Throughput Guarantees Through Maximal Scheduling in Wireless Networks

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Abstract

We address the question of providing throughput guarantees through distributed scheduling, which has remained an open problem for some time. We consider a simple distributed scheduling strategy, *maximal scheduling*, and prove that it attains a guaranteed fraction of the maximum throughput region in arbitrary wireless networks. The guaranteed fraction depends on "interference degree" of the network which is the maximum number of sessions that interfere with any given session in the network and do not interfere with each other. Depending on the nature of communication, the transmission powers and the propagation models, the guaranteed fraction can be lower bounded by the maximum link degrees in the underlying topology, or even by constants that are independent of the topology. The guarantees also hold in networks with multicast communication and an arbitrary number of frequencies. We prove that the guarantees are tight in that they can not be improved any further with maximal scheduling.

I. Introduction

Maximizing the network throughput by appropriately scheduling sessions is a key design goal in wireless networks. Tassiulas *et al.* characterized the maximum attainable throughput region and also provided a scheduling strategy that attains this throughput region in any given wireless network [6]. The policy, however, is centralized and can have exponential complexity depending on the network topology considered. Later, Tassiulas [5] and Shah *et al.* [4] provided linear complexity randomized scheduling schemes that attain the maximum achievable throughput region; both scheduling strategies however require centralized control.

Designing a distributed scheduling policy that attains the throughput region in wireless networks has remained elusive. Recently, Lin *et al.* [2] proved that a distributed maximal matching scheduling strategy is guaranteed to attain at least half of this region for the node-exclusive spectrum sharing model. In the node-exclusive spectrum sharing model, the only scheduling constraint is that a node cannot communicate with multiple nodes simultaneously. This specific interference model holds only when every node has a unique frequency in its two-hop neighborhood.

Different wireless networks have significantly different interference constraints. Bluetooth networks satisfy the node-exclusive spectrum sharing model. On the other hand, IEEE 802.11 networks have limited number of frequencies that may not permit the allocation of unique frequencies in a two-hop neighborhood. Furthermore, the interference regions of nodes involved in transmissions may vary widely depending on the signal propagation conditions, and may be different for different transmitter-receiver pairs. A basic question that remains open is whether a distributed scheduling strategy can attain a guaranteed fraction of the maximum achievable throughput region for arbitrary interference models. Our investigation takes a step forward in solving this open problem.

Our contribution is to characterize the maximum throughput region attained by a distributed scheduling strategy under arbitrary topologies and interference models. The simple scheduling policy we consider, referred to as *maximal scheduling*, only ensures that if a transmitter u has a packet to transmit to a receiver v, either (u, v) or a transmitter-receiver pair that can not simultaneously transmit with (u, v) is scheduled for transmission; the scheduling is otherwise arbitrary. Our investigation of this maximal scheduling policy has been motivated by the following observations. In the specific node-exclusive spectrum sharing model,

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the maximal scheduling policy becomes the maximal matching policy considered by Lin *et. al.*, and is therefore guaranteed to attain at least half of the maximum throughput region [2]. Dai *et. al.* [1] has also obtained a similar guarantee for the maximal matching policy in input-queued switches where the scheduling constraints are similar to that in the node-exclusive spectrum sharing model. Last but not the least, the simplicity and localized nature of maximal scheduling imply that it can be readily implemented in a distributed manner with low overhead and computation cost. It is therefore interesting and important to examine whether maximal scheduling can provide any throughput guarantee under arbitrary interference models and topologies.

