a channel can be simultaneously used by all cells in an independent set. Thus one can think of assigning channels to maximal independent sets rather than to cells. Whenever a channel is assigned to a maximal independent

[1] The number of channels and the offered traffic are made arbitrarily large while keeping the ratio finite.

set, all cells in the set receives the channel. Here, X_j denotes the fraction of channels allocated to the jth maximal independent set. Thus $\sum_{j=1}^{M} X_j a_{ij}$ denotes the fraction of channels allotted to the maximal independent sets which contain cell i, and this is the channel allocation of cell i. Clearly the carried traffic in cell i, x_i is upper bounded by this quantity. The first constraint represents this condition. The second constraint states that the carried traffic in a cell is upper bounded by the offered traffic. The remaining constraints are intuitive. This linear program has $M + N$ nonnegative variables and $2N + 1$ constraints.

It was proved in [4] that if the offered traffic intensity, r, is less than or equal to a certain quantity r_0, which depends on the cellular system and the traffic pattern and there is no adjacent channel use constraint, then there exists a channel assignment algorithm which achieves arbitrarily low blocking probabilities, if the number of available channels is sufficiently large. For $r > r_0$, no channel assignment algorithm can produce zero blocking for any number of channels. The quantity r_0 has been termed the *capacity* of the system. The capacity of a system is a measure of the reuse offered by the system as informally speaking each channel can carry r_0 calls simultaneously in the system on an average. It can also serve as a good operating load. The capacity, r_0 is given by the optimal value of the objective function of the following linear program:

$$\max \left\{ \begin{array}{c} r : \sum_{j=1}^{M} X_j = 1, X_j \geq 0, \sum_{j=1}^{M} X_j a_{ij} \geq p_i r, \\ i = 1, \ldots, N. \end{array} \right\} \text{(LP2)}$$

We denote this linear program by LP2. Intuitively, r_0 is the largest value of offered traffic r for which the offered traffic in any cell i, $p_i r$ does not exceed the resource allocated to cell i, $\sum_{j=1}^{M} X_j a_{ij}$. This linear program has $M + 1$ nonnegative variables and $N + 1$ constraints.

The carried traffic $T(r)$ and the system capacity r_0 are important performance metrics of a cellular system. Now, [2] computes these metrics in presence of adjacent channel use constraints. However, the strategies presented there are computationally complex. An important contribution of this paper is to approximate these computations using techniques which are computationally simple. For this purpose, we will use the optimum solutions of LP1 and LP2 and a further graph theoretic technique. First, we explain the exact computations presented in [2].

Mathematical Model for systems with Co-channel and Adjacent channel use Constraints: Imperfect receiver filter responses impose restrictions on simultaneous use of adjacent channels in nearby cells. By "adjacent" channels we mean consecutive carrier frequencies in a FDMA system. In the case of FDMA/TDMA systems, we assume all the time slots in a frequency channel are allocated to the same cell so that we can treat each carrier frequency as a channel. In the rest of this paper, we assume that a channel is a carrier frequency. The n channels are numbered $1, 2, \ldots, n$ with adjacent channels given consecutive numbers. The adjacent channel use constraints can be modeled by an $N \times N$ matrix B, such that

$$b_{ij} = \left\{ \begin{array}{ll} 1, & \text{if the } i\text{th and the } j\text{th cells can use} \\ & \text{adjacent channels simultaneously, and} \\ 2, & \text{otherwise.} \end{array} \right.$$

The matrix B can be determined from the transmission quality requirements and the interference produced by adjacent channels. For example, if the filters are perfect, then the adjacent channels do not interfere with each other. As such, any two cells can use adjacent channels simultaneously. In this case, $b_{ij} = 1$ for all i, j. Typically the cells within a certain distance d can not use adjacent channels simultaneously. Thus, $b_{ij} = 1$ if cells i and j are separated by a distance greater than some d and $b_{ij} = 2$ otherwise. The value of d depends on the interference caused by adjacent channels, and the transmission quality requirements. Higher the interference or lesser the acceptable interference threshold, higher the value of d. Typical values of d are 0 and $\sqrt{3}P$ in regular hexagonal cellular systems, where P is the cell radius. If $d = 0$, adjacent channels cannot be used simultaneously in the same cell and if $d = \sqrt{3}P$ adjacent channels cannot be used simultaneously in the same cell and in adjacent cells.

Now, [2] presents asymptotically tight upper bounds on the carried traffic intensity and the system capacity in presence of co-channel and adjacent channel use constraints. We introduce some notations for describing the results. The state of a channel is an N-tuple whose elements are either 0 or 1. The ith element is 1 iff the channel is carrying a call in the ith cell. The state of a channel (other than the one represented by the all zero N-tuple) represents an independent set of the underlying hypergraph modeling the co-channel reuse constraints. Let τ denote the set of states of a channel. (If the co-channel reuse constraints are modeled by a hypergraph, the elements of τ, with the exception of the all zero N-tuple, have a one-to-one correspondence with the set of independent sets of the hypergraph.) The hyperstate of a channel i, $1 \leq i < n$, is (p, q), if the channel i is in state p and the channel $i + 1$ is in state q. $\Omega \subseteq \tau \times \tau$ is the set of hyperstates of a channel.[2] $d_{i\omega} = 1$ iff a channel carries a call in cell i in the hyperstate $\omega \in \Omega$. $t_\omega = \sum_{i=1}^{N} d_{i\omega}$. Let $|\Omega| = R$. $\Omega_U(\eta) = \{\omega : \omega = (\eta, \eta') \in \Omega, \eta' \in \tau\}$, and $\Omega_L(\eta) = \{\omega : \omega = (\eta', \eta) \in \Omega, \eta' \in \tau\}$, $\eta \in \tau$. $\Omega_U(\eta), \Omega_L(\eta) \subseteq \Omega$. An asymptotically tight upper bound, $T_A(r)$ [3] on the total carried traffic intensity in the system in the presence of co-channel and adjacent channel use constraints can be computed as follows.

$$T_A(r) = \max \left\{ \sum_{\omega \in \Omega} t_\omega s_\omega : s = (s_1, s_2, \ldots, s_R) \in S(r) \right\}, \text{where}$$

$$S(r) = \left\{ \begin{array}{lll} (s_1, \ldots, s_R) : & & \\ \sum_{\omega \in \Omega} d_{i\omega} s_\omega & \leq p_i r, & i = 1, 2, \ldots N, \\ s_\omega & \geq 0, & \omega \in \Omega, \\ \sum_{\omega \in \Omega_U(\eta)} s_\omega & = \sum_{\omega \in \Omega_L(\eta)} s_\omega, & \eta \in \tau, \\ \sum_{\omega \in \Omega} s_\omega & = 1, & \end{array} \right\}$$

(Here $s_1, s_2, \ldots$ are variables in the linear program).

Note that $T_A(r)$ can be determined by computing a linear program with R nonnegative variables and $N + 1 + |\tau|$ constraints. We denote this linear program as LP3.

[2] Ω is determined by τ and the matrix B.

[3] Unlike $T(r)$ which is the upper bound for all n, $T_A(r)$ is the upper bound only in the asymptotic case.

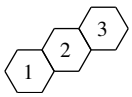

Fig. 1. A system with 3 cells

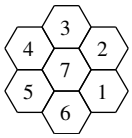

Fig. 2. A system with 7 cells

Next, we describe the computation of the network capacity, r_0^A in presence of both co-channel and adjacent channel use constraints. Recall that network capacity r_0^A has the property that there exists a channel assignment algorithm which achieves arbitrarily low blocking probabilities, if the number of available channels is sufficiently large and the offered traffic r is less than or equal to r_0^A. For $r > r_0^A$, no channel assignment algorithm can produce zero blocking for any number of channels. Now, [2] shows that

$$r_0^A = \max r : \left\{ \begin{array}{lll} \sum_{\omega\in\Omega} d_{i\omega} s_\omega & \geq & p_i r \\ & & i = 1, 2, \ldots N, \\ s_\omega & \geq & 0 \qquad\qquad \omega \in \Omega, \\ \sum_{\omega\in\Omega_U(\eta)} s_\omega & = & \sum_{\omega\in\Omega_L(\eta)} s_\omega \quad \eta \in \tau, \\ \sum_{\omega\in\Omega} s_\omega & = & 1. \end{array} \right\}$$

Note that r_0^A can be determined by computing a linear program with $R+1$ nonnegative variables and $N + |\tau| + 1$ constraints. We denote this linear program as LP4.

In general both R and $|\tau|$ are very large. When co-channel reuse constraints are modeled by a hypergraph, and adjacent channels not used in the same cell ($d = 0$), $R \sim 10$, $|\tau| \sim 10$ for a system with 3 cells (refer figure 1), $R \sim 100$, $|\tau| \sim 10$ for a system with 7 cells (refer figure 2), $R \sim 10^6$, $|\tau| \sim 10^3$ for a system with 19 cells (refer figure 3) and $R \sim 10^9$, $|\tau| \sim 10^6$ for a system with 37 cells(refer figure 4). Thus both these linear programs are computationally in-tractable for systems of reasonable size.

We will not use LP3 and LP4 any further in this paper except for computing the exact values of r_0^A and $T_A(r)$ for comparison with the corresponding approximations. In Sections IV and V we derive lower bounds on r_0^A and $T_A(r)$ which we denote by r_0^{AL} and $T_{AL}(r)$ respectively, using the solutions to the linear programs LP1 and LP2 for computing $T(r)$ and r_0 and a further graph theoretic approach. These lower bounds are computationally much simpler as $M \ll R$ and $N \ll |\tau| \ll R$. In general, $T_{AL}(r)$ and r_0^{AL} are quite close to $T_A(r)$ and r_0^A

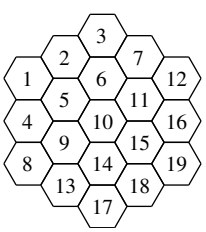

Fig. 3. A system with 19 cells

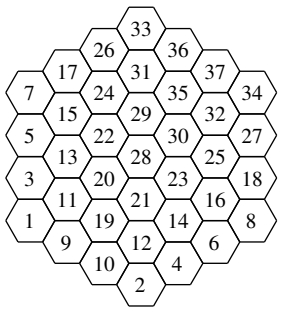

Fig. 4. A system with 37 cells

respectively. Note that upper bounds on $T_A(r)$ and r_0^A are already known in the form of $T(r)$ and r_0 respectively. We first present some heuristics for designing fixed channel assignment algorithms with a minimum number of channels satisfying both co-channel and adjacent channel use constraints in the next section (Section III). The design strategy will introduce a graph theoretic approach which will be used subsequently to approximate r_0^A and $T_A(r)$.