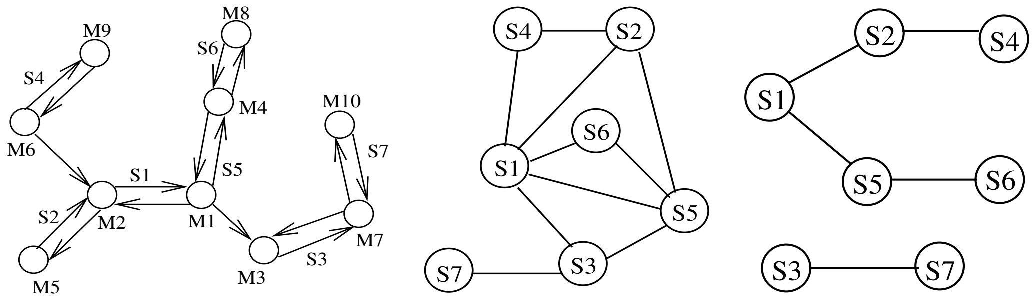
We prove that the maximum throughput region attained by maximal scheduling is significantly different for different interference models. We first consider a "bidirectional equal power" interference model in which the network has a single frequency, and all communications use the same power and involve bidirectional message exchanges (e.g., RTS, CTS, data, ACK exchanges in IEEE 802.11). Using a combination of Lyapunov theory and geometric packing, we prove that in this interference model, maximal scheduling is guaranteed to attain at least $1/8$th of the maximum throughput region. This result therefore guarantees that as in the node-exclusive spectrum sharing model, a distributed scheduling can attain a constant fraction of the maximum throughput region in this case as well. Furthermore, we show that the guarantee can not be improved any further in this case as there exists topologies for which maximal scheduling will attain at most $1/8$th of the maximum throughput region.

We next consider a "unidirectional equal power" interference model in which all communications involve unidirectional message exchanges. The network still has a single frequency and all communications use the same power. In this case, however, the performance of maximal scheduling can become arbitrarily bad. More precisely, given any constant Z, there exists topologies in which maximal scheduling will attain less than $1/Z$ of the maximum throughput region. On the other extreme, as discussed before, in the node-exclusive spectrum sharing model, maximal scheduling is guaranteed to attain at least half of the maximum throughput region [2]. We also demonstrate that in this case there exists topologies in which maximal scheduling, and hence maximal matching, will attain at most $1/2$ of the maximum throughput region.

We conclude that a slight variation in the interference constraints may significantly alter the throughput region attained by maximal scheduling (and possibly by other distributed scheduling strategies as well). We can not therefore draw conclusions about the performance of maximal scheduling for arbitrary interference constraints from the results in a few representative scenarios. Also, given that large number of interference models arise, case by case investigations may not be feasible. We therefore proceed to design a framework for characterizing the throughput region of maximal scheduling in arbitrary wireless networks.

We characterize the fraction of the maximum throughput region attained by maximal scheduling in any given topology and interference model. Let $K(\mathcal{N})$ be the maximum interference degree in an arbitrary wireless network $\mathcal{N}$, where the "interference degree" of any transmitter-receiver pair (u, v) is the maximum number of transmitter-receiver pairs that interfere with (u, v) but do not interfere with each other. We prove that maximal scheduling is guaranteed to attain at least $1/K(\mathcal{N})$ of the maximum throughput region in the given network $\mathcal{N}$. Also, there exists an arrival process in the given network $\mathcal{N}$ for which maximal scheduling will attain at most $1/K(\mathcal{N})$ of the maximum throughput region. Given a network, the maximum interference degree may be computed using geometric or graph-theoretic techniques. These results therefore allow us to obtain performance guarantees for maximal scheduling for arbitrary node locations, propagation conditions, interference models and channel allocations.

The paper is organized as follows. We describe the system model and the maximal scheduling policy in Section II. We describe some example communication and interference models in Section III. We characterize the throughput regions of maximal scheduling for some representative interference models in Section IV, and for arbitrary wireless networks with unicast sessions in Section V. In Section VI, we generalize the analytical results and the framework so as to include multicast and multichannel networks. We conclude in Section VII. We present the proofs in appendix.



(a) Network $\mathcal{N}'$ (b) (c)

Fig. 1. Panel (a) shows a directed graph with $V = \{M1, \ldots, M10\}$. The arrows between the nodes indicate the directed links. There are 7 sessions: $S1, \ldots, S7$. Nodes $M2, M5, M3, M6, M1, M8$ and $M10$ are the transmitters of sessions $S1, S2, S3, S4, S5, S6$ and $S7$, respectively. Node $M2$ has 3 neighbors: $M1$, $M5$, $M6$. Nodes $M1$ and $M2$ have degree 5; hence the degrees of edges $(M1, M2)$ and $(M2, M1)$ are 10. Here, $\delta_G = 10$. Both the out-degree of $M1$ and in-degree of $M2$ are 3. Thus, the directed degree of $(M1, M2)$ is 6. Here, $\Delta_G = 6$. Sessions $S5$ and $S6$ interfere with each other, as $M4$ has a single transceiver.
Panels (b) and (c) show the interference graphs for the network shown in (a) under bidirectional and unidirectional communication models, respectively. As panels (b) and (c) show, the interference sets of $S6$ are $\{S1, S5\}$ and $\{S5\}$ under the bidirectional and unidirectional communication models, respectively.

II. System Model

We consider scheduling at the MAC layer in a wireless network. We assume that time is slotted. The topology in a wireless network can be modeled as a directed graph $G = (V, E)$, where V and E respectively denote the sets of nodes and links. A link exists from a node u to another node v if and only if v can receive u's signals. The link set E depends on the transmission power levels of nodes and the propagation conditions in different directions.

We now introduce terminologies that we use throughout the paper. Some of these are well-known in graph theory; we mention these for completeness.

Definition 1: A node i is a *neighbor* of a node j, if there exists a link from i to j, i.e., $(i, j) \in E$.

The *degree of a node* u is the number of links in E originating from or ending at u. The *degree of a link* $e = (u, v)$ is defined as the sum of the degrees of u and v. The *maximum link degree* in G, δ_G, is the maximum degree of any link in E.

The *out-degree* of a node u is the number of links in E originating from u. The *in-degree* of a node u is the number of links in E ending at u. The *directed degree of a link* $e = (u, v)$ is defined as the sum of the out-degree of u and in-degree of v. The *maximum directed link degree* in G, Δ_G, is the maximum directed degree of any link in E.

At the MAC layer, each session traverses only one link. If a session i traverses link (u, v) then u and v are i's transmitter and receiver respectively, and the session is completely specified by the $3-$tuple, (i, u, v). Multiple sessions may traverse the same link. Without loss of generality, we assume that every node in V is either the transmitter or the receiver of at least one session. If this assumption does not hold, we can consider G to be a subgraph obtained from the original topology by removing the nodes that are not the end points of sessions.

Definition 2: A session i *interferes* with session j if j can not successfully transmit a packet when i is transmitting.

In Section III, we will describe broad classes of communication and interference models and how to obtain the pairwise interference relations in each case.

A wireless network $\mathcal{N}$ can be described by the topology $G = (V, E)$, the $3-$tuple specifications of the sessions and the pair-wise interference relations between the sessions. We consider a network with N sessions.

Definition 3: The *interference set* of a session i, S_i, is the set of sessions j such that either i interferes with j or j interferes with i.
Note that if $j \in S_i$, then $i \in S_j$.

Definition 4: The *interference graph* $I^{\mathcal{N}} = (V_I^{\mathcal{N}}, E_I^{\mathcal{N}})$ of a network $\mathcal{N}$ is an undirected graph in which the vertex set $V_I^{\mathcal{N}}$ corresponds to the set of sessions in $\mathcal{N}$ and there is an edge between two vertices i and j if $j \in S_i$.

We elucidate these definitions through examples in Fig. 1.