III. A Heuristic Approach for the Determination of the Number of Channels Required for a Given Fixed Channel Allocation

A channel allocation algorithm in which groups of channels are allocated to cells *a priori* and a cell accepts a requested call only if it has a free channel amongst those allocated to it is known as a *fixed channel allocation* algorithm. Though much more sophisticated algorithms, e.g., *dynamic channel allocation* schemes, have been devised which sometimes out-perform the fixed channel allocation schemes, fixed channel allocation algorithms are still the ones in actual use because they are easy to implement. For a fixed channel allocation scheme, the number of channels required for cell i, n_i can be determined from the traffic offered to cell i and the acceptable blocking probability, using Erlang-B formula. Given the channel requirements for each cell, we may be interested in $n_{\min}$, the minimum number of channels necessary to make the allocation, satisfying the co-channel reuse and the adjacent channel use constraints imposed by the transmission quality requirements. This will be less than $\sum_{i=1}^N n_i$ in general because of channel reuse. We have not found any efficient algorithm for this purpose in the literature when channel allocation needs to satisfy both co-channel and adjacent channel use constraints. We suggest a heuristic approach here.

We first describe the concept of graphs $F(\tilde{X})$, $F'(\tilde{X})$ induced by a M-tuple $\tilde{X} = (X_1, X_2, \ldots, X_M)$. $\tilde{X}$ be any M-tuple such that X_j is a nonnegative rational number and M is the number of maximal independent sets of the hypergraph modeling the co-channel reuse constraints of the cellular system. Let $X_{k_1}, X_{k_2}, \ldots, X_{k_V}$ be the ones with nonzero values. Let $X_{k_1} : X_{k_2} : \ldots : X_{k_V} = m_{k_1} : m_{k_2} : \ldots : m_{k_V}$, where $m_{k_1}, m_{k_2}, \ldots, m_{k_V}$ are relatively prime positive integers. If $X_j = 0$, $m_j = 0$. It is convenient to think of m_{k_l} as being (proportional to) the number of channels that are to be assigned to maximal independent set X_{k_l}. Form a graph $F(\tilde{X}) = (V_F, E_F)$[4]

[4]In our notation in any graph $G = (V, E)$, V is the vertex set and E is the

as follows. $V_F = \{v_{k_1}, v_{k_2}, \ldots, v_{k_V}\}$. Vertices v_{k_l} and v_{k_t} are joined by an edge iff $b_{ij} = 1$ for each cell i in the k_lth maximal independent set and each cell j in the k_tth maximal independent set of the hypergraph modeling the co-channel reuse constraints for the cellular system, i.e., if any two cells such that one is in the k_lth maximal independent set and the other in the k_tth maximal independent set can use adjacent channels simultaneously. Next form a graph $F'(\tilde{X}) = (V_{F'}, E_{F'})$. $V_{F'} = \{v_{k_1 1}, \ldots, v_{k_1 m_{k_1}}, v_{k_2 1}, \ldots, v_{k_2 m_{k_2}}, \ldots, v_{k_V 1}, \ldots, v_{k_V m_{k_V}}\}$. Vertices $v_{k_l a}$ and $v_{k_t b}$ have an edge between them iff the vertices v_{k_l} and v_{k_t} have an edge between them in $F(\tilde{X})$. In effect, each vertex in $F(\tilde{X})$ represents a maximal independent set (which is to be assigned at least one channel) and to obtain $F'(\tilde{X})$ each vertex in $F(\tilde{X})$ is replaced by m vertices where m is (proportional to) the number of channels to be assigned to the maximal independent set represented by that vertex. We say that $F(\tilde{X})$ and $F'(\tilde{X})$ have been induced by the M-tuple $\tilde{X} = (X_1, X_2, \ldots, X_M)$.

Theorem 1: Let $F(\tilde{X})$ and $F'(\tilde{X})$ be the graphs induced by some M-tuple $\tilde{X} = (X_1, X_2, \ldots, X_M)$. Let $F'(\tilde{X})$ become hamiltonian[5] upon the addition of some p edges. Let $q(p + \sum_{j=1}^{M} m_j) - 1_{\{pq>0\}}$[6] channels be available. Then there exists a fixed channel allocation algorithm which allocates $q\sum_{j=1}^{M} m_j a_{ij}$ channels to the ith cell, for $1 \leq i \leq N$, for any nonnegative integer q.

Note that the proof exhibits a channel allocation algorithm which attains the purpose.

Proof: Let $q(p + \sum_{j=1}^{M} m_j) - 1_{\{pq>0\}}$ channels be available. If $q = 0$ the lemma is trivially true. Let $q > 0$. We say that we allocate a channel to a maximal independent set when we allocate the channel to each cell in the maximal independent set. First we show that it is possible to allocate qm_j channels to the jth maximal independent set without violating the co-channel reuse and adjacent channel use constraints. Let the hamilton cycle of vertices of F' formed upon the addition of p edges to F' be $v_{i_0,j_0}, v_{i_1,j_1}, \ldots, v_{i_{U-1},j_{U-1}}, v_{i_0,j_0}$. Note that $U = \sum_{j=1}^{M} m_j$. Some of the consecutive vertices in this cycle are linked by an edge in F' and some by an added edge if $p > 0$. If $p > 0$, without loss of generality, $v_{i_{U-1},j_{U-1}}$ and $v_{i_0 j_0}$ are joined by an added edge. Allocate channels to the maximal independent sets as per the following algorithm:

1. $a = 0, b = 0, c = 1$.
2. Give channel numbered b to the i_ath maximal independent set.
3. If $a = U - 1, c \to c + 1$. If $c > q$ stop.
4. If v_{i_a,j_a} and $v_{i_{(a+1) \bmod U}\, j_{(a+1) \bmod U}}$ are linked by an edge in F', then $b \to b + 1$; else $b \to b + 2$. $a \to (a + 1) \bmod U$. Go to step 2.

Observe that we are moving along the hamilton cycle q times and each time the vertex $v_{i_l j_l}$ is encountered the i_lth maximal independent set is given a channel. Since we are allocating different channels to different maximal independent sets, the co-channel reuse constraints are satisfied. Two cells i and j can get adjacent channels only if i is in a maximal independent set k_1 and j in k_2, such that the vertices corresponding to the two maximal independent sets k_1 and k_2 are linked by an edge in F' and hence in F (v_{de} and v_{fg} have an edge between them in F' iff v_d and v_f have an edge between them in F.). This means that any cell i in the maximal independent set k_1 and j in k_2 are permitted to use adjacent channels simultaneously. Thus the adjacent channel use constraints are satisfied. The vertex $v_{i_l j_l}$ is encountered q times while traversing the hamilton cycle q times giving q different channels to the i_lth maximal independent set, for each i_l such that $X_{i_l} > 0$ and $j_l \in \{1, 2, \ldots, m_{i_l}\}$. Thus each of the following vertices $v_{i_l 1}, v_{i_l 2}, \ldots, v_{i_l, m_{i_l}}$ give q channels to the i_lth maximal independent set, giving it qm_{i_l} channels, if $X_{i_l} > 0$. If $X_j = 0$, the jth maximal independent set gets $qm_j = 0$ channels. If $p = 0$ no channel has been skipped.[7] If $p > 0$, $p - 1$ channels are skipped the last time the hamilton cycle is traversed and p channels are skipped each of the other $q - 1$ times the hamilton cycle is traversed, because of the p added edges. Thus $q(p + \sum_{j=1}^{M} m_j) - 1_{\{pq>0\}}$ channels have been used to allocate qm_j channels to the jth maximal independent set satisfying the co-channel reuse and adjacent channel use constraints. This fixed channel allocation algorithm allocates $q\sum_{j=1}^{M} m_j a_{ij}$ channels to the ith cell. ■

Consider the following integer linear program:

$$\left.\begin{array}{l} \text{Minimize } \sum_{j=1}^{M} W_j \\ \sum_{j=1}^{M} W_j a_{ij} \geq n_i, \quad i = 1, 2, \ldots N \\ W_j \geq 0, \quad j = 1, 2, \ldots, M, \quad W_j \text{ integer.} \end{array}\right\} \text{(ILP)}$$

Let n be the optimal value of the objective function. Consider graphs $G = F(W_1^O, \ldots, W_M^O)$ and $G' = F(W_1^O, \ldots, W_M^O)'$ induced by the optimal solution $(W_1^O, \ldots, W_M^O)$. The following Corollary follows from Theorem 1.

Corollary 1: Let G' become hamiltonian on the addition of p edges. Then $n + (qp - 1)^+$ channels are sufficient for the fixed channel allocation algorithm which allocates n_i channels to the ith cell, for $1 \leq i \leq N$, where q = g.c.d. of nonzero W_j^Os, if there exists at least one nonzero W_j^O and $q = 0$ otherwise.