We now describe the arrival process. Let $A_i(n)$ be the number of packets that session i generates in interval $(0, n)$, $i = 1, \ldots, N$. We assume that any packet arriving in a slot arrives at the beginning of the slot, and may be transmitted in the slot. The arrival process $\{A_i(.), i = 1, \ldots, N\}$ satisfies a strong law of large numbers (SLLN). Thus, there exists non-negative real numbers $\lambda_i, i = 1, \ldots, N$ such that with probability 1,

$$\lim_{n\to\infty} A_i(n)/n = \lambda_i,\ \ i = 1, \ldots, N. \tag{1}$$

The condition (1) on the arrival processes is mild. Several arrival processes including all jointly stationary and ergodic arrival processes satisfy (1).

Definition 5: The *arrival rate* of session i is λ_i, $i = 1, \ldots, N$. The *arrival rate vector* $\vec{\lambda}$ is an $N-$dimensional vector whose components are the arrival rates.

Definition 6: A *scheduling policy* is an algorithm that decides in each slot the subset of sessions that would transmit packets in the slot.
Clearly, a subset S of sessions can transmit packets in any slot if no two sessions in S interfere with each other and every session in S has a packet to transmit. Every packet has length 1 slot. Thus, if a session is scheduled in a slot, it transmits a packet in the slot.

Let $D_i(n)$ be the number of packets that session i transmits in interval $(0, n)$, $i = 1, \ldots, N$. Clearly the transmissions depend on the scheduling policy.

Definition 7: The network is said to be *stable* if with probability 1,

$$\lim_{n\to\infty} D_i(n)/n = \lambda_i,\ \ i = 1, \ldots, N. \tag{2}$$

Thus, a network is stable if the arrival and departures rates are equal for each session.

Definition 8: The *throughput region* of a scheduling policy is the set of arrival rate vectors $\vec{\lambda}$ such that the network is stable under the policy for any arrival process that satisfies (1) and has arrival rate vector $\vec{\lambda}$.

Definition 9: An arrival rate vector $\vec{\lambda}$ is said to be *feasible* if it is in the throughput region of some scheduling policy.

Definition 10: The *maximum throughput region* Λ is the set of feasible arrival rate vectors.
Note, Λ depends on the network $\mathcal{N}$.

Example 1: Consider the network shown in Fig. 2(a). Consider a scheduling policy π_1, that serves session $t \bmod 9 + 1$ in slot t, where "mod" is a modulo operator. Under π_1, each session $i \in \{1, \ldots, 9\}$ can transmit at the rate of at most $1/9$. Thus, the throughput region of π_1, Λ^{π_1}, is characterized as follows:

$$\Lambda^{\pi_1} = \{(\lambda_1, \ldots, \lambda_9)\ :\ \lambda_i \le 1/9\ \forall i\}.$$

In this case, the maximum throughput region Λ is given by

$$\Lambda = \left\{(\lambda_1, \ldots, \lambda_9)\ :\ \lambda_1 + \max_{2\le i\le 9}\{\lambda_i\} \le 1\right\}.$$

Therefore, in this example, scheduling policy π_1 achieves only a small fraction of the maximum throughput region.

We now describe the "maximal scheduling" policy we consider. This policy schedules a subset S of sessions such that (i) every session in S has a packet to transmit, (ii) no session in S interferes with any other session in S, (iii) if a session i has a packet to transmit, then either i or a session in

S_i, is included in S. Clearly, many subsets of sessions satisfy the above criteria in each slot, e.g., in Fig. 1(b), $\{S1, S7\}, \{S2, S3, S6\}$ satisfy the above criteria in any slot in which all sessions have packets to transmit. Maximal scheduling can select any such subset. If each session knows its interference set, maximal scheduling can be implemented in distributed manner using standard algorithms [3]. In most cases of practical interest, sessions can determine their interference sets using local message exchange.

III. Interference Models

The pairwise interference relations between the sessions depend on topology $G = (V, E)$ and the nature of communication. The topology G is determined by the transmission powers, propagation conditions and node locations. Communication can either be bidirectional or unidirectional. In the former, when a session is scheduled, both the transmitter and the receiver transmit sequentially. For example, the transmitter may transmit data and control messages while the receiver may transmit control messages. Such bidirectional communications occur in IEEE 802.11. Thus, there must be links in both directions between a session's transmitter and receiver. In unidirectional communication, when a session is scheduled, it transmits packets from only the transmitter to the receiver. For example, unidirectional communication occurs in IEEE 802.11 when control messages are disabled (e.g., in broadcast mode). Different combinations of these conditions lead to different interference relations. We next characterize the pair-wise interference relations for some of these combinations.

We assume that each node has a single transceiver. Thus a node can be involved in at most one transmission. In other words, sessions that have a node in common interfere with each other. We initially assume that all transmissions use the same frequency. Thus, node j can not receive any packet successfully if more than one of its neighbors are transmitting simultaneously (we do not assume any capture). Thus, a transmission on link $(i, j) \in E$ is successful in a slot if and only if no neighbor of j other than i transmits in the slot. For example, in Fig. 1(a), transmission along $(M5, M2)$ is successful if $M1$ and $M6$ do not transmit. For bidirectional communication, when a session (i, u, v) is scheduled, transmissions proceed along both (u, v) and (v, u). For unidirectional communication, when a session (i, u, v) is scheduled, transmissions proceed only along (u, v). The above constraints provide the interference relations for both the bidirectional and unidirectional communication models.

In the *bidirectional communication model*, a session i interferes with session j if i and j have a common end point, or one end point (transmitter or receiver) of j is a neighbor of an end point of i. For example, in Fig. 1(a), $S1, S5, S7$ interfere with $S3$. This is also clearly evident from Fig. 1(b). In the *unidirectional communication model*, session i interferes with session j if i and j have a common end point, or j's receiver is a neighbor of i's transmitter. For example, in Fig. 1(a), only $S7$ interferes with $S3$. Observe that the interference relations may be asymmetric, i.e., i may interfere with j but j may not interfere with i. For example, under bidirectional communication model, in Fig. 1(a), $S1$ interferes with $S3$ but $S3$ does not interfere with $S1$.

We now describe several important special cases. First assume that the propagation conditions are identical in all directions. Each node transmits at a fixed power level which can be different for different nodes. The power level of a node u determines its transmission range, and all nodes within u's transmission range receive u's signal. Thus, the link set E has the following structure: a link exists from u to v if and only if the distance between u and v is less than or equal to u's transmission range. In the bidirectional communication model, session i interferes with session j if one end point of j is within the transmission range of an end point of i. In the unidirectional communication model, session i interferes with session j if j's receiver is within the transmission range of i's transmitter.

Let us further assume that all nodes transmit at the same power. Thus, all nodes have the same transmission range d which is determined by the transmission power. Now, the link set E has the following structure: a link exists from u to v if and only if the distance between u and v is less than d. Now, in the bidirectional communication model, a session i interferes with session j if one end point of j is within distance d from an end point of i (*bidirectional equal power model*). In the unidirectional interference model, a session i interferes with session j if j's receiver is within distance d from i's transmitter (*unidirectional equal power model*). Refer to Fig. 2(a) and (b) for examples of both cases. Note that now the interference relation is symmetric in the bidirectional communication model, i.e., if node i interferes with node j, then node j also interferes with node i. However, interference relationships could still be asymmetric in the unidirectional communication model.

We also consider multi-channel networks. We first assume that the network has a large number of frequencies such that every node has a unique frequency in its two-hop neighborhood. Now, for both bidirectional and unidirectional communications, only the sessions that have common end point interfere. This model arises in Bluetooth communications, and is commonly referred to as the *node-exclusive spectrum sharing model* (Fig. 3). In Section VI, we also discuss the more general case where the network has an arbitrary number of frequencies.