We summarize the heuristic approach as follows:

1. Compute the ILP. Let n be the optimal value of the objective function. Let $(W_1^O, W_2^O, \ldots, W_M^O)$ be the optimal solution. q = g.c.d. of nonzero W_i^O, if there exists at least one nonzero W_j^O, and $q = 0$ otherwise.
2. Form G and G' from the optimal solution as discussed above, i.e., $G = F(W_1^O, \ldots, W_M^O)$ and $G' = F(W_1^O, \ldots, W_M^O)'$.
3. Compute the minimum number of edges $p_{\min}$ required to make G' hamiltonian, or an approximation to it, say p. We discuss how to compute of $p_{\min}$ in Section IV.
4. Use $n + (pq - 1)^+$ channels to make the fixed channel allocation. The closer p is to $p_{\min}$, fewer the number of additional edges required to make G' hamiltonian, and lesser the number of channels used.
5. The algorithm for making the required fixed channel allocation using $n + (pq - 1)^+$ channels follows from the proof of

edge set.

[5] A cycle in a graph is a sequence of vertices such that consecutive vertices in the cycle have an edge between them and no vertex in the sequence occurs more than once except the first (or the last) which occurs twice, once in the first position and again in the last position. A graph is hamiltonian iff it has a cycle consisting of all vertices. Such a cycle is known as a hamilton cycle.

[6] $1_{\{pq>0\}} = 1$ iff $pq > 0$ and $1_{\{pq>0\}} = 0$ otherwise.

[7] A channel is said to be "skipped" if it is not allocated to any maximal independent set.

Theorem 1. Find the channels allocated to the jth maximal independent set by the algorithm described in the proof to allocate $W_j^O = qm_j$ channels to the jth maximal independent set. All the channels allocated to the jth maximal independent set are allocated to each of the cells in the jth maximal independent set. This gives the actual channel allocation to the cells.

If pq is small compared to n, then the number of channels obtained from our heuristic will be close to the minimum since n channels are necessary for making the fixed channel allocation even while satisfying only the co-channel reuse constraints. We illustrate the actual channel allocation to the cells in the following examples:

Example III.1: Consider the system with 7 cells shown in figure 2. Let the co-channel reuse constraint be that the maximum interference should not exceed 0.15 units. A channel may be used simultaneously in any number of cells, as long as the total interference does not exceed 0.15. Assume that the cell radius is $1/\sqrt{3}$. Distance between adjacent cells is 1 unit, and distance between non-adjacent two-hop away cells like 1 and 3 is $\sqrt{3}$. Assume that the interference produced in cell u by the simultaneous use of a channel in cell v is $d(u,v)^{-4}$. Thus adjacent cells can not use the same channel simultaneously (interference will be 1). No combination of 3 or more cells can use a channel simultaneously, e.g., the total interference produced in cell 3 if cells $1, 3, 5$ use the same channel simultaneously is $2/9$ ($d(1,3)^{-4} + d(3,5)^{-4} = 2/9$). However, cells 1 and 3 can simultaneously use the same channel, when cell 5 is not using it. The hypergraph modeling this co-channel reuse constraint has 10 maximal independent sets: $\{1,3\}$, $\{1,4\}$, $\{1,5\}$, $\{2,4\}$, $\{2,5\}$, $\{2,6\}$, $\{3,5\}$, $\{3,6\}$, $\{4,6\}$, $\{7\}$. We need to allocate 4 channels to each cell. Thus $n_i = 4$, for $1 \leq i \leq 7$. The ILP gives nonzero values to only the following maximal independent sets: $a \rightarrow \{1,3\}, b \rightarrow \{2,4\}, c \rightarrow \{3,5\}, d \rightarrow \{4,6\}, e \rightarrow \{1,5\}, f \rightarrow \{2,6\}, g \rightarrow \{7\}$. $a = b = c = d = e = f = 2$, $g = 4$, $n = 16$, $q = 2$, $m_a = m_b = m_c = m_d = m_e = m_f = 1$ and $m_g = 2$. We shall consider two different adjacent channel use constraints.

Case 1: Adjacent channels cannot be used simultaneously only in the same cell. The graphs G and G' have been shown in figure 5. G' is hamiltonian. Thus $p_{\min} = 0$ and 16 channels are sufficient. $a\ g1\ b\ c\ d\ g2\ e\ f\ a$ is a hamilton cycle in G'. Proceeding as per the algorithm in Theorem 1 with $q = 2$, a gets channels $1, 9$, $g1$ gets $2, 10$, b gets $3, 11$, c gets $4, 12$, d gets $5, 13$, $g2$ gets $6, 14$, e gets $7, 15$ and f gets $8, 16$. Thus cell 1 gets channels $\{1, 7, 9, 15\}$, cell 2 gets $\{3, 8, 11, 16\}$, cell 3 gets $\{1, 4, 9, 12\}$, cell 4 gets $\{3, 5, 11, 13\}$, cell 5 gets $\{4, 7, 12, 15\}$, cell 6 gets $\{5, 8, 13, 16\}$, and cell 7 gets $\{2, 6, 10, 14\}$.

Case 2: Again assume that adjacent channels cannot be used simultaneously in the s ame cell and also that cell 7 cannot use a channel if its adjacent channel is being used in either cell 2, 4 or 6. Refer to figures 6(a) and (b) for G and G' respectively. G' is not hamiltonian. However it becomes hamiltonian when edges are added between the pairs of vertices $(g1, b)$ and $(d, g2)$. Thus $p_{\min} \leq 2$ and 19 channels are sufficient. $b\ c\ d\ g2\ e\ f\ a\ g1\ b$ is a hamilton cycle in G' with the added edges. Proceeding as per the algorithm in Theorem 1 with $q = 2$, b gets channels $1, 11$, c gets $2, 12$, d gets $3, 13$, $g2$ gets $5, 15$, e gets $6, 16$, f gets $7, 17$, a gets $8, 18$ $g1$ gets $9, 19$. Thus cell 1 gets channels $\{6, 8, 16, 18\}$, cell 2 gets $\{1, 7, 11, 17\}$, cell 3 gets $\{2, 8, 12, 18\}$, cell 4 gets $\{1, 3, 11, 13\}$, cell 5 gets $\{2, 6, 12, 16\}$, cell 6 gets $\{3, 7, 13, 17\}$, and cell 7 gets $\{5, 9, 15, 19\}$.

For large systems the ILP is difficult to evaluate. The integer constraints in the ILP may be relaxed to get an approximate solution. Thus, $n_{\text{approx}} = \sum_{j=1}^{M} \lceil W_j^O \rceil$ can then be taken as an approximation to n and $(\lceil W_1^O \rceil, \lceil W_2^O \rceil, \ldots, \lceil W_M^O \rceil)$ can be used in place of the optimal solution to the ILP. Clearly $n_{\text{approx}} + (pq - 1)^+$ channels, with p obtained from the corresponding G and G' and q from the $(\lceil W_1^O \rceil, \lceil W_2^O \rceil, \ldots, \lceil W_M^O \rceil)$, is sufficient for the required fixed channel allocation. In general n_{approx} is a good approximation to n, the optimal value of the objective function of the ILP.

IV. Approximation to r_0^A

We will present an approximation strategy for r_0^A. We use a two step procedure here. We start with the optimal solution of LP2 which satisfies the co-channel constraints. Next, we modify this optimal solution with the objective of satisfying the adjacent channel constraints, using a graph theoretic technique similar to that of Section III. We describe the approach in details next.

Let G and G' be the graphs induced by a rational M-tuple $(X_1^O, X_2^O, \ldots, X_M^O)$ which forms an optimal solution of LP2. (Clearly any extreme point of the feasible region corresponds to rational values of $X_1^O, X_2^O, \ldots, X_M^O$ because the constraints involve rational numbers only. We know at least one of the extreme points gives the optimal solution, if there exists an optimal solution [8]. LP2 always has an optimal solution [4]. Thus there always exists a rational M-tuple $(X_1^O, X_2^O, \ldots, X_M^O)$ which forms an optimal solution of LP2.) γ_G (the number of vertices in G) $= V \leq N$ and $\gamma_{G'} = \sum_{j=1}^{M} m_j$.

Theorem 2: Let G' become hamiltonian upon the addition of some p edges. Then $r_0^A \geq r_0/(1 + p/\sum_{j=1}^{M} m_j)$.

Proof: Let there be n channels. Then $n = q(p + \sum_{j=1}^{M} m_j) + t$ where $0 \leq t < (p + \sum_{j=1}^{M} m_j)$ and q is a nonnegative integer (Euclidean division theorem.). By Theorem 1 we have a fixed channel allocation algorithm that allocates $q\sum_{j=1}^{M} m_j a_{ij}$ channels to the ith cell, for $1 \leq i \leq N$. Using the ATP and the independence of offered traffic from cell to cell it can be shown that this fixed channel allocation algorithm carries a traffic intensity of $\min\left(p_i r, \frac{\sum_{j=1}^{M} m_j a_{ij}}{p + \sum_{j=1}^{M} m_j}\right)$ in the asymptotic limit ($n \rightarrow \infty$) in cell i (refer to the Appendix for the proof). Let $r \leq r_0/(1 + p/\sum_{j=1}^{M} m_j)$. Then

$$
\begin{aligned}
p_i r &\leq p_i r_0/(1 + p/\textstyle\sum_{j=1}^{M} m_j) \\
&\leq \frac{\sum_{j=1}^{M} X_j^O a_{ij}}{(1+p/\sum_{j=1}^{M} m_j)} && \text{from the third set of} \\
& && \text{inequalities in LP2} \\
&= \frac{\frac{\sum_{j=1}^{M} m_j a_{ij}}{\sum_{k=1}^{M} m_k}}{(1+p/\sum_{j=1}^{M} m_j)} && \text{since } X_j^O = \frac{m_j}{\sum_{k=1}^{M} m_k} \\
&= \frac{\sum_{j=1}^{M} m_j a_{ij}}{(p+\sum_{j=1}^{M} m_j)}.
\end{aligned}
$$

Thus in the asymptotic limit the carried traffic intensity in the ith

cell is
$\lim_{n\to\infty} C(p_i r n, q\sum_{j=1}^{M} m_j a_{ij})/n = \min(p_i r, \sum_{j=1}^{M} m_j a_{ij}/(p+\sum_{j=1}^{M} m_j)) = p_i r$. Thus the blocking probability is 0 in the asymptotic limit for this fixed channel allocation algorithm if $r \le r_0/(1+p/\sum_{j=1}^{M} m_j)$. Thus $r_0^A \ge r_0/(1+p/\sum_{j=1}^{M} m_j)$, since r_0^A is the largest value of r for which a channel assignment algorithm that has asymptotically zero blocking exists. ■

Let $r_0^{AL} = r_0/(1+p/\sum_{j=1}^{M} m_j)$. Clearly the lower bound is the tightest if $p = p_{\min}$, where $p_{\min}$ is the minimum number of edges which must be added to G' to make it hamiltonian. In general, the problem of finding $p_{\min}$ is NP-complete [9] but $p_{\min}$ can be found easily in the following special cases.