We observe that the pairwise interference relations are significantly different in each of the cases discussed above. There is however one important similarity. If session i interferes with another session j, the distance between the transmitters of i and j is at most three hops. Thus, a session can use local message exchange to determine its interference set. Hence, maximal scheduling can be implemented in distributed manner in each of these cases. But, given the significant difference between the interference relations, it is not clear how similar the performance of maximal scheduling will be in these different cases. In the next section, we assess this difference by characterizing the throughput region of maximal scheduling in a few representative scenarios.

IV. Performance of Maximal Scheduling for Specific Interference Models

We characterize the throughput regions of maximal scheduling Λ^{MS} for some representative interference models. We focus on the bidirectional equal power (Subsection IV-A) and unidirectional equal power models (Subsection IV-B). We subsequently compare the throughput regions obtained in these cases with that in the well-investigated node-exclusive spectrum sharing model (Subsection IV-C). We conclude that the throughput regions are significantly different in different cases.

A. Throughput region of maximal scheduling for bidirectional equal power model

Lemmas 1 and 2 show that in the bidirectional equal power model, Λ^{MS}, is $1/8$th of the maximum throughput region Λ.

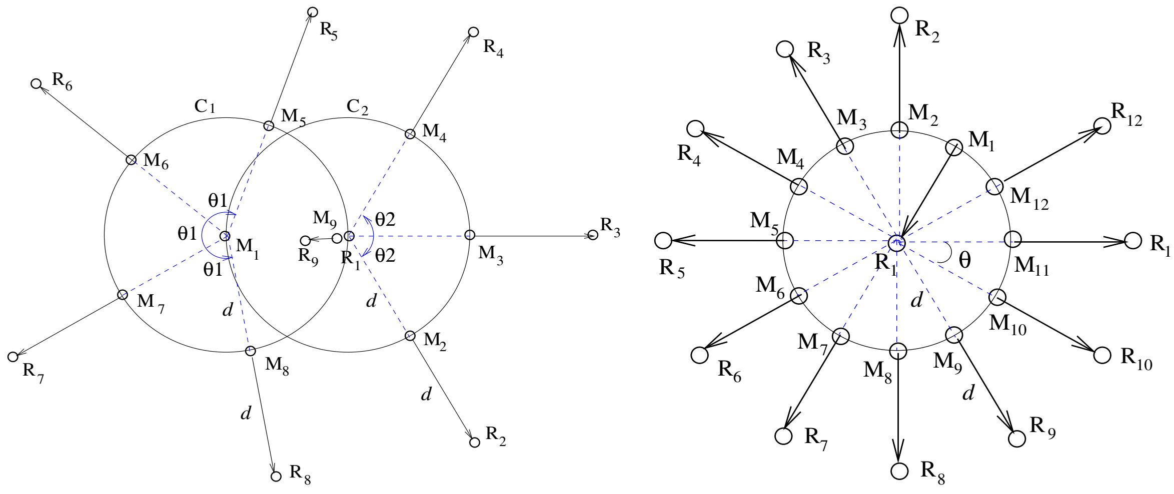
Lemma 1: For the bidirectional equal power model, if $\vec{\lambda} \in \Lambda$, $\vec{\lambda}/8 \in \Lambda^{MS}$.

Proof: Proof is presented in Appendix III. ■

We describe the intuition behind the result. Let arrival rate vector $\vec{\lambda} \in \Lambda$. Then, from (2), under some scheduling policy the packet arrival rate λ_j for each session j equals j's departure rate. Thus, for each session i, the sum of its arrival rate and the arrival rates of the sessions in its interference set S_i must equal the sum of the corresponding departure rates.

In each slot, either i or one or more sessions in S_i may transmit packets, but i can not simultaneously transmit with any session in S_i. We prove using geometry that for any i at most 8 sessions in S_i can simultaneously transmit packets (refer to Appendix III). Thus in any slot at most 8 packets can be transmitted by sessions in $\{i\} \cup S_i$. Thus, the sum of the departure rates of sessions in $\{i\} \cup S_i$, and hence the sum of the corresponding arrival rates, is at most 8. Thus, when the arrival rate vector is $\vec{\lambda}/8$ instead of $\vec{\lambda}$, the sum of the arrival rates of sessions in $\{i\} \cup S_i$ is at most 1.

Let the arrival rate vector be $\vec{\lambda}/8$, and let maximal scheduling be used. For any session i, maximal scheduling always serves 1 packet from either i or a session in S_i in any slot in which i has a packet to transmit. Thus, whenever i has a packet to transmit, the sum of the departure rates for these sessions is 1, which is greater than or equal to the sum of the arrival rates of these sessions. Now, since the departure rate of any session j cannot exceed j's arrival rate, for all i, the sum of the departure rates from the sessions in $\{i\} \cup S_i$ equals the sum of the corresponding arrival rates. It follows that the departure rate of each session i equals i's arrival rate. Thus, the system is stable. Hence $\vec{\lambda}/8 \in \Lambda^{\text{MS}}$.



(a) (b)

Fig. 2. Fig. (a) shows a network with interference constraints given by the bidirectional equal power model and transmission range d. There are 9 sessions: $1, \ldots, 9$. Session i has transmitter M_i and receiver R_i. The interference area of session 1 is the union of circles C_1 and C_2. Here, $\theta_1 = 70$ deg, and $\theta_2 = 61$ deg. Distance between (i) M_i and R_i is d for every $i = 1, \ldots, 8$, (ii) M_9 and R_9 is $\epsilon > 0$, where ϵ is a small positive number, (iii) M_1 and M_i is d for every $i = 2, \ldots, 9$, (ii) M_j and M_k is greater than d for every $j, k \in \{2, \ldots, 9\}, j \neq k$ and (iv) M_9 and R_1 is ϵ. *Thus, session 1 interferes with all the other* 8 *sessions, but none of the other sessions interfere with each other.* Fig. (b) shows a network with interference constraints given by the unidirectional equal power model and transmission range d. There are 12 sessions: $1, \ldots, 12$. Session i has transmitter M_i and receiver R_i. The distance between M_i and R_i, and R_1 and M_i is d for every i. Thus, session 1 interferes will all the other 11 sessions, but none of the other sessions interfere with each other. We refer to sessions $2, \ldots, 12$ as non-interfering sessions. Here, θ is $\pi/6$. Note that $2\pi/\theta - 1$ non-interfering sessions can be accommodated. Thus, for any given Z, $Z + 1$ non-interfering sessions can be accommodated by choosing $\theta = 2\pi/(Z + 2)$.

We now describe why for any i at most 8 sessions in S_i can simultaneously transmit packets. From the interference constraints, at least one end point of each session in S_i must be within a distance d from either i's transmitter or i's receiver. Also, the distance between i's transmitter and receiver is at most d. Thus, at least one end point of each session in S_i must be in the union of two circles of radius d and centered around i's transmitter and receiver respectively (Fig. 2(a)). We refer to the area in this union as i's *interference area*. We prove using geometric arguments that at most 8 points can be present in this interference area such that the distance between any two points exceeds d. Clearly, if sessions j and k need to simultaneously transmit packets, the distance between an end point of j and an end point of k must exceed d. The result follows.

Lemma 2: Consider an arbitrary positive constant Z such that $Z < 8$. For the bidirectional equal power model, there exists a network $\mathcal{N}$ and an arrival rate vector $\vec{\lambda}$, such that $\vec{\lambda} \in \Lambda$ in $\mathcal{N}$, but $\vec{\lambda}/Z \notin \Lambda^{MS}$ in $\mathcal{N}$.