1. G is a complete[8] graph. Let G' have $\gamma_{G'}$ vertices. Then

$$p_{\min} = \begin{cases} (\max_j m_j - \sum_{\substack{j=1 \\ j\neq \arg\max_k m_k}}^{M} m_j)^+, & \text{if } \gamma_{G'} \ge 3, \\ 0, & \text{if } \gamma_{G'} = 2 \\ 1, & \text{if } \gamma_{G'} = 1. \end{cases}$$

The proof goes as follows. Let $\gamma_{G'} \ge 3$, and let $\max_j m_j \le \sum_{\substack{j=1 \\ j\neq \arg\max_k m_k}}^{M} m_j$. Let v_{jk} be a vertex in G'. $\gamma_{G'} = \sum_{i=1}^{M} m_i$. The degree of a vertex v in a graph G (denoted by $d_G(v)$) is the number of edges incident on v. $d_{G'}(v_{jk}) = \sum_{\substack{i=1 \\ i\neq j}}^{M} m_i = \gamma_{G'} - m_j$.

$$\begin{aligned} \max_i m_i &\le \sum_{\substack{i=1 \\ i\neq \arg\max_l m_l}}^{M} m_i \\ 2\max_i m_i &\le \sum_{\substack{i=1 \\ i\neq \arg\max_l m_l}}^{M} m_i + \max_i m_i \\ 2\max_i m_i &\le \sum_{i=1}^{M} m_i \\ \max_i m_i &\le \frac{\sum_{i=1}^{M} m_i}{2} \\ m_j &\le \frac{\gamma_{G'}}{2} \quad \text{for all } j \\ d_{G'}(v_{jk}) &\ge \frac{\gamma_{G'}}{2}. \end{aligned}$$

Thus G' is hamiltonian ([10], page 54, Theorem 4.3). Thus $p_{\min} = 0$ and the relation holds in this case.

Let $\max_j m_j > \sum_{\substack{j=1 \\ j\neq \arg\max_k m_k}}^{M} m_j$. Let $i = \arg\max_k m_k$. $v_{i1}, v_{i2}, \ldots, v_{im_i}$ must be part of the hamilton cycle. Clearly v_{ia}, v_{ib} do not have an edge between them in G', since G is simple. Thus an extra edge must be added between v_{ia} and v_{ib} if they are consecutive vertices in the cycle. There must be at least $m_i - \sum_{\substack{j=1 \\ j\neq i}}^{M} m_j$ pairs v_{ia} and v_{ib} such that the vertices of the pair occupy consecutive positions in the cycle. Thus $\max_j m_j - \sum_{\substack{j=1 \\ j\neq \arg\max_k m_k}}^{M} m_j$ edges must be added to make G' hamiltonian. By placing one v_{jk} for some k and some $j \neq i$ between v_{il} and v_{il+1} till the v_{jk}s are exhausted, exactly $m_i - \sum_{\substack{j=1 \\ j\neq i}}^{M} m_j$ pairs v_{ia} and v_{ib} will be there such that the vertices of the pair occupy consecutive positions in the cycle. Thus only $\max_j m_j - \sum_{\substack{j=1 \\ j\neq \arg\max_k m_k}}^{M} m_j$ edges need be added to make G' hamiltonian. Thus $p_{\min} = \max_j m_j - \sum_{\substack{j=1 \\ j\neq \arg\max_k m_k}}^{M} m_j$ in this case.

Let $\gamma_{G'} = 1$. Since G is simple, G' must be simple and thus 1 edge must be added to make G' hamiltonian. Let $\gamma_{G'} = 2$. Note that $\gamma_G \le \gamma_{G'}$. If $\gamma_G = 1$, and $V_G = v_l$, $m_l = 1$. ($X_l^O > 0$, $X_i^O = 0$, if $i \neq l$, $\sum_{i=1}^{M} X_i^O = 1 \Rightarrow X_l^O = 1$, $m_l = 1$.) This means $\gamma_{G'} = 1$ which is a contradiction. Thus $\gamma_G = 2$. Let $V_{G'} = \{v_{i1}, v_{j1}\}$. Since G is complete, so is G'. Thus there is an edge between v_{i1} and v_{j1}. As per our definition, $v_{i1}v_{j1}v_{i1}$ is a hamilton cycle. Thus no edge need be added to G' to make it hamiltonian. Thus $p_{\min} = 0$ if $\gamma_{G'} = 2$.

G is complete for many systems. For example, G is complete for a linear array of cells, where any two cells can use the same channel simultaneously iff they are separated by a distance $D \ge uZ$, where Z is the center to center distance between adjacent cells, and the adjacent channel use constraint is that the same cell cannot use adjacent channels simultaneously, and $p_i = p_{i+u}$ for all i. Thus $p_{\min}$ can be computed easily for these systems.

2. G is completely disconnected, i.e., G has no edge. Thus G' is also completely disconnected. Thus $p_{\min} = \gamma_{G'} = \sum_{j=1}^{M} m_j$. We have found that G is in general disconnected if adjacent channels cannot be used simultaneously in the same or adjacent cells.

3. There are certain sufficient conditions involving the degree sequence and the number of edges in G' ([10], [11]) for G' to be hamiltonian. If these are satisfied, then $p_{\min} = 0$. Often a good approximation to $p_{\min}$ can be found from these sufficiency conditions.

If none of the above cases apply then we need to resort to other means to find $p_{\min}$. The problem of finding $p_{\min}$ can be reduced to a traveling salesman problem [10]. The traveling salesman problem is an NP-complete problem but there exist techniques which yield good approximations to the required result [9], [10].

We have used the following seemingly crude method which surprisingly gives small values of p in a very short time for all the cases we have studied and whose results we shall present in tables II and III. We first briefly describe our method. Finding a hamilton cycle in G' is equivalent to finding a closure-possible walk[9] in G ($\gamma_G \le N$) traversing vertex v_j m_j times. The length of the walk must be $\gamma_{G'}$. We have observed that a simple *branch and backtrack*[10] technique gives a walk of length, say T, very fast, where T is fairly close to $\gamma_{G'}$. Moreover this walk traverses no vertex v_j more than m_j times but after the length of the walk increases to T any further increase takes place very slowly. We stop the branch and backtrack process after a walk which is T vertices long is obtained. If this walk traverses the vertex v_j n_j times, we try to insert $m_j - n_j$ v_js in the walk that has already been obtained. If we cannot do so, we add extra edges which

[8] A graph is simple if no edge joins a vertex to itself.A graph is complete if it is simple and any two distinct vertices in the graph has an edge between them [10].

[9] A walk in a graph is a sequence of vertices such that the consecutive vertices in the sequence have an edge between them. A walk is closure-possible if the first and the last vertices have an edge between them.

[10] This procedure traverses along the graph without visiting any vertex v_j more than m_j times as long as it can, and when it cannot it backtracks along the traced path, till it can proceed along some branch which has not yet been visited.

increase the value of p we can get. The number T is decided upon after a few observations. We illustrate the branch and backtrack procedure by the following example.

Example IV.1: Let the graph G consist of W vertices numbered $1, 2, \ldots, W$ where W is quite large. The edges in G are as follows: i and $i+1$ are joined by an edge for all $i < W-1$. W has an edge with 1 and 2. $W-1$ has an edge with 1. There is no other edge in G. $m_i = 1$ for all i. The only closure-possible walk covering each vertex once is $1\ W\ 2\ 3 \ldots W-1$. (We consider cyclic shifts such as $1\ W\ 2\ 3 \ldots W-1$ and $W\ 2\ 3 \ldots W-1\ 1$ as being equivalent.) However the branch and backtrack procedure may move along the branch $1\ 2\ 3 \ldots W-1$. Thus a walk of length $W-1$ is obtained very fast but then on, the process backtracks in order to insert W, and this may take a long time. If the procedure is terminated when the walk of length $W-1$ is obtained, then W can be very easily inserted between 1 and 2 and thus a closure-possible walk is obtained and $p = 0$. If W could not be inserted i.e., if it did not have edges joining it to 1 and 2, then 2 edges could have been added, one joining W to 1 and the other to 2, and a closure-possible walk could have been obtained, giving $p = 2$. This value of p is small, particularly if W is large and thus we get a good approximation to $p_{\min}$ in a very short time.

If the graph G' does not give a small value of p fast with the above method with a particular optimal solution, the graph G' obtained from some other rational optimal solution may be tried. This often helps when there are multiple optimal solutions (degeneracy).

We now present our results for systems with $3, 7, 19, 37$ regular hexagonal cells (refer figures 1 to 4). We have assumed that the co-channel reuse in the system is constrained by the maximum tolerable interference. (Any two calls in the system using the same channel simultaneously interfere with each other. The interference diminishes with increase in distance between the callers.) The assumed model of interference is as follows:

- Interference produced in cell u due to the usage of the same channel in cell v equals $d(u,v)^{-4}$, where $d(u,v)$ is the center-to-center distance between cells u and v.
- Total interference produced in cell u = interference produced by all other cells using the same channel $= \sum_{v \in C(u)} d(u,v)^{-4}$, where $C(u)$ is the set of cells using the same channel as u, barring u. An *additive* model of interference is thus assumed.
- The cell radius is assumed to be $1/\sqrt{3}$ or equivalently the distance between adjacent cells is taken to be 1.
- Let the requisite transmission quality be that the maximum interference must be less than or equal to some given threshold. This limits co-channel reuse.

This model for interference is the same as that used in [4]. The co-channel reuse constraints have been modeled by a hypergraph. We illustrate the computation of the approximation to r_0^A by the following example.

Example IV.2: Consider the system with 7 cells shown in figure 2 described in Example III.1. The same model for interference is assumed as in Section III. The co-channel reuse constraints, and hence the maximal independent sets of the hypergraph modeling the co-channel reuse constraints, remain the same as in Example III.1. LP2 gives nonzero values to only the following maximal independent sets: $a \to \{1,3\}, b \to \{2,4\}, c \to \{3,5\}, d \to \{4,6\}, e \to \{1,5\}, f \to \{2,6\}, g \to \{7\}$. $a = b = c = d = e = f = 1/8$, $g = 1/4$, $m_a = m_b = m_c = m_d = m_e = m_f = 1$, and $m_g = 2$. We shall consider two different adjacent channel use constraints as in Example III.1:

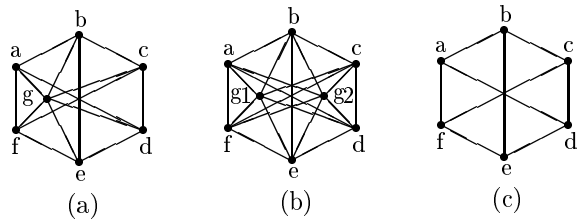

Fig. 5. The graphs for the system with 7 cells when use of adjacent channel is prohibited in the same cell.

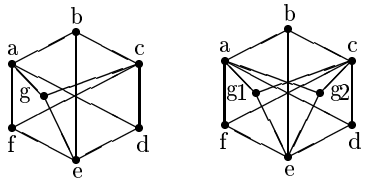

Fig. 6. The graphs for the system with 7 cells when adjacent channel use is prohibited in the same cell, and also cell 7 cannot use a channel if its adjacent channel is being used in cell 2, 4 or 6.

Case 1: Adjacent channels cannot be used simultaneously only in the same cell. Thus $b_{ii} = 2$ for all i and $b_{ij} = 1$, if $i \neq j$. The graphs G and G' have been shown in figure 5(a) and (b) respectively. G' is hamiltonian. Thus $p_{\min} = 0$ and $r_0^{AL} = r_0 = 1.75$. Clearly $r_0^{AL} = r_0^A$ here. $a\ g1\ b\ c\ d\ g2\ e\ f\ a$ is a hamilton cycle in G'.

Case 2: Assume adjacent channels cannot be used simultaneously in the same cell and also that cell 7 cannot use a channel if its adjacent channel is being used in either cell 2, 4 or 6. Thus

$$b_{ij} = \begin{cases} 2, & \text{if } i = j, \text{ or } (i,j) \in \{(2,7), \\ & \qquad (4,7), (6,7), (7,2), (7,4), (7,6)\}, \\ 1, & \text{otherwise.} \end{cases}$$

Refer to figure 6(a) and (b) for G and G' respectively. G' is not hamiltonian. However it becomes hamiltonian when edges are added between $g1, b$ nd $d, g2$. Thus $p_{\min} \leq 2$ and $r_0^{AL} = 0.8 r_0 = 1.4$. Using $r_0^{AL} \leq r_0^A \leq r_0$, $1.4 \leq r_0^A \leq 1.75$. $b\ c\ d\ g2\ e\ f\ a\ g1\ b$ is a hamilton cycle in G' with the added edges.

We will study two cases of adjacent channel use constraints. The first prevents the use of adjacent channels in the same cell simultaneously but allows any other form of adjacent channel use. All non-diagonal elements of the corresponding B matrix are 1 and the diagonal elements are 2. The second adjacent channel use constraint prevents the simultaneous use of adjacent channels in cells separated by a distance ≤ 1, i.e., $b_{ij} = 1$ iff $d(i,j) > 1$ and $b_{ij} = 2$ otherwise.

We have also studied two different traffic patterns: Uniform traffic pattern (UT) and the Non-uniform traffic pattern (NUT). In the former the same amount of traffic is offered in each cell, i.e., $p_i = 1/N$, for all i. In the latter the maximum traffic is offered in the central cell, one half of that in the next ring of cells, one third of that in the next ring, and so on. Non-uniform

TABLE III

THE LOWER BOUNDS ON CAPACITIES OBTAINED USING OUR METHOD IN THE CASE WHERE ADJACENT CHANNEL USE IN THE SAME CELL ALONE IS PROHIBITED.

No. of cells	Cochannel Interf. Threshold	r_0^{AL}/r_0 UT	r_0^{AL}/r_0 NUT	r_0^{AL} UT	r_0^{AL} NUT	r_0 UT	r_0 NUT
3	(0,1)	1.0		1.50		1.50	
7	0.15	1.0		1.75		1.75	
19	0.02	1.0	1.0	2.11	1.85	2.11	1.85
19	0.1	1.0	1.0	3.10	2.56	3.10	2.56
19	0.2	1.0	1.0	4.22	3.20	4.22	3.2
19	0.3	1.0	1.0	4.65	3.84	4.65	3.84
19	0.375	1.0	1.0	5.04	4	5.04	4
19	0.4	0.99	1.0	5.04	4	5.10	4
19	0.5	1.0	1.0	5.73	4	5.73	4
19	0.6	1.0	1.0	6.00	4	6.00	4
19	2/3	1.0	1.0	6.33	4	6.33	4
37	0.1	1.0		5.13		5.13	
37	0.12	1.0		5.69		5.69	
37	0.39	1.0		9.25		9.25	
37	0.5	1.0		9.25		9.26	
37	0.6	1.0		10.74		10.74	
37	0.7	1.0		10.96		10.98	
37	0.74	1.0		12.33		12.33	

traffic pattern has been studied for the system with 19 cells only. For this system,

$$p_i = \begin{cases} 1/24, & i \in \{1, \ldots, 4,\ 7,\ 8,\ 12,\ 13,\ 16, \ldots, 19\}, \\ 1/16, & i \in \{5, 6, 9, 11, 14, 15\}, \text{ and} \\ 1/8, & i = 10. \end{cases}$$

This model of non-uniform traffic pattern may be representative of cities where more traffic is offered in the center and less in the outskirts.

The results for the first case have been tabulated in Table II and Table III and for the second case in Table IV and Table V. The number of vertices in G and G' have not been listed for the second case because the graph G is completely disconnected in all these cases and we get $p_{\min}$ and hence r_0^{AL} from the special case (2). The number of vertices in G and G' are anyway the same as the corresponding ones in Table II.

In Table II we have not listed the number of vertices of G and G' for the 19-cell system with nonuniform traffic pattern for interference threshold ≥ 0.4 because an optimal solution of LP2 in each of these cases is same as that for an interference threshold of 0.375. Thus the graphs G and G' are also the same as that for an interference threshold of 0.375. We have listed the number of variables, constraints, and so on, in the linear program used in [2] in one of the cases for the 37 cell system. The numbers are even larger for the other cases for the 37 cell system when the interference threshold is higher.

The following observations may be made from the tabulated data:

1. We know that $r_0^{AL} \leq r_0^A \leq r_0$. Thus r_0^{AL} is a good approximation to r_0^A if $r_0^{AL}/r_0 = (1 + p/\sum_{j=1}^{M} m_j)^{-1}$ is close to 1, and is the same as r_0^A if $r_0^{AL}/r_0 = 1$. Thus Table III indicates that r_0^{AL} is a very good approximation to r_0^A and often gives the exact r_0^A in the first case, i.e., when adjacent channels cannot be used simultaneously in the same cell only. Also the proximity of r_0^{AL}/r_0 to 1 indicates that $r_0^A \approx r_0$ in this case. Table II indicates that the computation of the exact value of r_0^A as per [2] may be impossible in this case even for the system with 19 cells. We could compute the approximations for the system with 19 cells using no more than 0.4 seconds of system time on an IBM SP2 machine. Our computations took less than 0.5 minutes of system time on an IBM SP2 for the system with 37 cells.

2. Consider Table V where we list the results for the case where adjacent channels cannot be used simultaneously in the same cell and in adjacent cells. The graph G is completely disconnected in this case and $p_{\min} = \gamma_{G'}$. Thus $r_0^{AL}/r_0 = (1 + p/\sum_{j=1}^{M} m_j)^{-1} = 0.5$. Thus the value of r_0^{AL}/r_0 does not guarantee that our approximation is good. However in this case the number of variables and constraints in the linear program used in [2] are not that large and we could compute r_0^A as per [2] for systems with $3, 7, 19$ cells. Comparison of r_0^{AL} with r_0^A indicates that r_0^{AL} is reasonably close to r_0^A in most of the cases but there is a significant difference in some of the cases. Better heuristics than those we have proposed here may be needed in such cases. The results indicate that r_0 (in this case $r_0 = 2r_0^{AL}$) is significantly higher than r_0^A. Our approximation is much easier

[10] From the hamilton cycle, $r_0^{AL} = (1 + p/\sum_{j=1}^{M} m_j)^{-1} r_0 = 4.5$. Using the fact that r_0^A increases with increase in allowable interference threshold and $r_0^A = 5.04$ for a lower value of the interference threshold, namely 0.375, we get a better lower bound. We have similarly improved the lower bound for the 37-cell system with an interference threshold of 0.5.

TABLE IV

A COMPARISON OF THE COMPLEXITIES OF DEORA'S METHOD [2] AND OUR METHOD IN THE CASE WHERE ADJACENT CHANNEL USE IN THE SAME CELL AND IN ADJACENT CELLS IS PROHIBITED.

		Deora's method[2]		Our method	
No. of cells	Cochannel Interference Threshold	Number of vars.	Number of constrs.	Number of vars	Number of constrs
3	(0 1)	11	8	3	4
7	0.15	63	24	11	8
19	0.1	7368	311	86	20
19	0.2	19097	832	249	20
19	0.3	24131	1237	242	20
19	0.375	25125	1322	188	20
19	0.4	25244	1396	194	20
19	0.5	25429	1431	167	20
19	0.6	25465	1443	146	20
19	2/3	25467	1444	140	20

TABLE V

THE LOWER BOUNDS ON CAPACITIES OBTAINED USING OUR METHOD IN THE CASE WHERE ADJACENT CHANNEL USE IN THE SAME CELL AND IN ADJACENT CELLS IS PROHIBITED.

No. of cells	Cochannel Interference Threshold	r_0^{Al}/r_0^A UT	r_0^{Al}/r_0^A NUT	r_0^{AL} UT	r_0^{AL} NUT	r_0^A UT	r_0^A NUT
3	(0 1)	1.0		0.75		0.75	
7	0.15	0.75		0.875		1.17	
19	0.1	0.65	0.80	1.55	1.28	2.37	1.6
19	0.2	0.76	0.80	2.11	1.6	2.78	2.0
19	0.3	0.73	0.96	2.33	1.92	3.17	2.0
19	0.375	0.80	1.0	2.52	2.0	3.17	2.0
19	0.4	0.81	1.0	2.55	2.0	3.17	2.0
19	0.5	0.90	1.0	2.85	2.0	3.17	2.0
19	0.6	0.95	1.0	3.0	2.0	3.17	2.0
19	2/3	1.0	1.0	3.17	2.0	3.17	2.0

TABLE II

A COMPARISON OF THE COMPLEXITIES OF DEORA'S METHOD [2] AND OUR METHOD IN THE CASE WHERE ADJACENT CHANNEL USE IN THE SAME CELL ALONE IS PROHIBITED.

		Deora's method[2]		Our method					
No. of cells	Cochannel Interf. Threshold	Number of vars.	constrs.	Number of vars	constrs	Number of vertices in G UT	G NUT	G' UT	G' NUT
3	(0, 1)	17	9	3	4	2		2	
7	0.15	201	25	11	8	7		8	
19	0.1	53515	311	85	20	17	18	98	74
19	0.2	344539	833	249	20	8	14	18	624998
19	0.3	681543	1238	242	20	18	17	147	50
19	0.375	802297	1373	188	20	17	11	49	500006
19	0.4	824137	1397	194	20	18		82	
19	0.5	856699	1432	167	20	6		10	
19	0.6	867139	1444	146	20	9		19	
19	2/3	867971	1445	140	20	3		3	
37	0.1	235796085	21310	4987	38	34		1111	

to compute in this case even for the system with 19 cells. For the system with 37 cells again the computation of the exact value of r_0^A as per [2] may be impossible.

V. APPROXIMATION TO $T_A(r)$

We now show how to approximate $T_A(r)$. The approach is similar to the that of the previous section. We start with the optimal solution of LP1 instead of LP2 here. The details follow.

Let LP1 have a rational optimal solution, $(X_1^O, \ldots, X_M^O)$. Now let G and G' be the graphs induced by the M-tuple $(X_1^O, X_2^O, \ldots, X_M^O)$. Clearly $X_k^O = m_k / \sum_{j=1}^M m_j$, for $1 \leq k \leq M$.

Theorem 3: Let G' become hamiltonian upon the addition of p edges. Then $T_A(r) \geq T(r)/(1 + p/\sum_{j=1}^M m_j)$.

Proof: This proof is similar to that of Theorem 2. Let there be n channels. Then $n = q(p + \sum_{j=1}^M m_j) + t$ for some t and q where $0 \leq t < (p + \sum_{j=1}^M m_j)$ and q is a nonnegative integer (Euclidean division theorem). By Theorem 1 we have a fixed channel allocation algorithm allocating $q \sum_{j=1}^M m_j a_{ij}$ channels to the ith cell for $1 \leq i \leq N$. Using the ATP and the independence of offered traffic it can be shown that this fixed channel allocation algorithm carries a traffic intensity of $\min\left(p_i r, \frac{\sum_{j=1}^M m_j a_{ij}}{p + \sum_{j=1}^M m_j}\right)$ in the asymptotic limit ($n \to \infty$) in cell i (refer to the Appendix for the proof). Let the total traffic intensity carried by this algorithm in the system be $T_{FCA}(r)$. Then

$$\begin{aligned} T_{FCA}(r) &= \sum_{i=1}^N \min(p_i r, \frac{\sum_{j=1}^M m_j a_{ij}}{p + \sum_{j=1}^M m_j}) \\ &\geq \sum_{i=1}^N \frac{\sum_{j=1}^M m_j}{p + \sum_{j=1}^M m_j} \min(p_i r, \frac{\sum_{j=1}^M m_j a_{ij}}{\sum_{j=1}^M m_j}) \\ &= \frac{\sum_{j=1}^M m_j}{p + \sum_{j=1}^M m_j} \sum_{i=1}^N \min(p_i r, \sum_{j=1}^M X_j^O a_{ij}) \end{aligned}$$

$$\text{since } X_j^O = m_j / \sum_{k=1}^M m_k$$

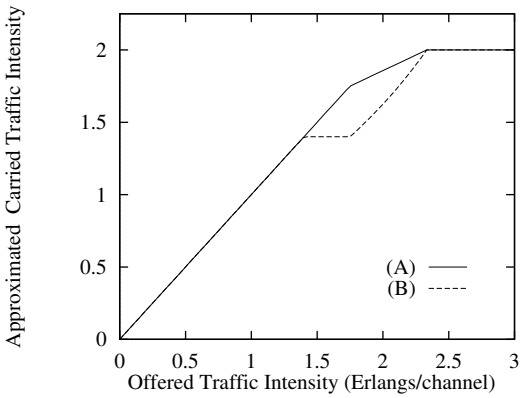

Fig. 7. The curve (A) gives $T_{AL}(r)$, $T(r)$ and $T_A(r)$ for Example II.3.1. The curve (A) also gives $T(r)$ for Example II.3.2, while the curve (B) gives $T_{AL}(r)$ for Example II.3.2. $T_A(r)$ is between curves (A) and (B) in this case.

$$= \frac{\sum_{j=1}^M m_j}{p + \sum_{j=1}^M m_j} T(r).$$

Thus $T_A(r) \geq T_{FCA}(r) \geq \frac{\sum_{j=1}^M m_j}{p + \sum_{j=1}^M m_j} T(r) = T_{AL}(r)$ where the last equality defines $T_{AL}(r)$ ■

Again the lower bound is the tightest if $p = p_{\min}$. The techniques for finding $p_{\min}$ in the special cases and the approximation to $p_{\min}$ in the general case discussed in Section IV apply here. Certain other observations simplify the computation of $T_A(r)$, e.g., $T_A(r) = r$ for $r \leq r_0^{AL}$. (This follows from the definition of r_0^A and the fact that $r_0^{AL} \leq r_0^A$.) Since $T_{AL}(r) = T(r)/(1 + p/\sum_{j=1}^M m_j) \leq T_A(r) \leq T(r)$, one can be assured that the approximation is doing well if p is small compared to $\sum_{j=1}^M m_j$. In general it is much easier to compute $T_{AL}(r)$ than $T_A(r)$ as per [2].

We would also like to point out that since $T_A(r) = r$, for $r \leq r_0^{AL} \leq r_0^A$, if r_0^{AL} is known then we can set $T_{AL}(r) = r$, for $r \leq r_0^{AL}$.

Example V.1: Consider the system with 7 cells described in Example III.1 (figure 2). The same model for interference is

assumed as in Section III The co-channel reuse constraint and hence the maximal independent sets of the hypergraph modeling the co-channel reuse constraint remains the same as in Example III.1. LP1 gives the following solution:

$$T(r) = \begin{cases} r, \text{ if } r \leq 7/4, \\ 1 + 3r/7, \text{ if } 7/4 < r \leq 7/3, \\ 2, \text{ if } r \geq 7/3. \end{cases}$$

$$\text{For } r \leq 7/4, \quad X_i(r) = \begin{cases} r/14, \text{ if } i \in \{a, b, \ldots, f\}, \\ r/7, \text{ if } i = g, \\ 0, \text{ otherwise.} \end{cases}$$

$$\text{For } 7/4 < r \leq 7/3, \quad X_i(r) = \begin{cases} r/14, \text{ if } i \in \{a, b, \ldots, f\}, \\ 1 - 3r/7, \text{ if } i = g, \\ 0, \text{ otherwise.} \end{cases}$$

$$\text{For } r \geq 7/3, \quad X_i(r) = \begin{cases} 1/6, \text{ if } i \in \{a, b, \ldots, f\}, \\ 0, \text{ otherwise.} \end{cases}$$

Throughut in this example we shall assume that $r \in Q$. We shall consider two different adjacent channel use constraints as in Example III.1:

Case 1: Adjacent channels cannot be used simultaneously only in the same cell. We have found in Example IV.2 that $r_0^A = r_0^{AL} = 7/4$. So $T_{AL}(r) = r$, for $r \leq 7/4$. Let $h = \text{LCM}(r, 14 - 6r)$. For $7/4 < r < 7/3$,

$$m_i = \begin{cases} h\frac{r}{\min(r, 14-6r)}, & \text{if } i \in \{a, b, \ldots, f\} \\ h\frac{14-6r}{\min(r, 14-6r)}, & \text{if } i = g \\ 0, & \text{otherwise.} \end{cases}$$

where $h = \text{LCM}(r, 14 - 6r)$

Since $m_i \neq 0$, $i \in \{a, b, \ldots, g\}$, G is given by figure 5(a). G' is always hamiltonian in this range. Let $m_i = x$, $i \in \{a, b, \ldots, f\}$ and $m_g = y$. $V'_G = \{a_1, a_2, \ldots, a_x, b_1, b_2, \ldots, b_x, \ldots f_1, f_2, \ldots, f_x, g_1, g_2, \ldots, g_y\}$. $a_1 b_1 \ldots f_1 a_2 b_2 \ldots f_2 \ldots a_x b_x \ldots f_x a_1$ is a cycle in G'. Now insert g_1 between a_1 and b_1, g_2 between b_1 and c_1, and so on in the above cycle till the g_is are exhausted. The g_is will be exhausted before one reaches $b_{\lceil x/3 \rceil}$ since $0 < y < 2x$ for $7/4 < r < 7/3$. This gives a hamilton cycle in G' since g_i and p_j, $p \in \{a, b, \ldots, f\}$ $1 \leq i \leq y$ and $1 \leq j \leq x$ are connected by an edge in G'. Thus $p_{\min} = 0$. $T_{AL}(r) = T(r)$ for $7/4 < r < 7/3$. For $r \geq 7/3$, $m_i = 1$, $i \in \{a, b, \ldots, f\}$ and $m_i = 0$ otherwise. G and G' are given by figure 5(c). G' is clearly hamiltonian. The hamilton cycle is $abcdefa$. Thus $T_{AL}(r) = T(r), r > 7/4$. Thus $T_{AL}(r) = T(r)$ for all r. Hence $T_A(r) = T(r)$ for all r. Refer to figure 7 for $T_{AL}(r)$, $T_A(r)$ and $T(r)$. Recall that we had observed earlier that r_0^{AL} is very close to r_0 and hence to r_0^A for the same adjacent channel use constraint as in this case and we find that $T_{AL}(r) = T_A(r) = T(r)$ for the system with 7 cells. Hence we expect that $T_{AL}(r)$ will track $T(r)$ and hence $T_A(r)$ very closely even for larger systems with the same adjacent channel use constraint as in this example.

Case 2: Again assume that adjacent channels cannot be used simultaneously in the same cell and also that cell 7 cannot use a channel if its adjacent channel is being used in either cell 2, 4 or 6. We know from Example IV.2 that $r_0^{AL} = 7/5$. Thus $T_{AL}(r) = r$ for $r \leq 7/5$. For $7/5 < r \leq 7/4$,

$$m_i = \begin{cases} 1, & \text{if } i \in \{a, b, \ldots, f\}, \\ 2, & \text{if } i = g, \\ 0, & \text{otherwise.} \end{cases}$$

Refer to figures 6(a) and (b) for G and G' respectively. G' is not hamiltonian. However it becomes hamiltonian when edges are added between the pairs of vertices $(g1, b)$ and $(d, g2)$. Thus $p_{\min} \leq 2$ and $T_{AL}(r) = 0.8T(r)$, for $7/5 < r \leq 7/4$. $bcdg2efag1b$ is a hamilton cycle in G' with the added edges. However we can get a better upper bound by observing that $T_{AL}(7/5) = 7/5$ and $T_{AL}(r) \geq T_{AL}(r')$, if $r \geq r'$. Using this $T_{AL}(r) = 7/5$, for $7/5 < r \leq 7/4$. For $7/4 < r < 7/3$, m_is are given by those in case 1 for the same range of r. G is given by figure 6(a). V'_G is also the same as that given in case 1 for the same range of r. G' is not, in general, hamiltonian. However G' can be made hamiltonian by adding edges between the pairs of vertices (g_1, b_1), (g_2, d_1), (g_3, b_2), (g_4, d_2) and so on till the g_is are exhausted. (All g_is will be exhausted because $0 < y < 2x$ here, where x, y are as defined in case 1 for the same range of r.) $b_1 c_1 d_1 g_2 e_1 f_1 a_1 g_1 b_2 c_2 d_2 g_4 e_2 f_2 a_2 g_3 \ldots b_l c_l d_l g_{2l} e_l f_l a_l g_{2l-1} \ldots b_x c_x d_x e_x f_x a_x b_1$ is a hamilton cycle in G' with the added edges. Thus for $7/4 < r < 7/3$, $p_{\min} \leq y$ and

$$\begin{aligned} T_{AL}(r) &= \frac{T(r)}{1 + \frac{y}{\sum_{i \in \{a, \ldots, g\}} m_i}}, \\ &= \frac{T(r)}{2 - 3r/7}, \quad \text{(using the values of } y \text{ and } m_i\text{s)} \\ &= \frac{7 + 3r}{14 - 3r}. \quad \text{(using the value of } T(r)\text{)} \end{aligned}$$

For $r \geq 7/3$, m_is are given by those in case 1 for the same range of r. G and G' are given by figure 5(c). G' is clearly hamiltonian. $abcdefa$ is the hamilton cycle. Thus $T_{AL}(r) = T(r)$. Summarizing,

$$T_{AL}(r) = \begin{cases} r, & \text{if } r \leq 7/5, \\ 7/5, & \text{if } 7/5 < r \leq 7/4, \\ \frac{7+3r}{14-3r} & \text{if } 7/4 < r < 7/3, \\ 2 & \text{if } r \geq 7/3. \end{cases}$$

VI. Approximation in the Presence of Arbitrary Restrictions on Simultaneous Use of any Two Different Channels

Simultaneous use of any two channels can produce interference even if they are not the same or adjacent. In most cases the filter responses are good enough so that the interference produced by non-adjacent channels is negligible. Nevertheless sometimes there may be restrictions on the simultaneous use of non-adjacent channels also. As mentioned before [2] formulates linear programs for the computation of $T_A(r)$ and r_0^A in the presence of any such restriction but these linear programs are in-tractable because of the large number of variables and constraints even for systems of moderate size, e.g., a system with 19 cells[11]. We can extend our approximations to this general case under certain special circumstances.

[11] If there are restrictions on the simultaneous use of c channels, $\Omega \subseteq \tau \times \tau \times \ldots \times \tau$, c times. Refer to the relevant discussion in Section I of the work in [2]. $|\tau|$ is large and $|\Omega|$ is very large making the number of variables extremely large.

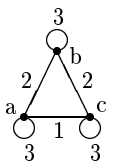

Fig. 8. The weighted D graph for a system with 3 cells. The numbers against the edges indicate their weights.

The co-channel reuse constraints are modeled by a hypergraph as before. The channels are numbered $1,2,\ldots,n$. The distance between channels numbered i,j is defined to be $|i-j|$. The numbering should be such that this distance is proportional to the actual separation between them in the radio spectrum. The adjacent channel use constraints are modeled by a matrix B, which is defined slightly differently from the corresponding definition in Section I. B is a $N \times N$ matrix, such that if cells i,j can use adjacent channels simultaneously, then $b_{ij} = 1$. Otherwise b_{ij} is the minimum distance between the channels that can be used simultaneously in cells i and j. Note that if the restriction is only on adjacent channel use, as assumed in Sections IV and III, then the elements of B are 1 and 2. (This agrees with the definition of B in Section IV.)

A graph D is induced by an M-tuple $(X_1, X_2, \ldots, X_M)$ as follows: Let $(X_1, X_2, \ldots, X_M)$ be any M-tuple, such that X_j is a nonnegative rational number and M is the number of maximal independent sets of the hypergraph modeling the co-channel reuse constraints of the cellular system. Let $X_{k_1}, X_{k_2}, \ldots, X_{k_V}$ be the ones with nonzero values. Let $X_{k_1} : X_{k_2} : \ldots : X_{k_V} = m_{k_1} : m_{k_2} : \ldots : m_{k_V}$, where $m_{k_1}, m_{k_2}, \ldots, m_{k_V}$ are relatively prime positive integers. If $X_j = 0$, $m_j = 0$. Form a weighted graph $D = (V_D, E_D)$ as follows: $V_D = \{v_{k_1 1}, \ldots, v_{k_1 m_{k_1}}, v_{k_2 1}, \ldots, v_{k_2 m_{k_2}}, \ldots, v_{k_V 1}, \ldots, v_{k_V m_{k_V}}\}$. There is an edge between any two vertices v_i and v_j (even if $v_i = v_j$). The weight of an edge between the vertices $v_{k_l a}$ and $v_{k_t b}$ is

$$\begin{aligned} w(v_{k_l a}, v_{k_t b}) &= \max_{1 \le i,j \le N} b_{ij} a_{ik_l} a_{jk_t} \\ &= \max_{\substack{1 \le i \le N, a_{ik_l} = 1 \\ 1 \le j \le N, a_{jk_t} = 1}} b_{ij}. \end{aligned}$$

Example VI.1: Consider the system with 3 cells shown in figure 1. Let the minimum separation between channels used simultaneously in the same cell be 3, i.e., cell 1 can not use channels $1,2$ or $1,3$ simultaneously, and so on. Adjacent cells cannot use adjacent channels simultaneously. Any other cells can use adjacent channels simultaneously. Let there be no co-channel reuse in the system, i.e., there are 3 maximal independent sets a, b, c each consisting of a single cell. $a = \{1\}, b = \{2\}, c = \{3\}$.

$$B = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

The graph D induced by $(2,2,2)$ is shown in figure 8.
We present the following results:

Theorem 4: Let D be the weighted graph induced by some M-tuple $(X_1, X_2, \ldots, X_M)$. Let D satisfy the triangle inequality.[12] Let there be a hamilton cycle in D of weight[13] L. Let $w_{\max}$ be the weight of an edge which has the maximum weight amongst all edges in the hamilton cycle. Let $(qL - w_{\max} + 1)^+$ channels be available. Then there exists a fixed channel allocation algorithm that allocates $q \sum_{j=1}^{M} m_j a_{ij}$ channels to the ith cell, for $1 \le i \le N$, for any nonnegative integer q.

Proof: This proof is along the same lines as that of Theorem 1. Again if $q = 0$ the lemma is trivially true ($w_{\max} \ge 1$). Let $q > 0$. Let the hamilton cycle of weight L be $v_{i_0 j_0}, v_{i_1,j_1}, \ldots, v_{i_{U-1},j_{U-1}}, v_{i_0 j_0}$. $U = \sum_{j=1}^{M} m_j$. Without loss of generality $w(v_{i_{U-1},j_{U-1}}, v_{i_0 j_0}) = w_{\max}$. If the algorithm given in the proof of Theorem 1 is slightly modified, it allocates qm_j channels to the jth maximal independent set satisfying the co-channel reuse and the adjacent channel use constraints, using $(qL - w_{\max} + 1)^+$ channels. The following modification is required in step 4:
$b \rightarrow b + w(v_{i_a,j_a}, v_{i_{(a+1) \bmod U} j_{(a+1) \bmod U}})$. $a \rightarrow (a+1) \bmod U$. Go to step 2.
Clearly this allocation satisfies the co-channel reuse constraints. Let cell i get channel k_1 and cell j get channel k_2. Let channel k_1 be allocated to the sth maximal independent set and channel k_2 be allocated to the tth maximal independent set. Thus cell i is in the sth maximal independent set and cell j is in the tth maximal independent set. Note that $|k_1 - k_2|$ is equal to the total weight of at least one of the paths between v_{k_1} and v_{k_2} on the hamilton cycle. Because of the triangle inequality, the weight of both the paths on the hamilton cycle is greater than or equal to $w(v_{k_1}, v_{k_2})$ and this weight is greater than or equal to b_{ij}, the minimum separation between the channels which can be simultaneously used in these cells. Thus this allocation satisfies the adjacent channel use constraints. This is the advertised fixed channel allocation. ■

The following Corollary follows from Theorem 4.

Corollary 2: Let a fixed channel allocation that allocates n_i channels to the ith cell be desired. Let D be the graph induced by an optimal solution of the ILP, $(X_1^O, X_2^O, \ldots, X_M^O)$. Let $q =$ g.c.d. of nonzero X_j^Os if there exists at least one nonzero W_j^O and let $q = 0$ otherwise. Let there be a hamilton cycle in D of weight L. $w_{\max}$ is the weight of an edge which has the maximum weight amongst all edges in the hamilton cycle. If D satisfies the triangle inequality, $(qL - w_{\max} + 1)^+$ channels are sufficient to achieve the required fixed channel allocation.

Theorem 5: Let D be the graph induced by a rational optimal solution of LP2, $(X_1^O, X_2^O, \ldots, X_M^O)$. Let there be a hamilton cycle in D of weight L. If D satisfies the triangle inequality, $r_0^A \ge r_0 \sum_{j=1}^{M} m_j / L$.

Proof: The proof follows from Theorem 4, Lemmas 1 and 2 along the same lines as the proof of Theorem 2 follows from Theorem 1, and Lemmas 1 and 2. ■

Remarks: $r_0 \sum_{j=1}^{M} m_j / L \le r_0^A \le r_0$. The lower bound is tightest if $L = L_{\min}$, where $L_{\min}$ is the weight of the minimum weight hamilton cycle in the graph D induced by $(X_1^O, X_2^O, \ldots, X_M^O)$, a rational optimal solution of LP2.

[12] $w(v_1, v_2) \le w(v_1, v_3) + w(v_3, v_2)$

[13] The weight of a hamilton cycle in a weighted graph is the sum of the weights of the edges in the cycle.

Theorem 6: Let D be the graph induced by a rational optimal solution of LP1, $(X_1^O, X_2^O, \ldots, X_M^O)$. Let there be a hamilton cycle in D of weight L. If D satisfies the triangle inequality, $T_A(r) \geq T(r) \sum_{j=1}^{M} m_j/L$.
The proof follows the same lines as that of Theorem 3.
Remarks: $T(r) \sum_{j=1}^{M} m_j/L \leq T_A(r) \leq T(r)$. Again the lower bound is the tightest if $L = L_{\min}$, where $L_{\min}$ is the weight of the minimum weight hamilton cycle in the graph D induced by $(X_1^O, X_2^O, \ldots, X_M^O)$, a rational optimal solution of LP1. Note that if the restriction is only on adjacent channel use, as assumed in Sections IV and III, then the elements of B are 1 and 2. (This agrees with the definition of B in Section IV.) This makes D always satisfy the triangle inequality in this case. Also the lower bounds with $L_{\min}$ given by Theorems 5 and 6 turn out to be the same as those given by Theorems 2 and 3 respectively with $p_{\min}$ in these cases. The same observation applies for the results of Corollaries 1 and 2.

The problem of finding the minimum weight hamilton cycle in D is the same as the traveling salesman problem which is NP-complete. However if D satisfies the triangle inequality, there are some polynomial time algorithms which will produce, within known bounds, an approximation to the traveling salesman problem [9]. Thus this technique is useful, whenever D satisfies the triangle inequality, to get an estimate of r_A^0 and $T_A(r)$, more so because the exact values of these quantities are in-tractable even for systems of reasonably small size. The generalization for the case in which D does not satisfy triangle inequality is complicated. This is a topic for future research.

We would like to mention that the results in this section present our preliminary research in this area. Nevertheless, these results indicate promising directions for extending the approximations towards the general case of arbitrary restrictions for simultaneous use of any two channels. We hope that this research will stimulate further interest in the arbitrary generalizations.

VII. Conclusion and Summary

We now summarize the contributions of this paper. There are various performance bounds for a cellular system which are useful from the theoretical as well as from the network operators' point of view. The computation of the exact values of these performance bounds in the presence of co-channel reuse and adjacent channel use constraints is difficult or rather impossible even for systems of reasonably small size. We have developed approximations to these performance bounds in the presence of co-channel reuse and adjacent channel use constraints which are computationally much simpler. These approximations track very closely the actual performance bounds in most cases. We have also presented good heuristics for the problem of finding the minimum number of channels necessary for achieving any given fixed channel allocation algorithm in the presence of co-channel reuse and adjacent channel use constraints.

We make one observation before conclusion: we have made all computations in two steps. The LPs or the ILP takes care of the co-channel reuse constraints and the graph theoretic approach takes care of the adjacent channel use constraint. The advantage of this modularization is that if for some reason the adjacent channel use constraints change but the co-channel reuse constraints remain the same, then only the graph theoretic computations need be repeated. This is helpful because for actual systems both the ILP and the LPs may be computationally quite intensive.

Appendix

I. Appendix

Lemma 1: Let the carried traffic in a one cell system be denoted by $C(r_n, h_n)$ when h_n channels are allocated to the cell and r_n is the offered traffic in the cell. $\lim_{n\to\infty} C(r_n, h_n)/n = \min(r, h)$ if $\lim_{n\to\infty} r_n/n = r$, $\lim_{n\to\infty} h_n/n = h$ and $h > 0$.

Proof: This proof follows easily from the ATP introduced in [4]. We give the proof here for the sake of completeness.

$$\begin{aligned}
\lim_{n\to\infty} \frac{C(r_n,h_n)}{n} &= \lim_{n\to\infty} \frac{h_n}{n} \lim_{n\to\infty} \frac{C(r_n,h_n)}{h_n} \\
&= h \lim_{h_n\to\infty} \frac{C(r_n,h_n)}{h_n} \\
&\quad (h_n \to \infty \text{ as } n \to \infty \text{ since } h > 0). \\
&= h \min(\lim_{h_n\to\infty} \tfrac{r_n}{h_n}, 1) \quad \text{by the ATP} \\
&= h \min(\tfrac{r}{h}, 1) \\
&= \min(r, h).
\end{aligned}$$

■

Lemma 2: Let $n = q(p + \sum_{j=1}^{M} m_j) + t$ channels be available where $0 \leq t < (p + \sum_{j=1}^{M} m_j)$ and q is a nonnegative integer. The fixed channel allocation algorithm which allocates $q \sum_{j=1}^{M} m_j a_{ij}$ channels to the ith cell carries a traffic intensity of $\min\left(p_i r, \frac{\sum_{j=1}^{M} m_j a_{ij}}{p+\sum_{j=1}^{M} m_j}\right)$ in the asymptotic limit ($n \to \infty$) in the ith cell.

Proof: As argued in [4], because of the independence of the offered traffic and the fixed channel allocation algorithm, each cell behaves as a one cell system with offered load $p_i r n$ and $q \sum_{j=1}^{M} m_j a_{ij}$ channels. Thus the carried traffic intensity in the ith cell is $C(p_i r n, q \sum_{j=1}^{M} m_j a_{ij})/n$. If $\sum_{j=1}^{M} m_j a_{ij} = 0$, $C(p_i r n, q \sum_{j=1}^{M} m_j a_{ij})/n = 0$, for each n. Thus $\lim_{n\to\infty} C(p_i r n, q \sum_{j=1}^{M} m_j a_{ij})/n = 0 = \min(p_i r, \sum_{j=1}^{M} m_j a_{ij}/(p+\sum_{j=1}^{M} m_j)$. Let $\sum_{j=1}^{M} m_j a_{ij} > 0$. Then $p_i r n/n \overset{n\to\infty}{\to} p_i r$ and $\lim_{n\to\infty} \frac{q}{n} = \frac{1}{(p+\sum_{j=1}^{M} m_j)}$ since $0 \leq t < (p + \sum_{j=1}^{M} m_j)$ and p, m_j, a_{ij}, M are fixed constants independent of n. Thus $\lim_{n\to\infty} \frac{q \sum_{j=1}^{M} m_j a_{ij}}{n} = \frac{\sum_{j=1}^{M} m_j a_{ij}}{p+\sum_{j=1}^{M} m_j} > 0$. Thus $\lim_{n\to\infty} C(p_i r n, q \sum_{j=1}^{M} m_j a_{ij})/n = \min(p_i r, \frac{\sum_{j=1}^{M} m_j a_{ij}}{p+\sum_{j=1}^{M} m_j})$ by Lemma 1. ■

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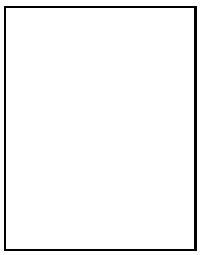

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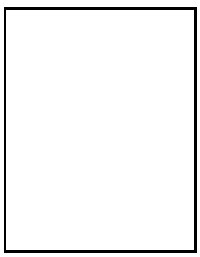

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