Proof: Proof is presented in Appendix IV-A. ■

We present the intuition behind the result. Using geometry, we first demonstrate that it is possible to obtain a network with 9 sessions where one session (session 1) interferes with all other sessions and none of the other sessions interfere with each other (Fig. 2(a)). In such a network, consider an arrival rate vector $\vec{\lambda}$ in which session 1 generates packets at the rate λ_1 and session i generates packets at the rate λ_2 for all $i \in \{2, \ldots 9\}$. Consider λ_1, λ_2 such that $\lambda_1 + \lambda_2 = 1$, $\lambda_2 = Z/8$. The system can be stabilized by scheduling session 1 in λ_1 fraction of slots and the other sessions in the remaining slots. Thus, $\lambda \in \Lambda$. Now consider arrival rate vector $\vec{\lambda}/Z$. Let session 1 generate packets at the rate λ_1/Z and session i generate packets at the rate λ_2/Z for all $i \in \{2, \ldots 9\}$. Furthermore, sessions $2, 3, \ldots 9$ generate packets in non-overlapping slots (Fig. 4). Thus, since $\lambda_2/Z = 1/8$, in each slot at least one session in $\{2, \ldots 9\}$ generates packets, and therefore has a packet to transmit. Let maximal scheduling schedule session 1 only when other sessions do not have packets. But then session 1 is never scheduled. Since session 1's arrival

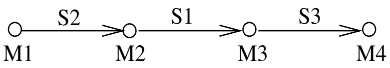


Fig. 3. Figure shows a network with 4 nodes $M1, \ldots, M4$ and 3 sessions $S1, S2$ and $S3$. Under node exclusive spectrum sharing model, $S1$ interferes with both $S2, S3$, but $S2$ and $S3$ do not interfere with each other.

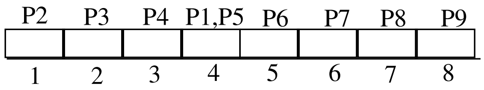


Fig. 4. Figure shows a periodic arrival process for the network in Fig. 2(a). The period is 8 slots. Session S_i generates a packet in the slot marked Pi, for each i. Here, $\lambda_i = 1/8$, for each i.

rate is positive, the system is not stable under maximal scheduling. Thus, $\vec{\lambda}/Z \notin \Lambda^{\text{MS}}$.

B. Throughput region of maximal scheduling for unidirectional equal power model

We now consider the unidirectional equal power model. We prove that maximal scheduling can not attain a constant fraction of the maximum throughput region.

Lemma 3: Consider an arbitrary positive constant Z. For the unidirectional equal power model, there exists a network $\mathcal{N}$, an arrival rate vector $\vec{\lambda}$, such that $\vec{\lambda} \in \Lambda$ in $\mathcal{N}$, but $\vec{\lambda}/Z \notin \Lambda^{MS}$ in $\mathcal{N}$.

Proof: Proof is presented in Appendix IV-B. ■

We present the intuition behind the result. We could obtain the throughput guarantee of $1/8$ in the bidirectional equal power model irrespective of the network primarily because for this model in any network the interference set of any session consists of at most 8 sessions that can transmit simultaneously. This no longer holds for the unidirectional equal power model. In fact, for unidirectional equal power model given any constant Z we can construct a network where the interference set of a session consists of $\lceil Z+1 \rceil$ sessions that can transmit simultaneously (Fig. 2(b)). We can prove that in such a network there exists an arrival rate vector $\vec{\lambda}$, such that $\vec{\lambda} \in \Lambda$, but $\vec{\lambda}/Z \notin \Lambda^{\text{MS}}$. The intuition behind the proof of this part is similar to that for Lemma 2.

C. Throughput region of maximal scheduling for node-exclusive spectrum sharing model

The throughput regions of maximal scheduling are significantly different for the bidirectional and unidirectional power models. We next mention the guarantees for a third interference model, the node-exclusive spectrum sharing model, and then contrast the guarantees in the three cases. We need the following concepts that are well-known in graph theory. Consider a graph $G' = (V, E')$ where E' consists of only those links in E that are traversed by sessions

Definition 11: A *matching* in G' is a set of links such that no two links have a common node.

Definition 12: A *maximal matching* is a matching in G' such that for any link e that is traversed by a session that has a packet to transmit, either e is in the matching or a link that has a common node with e is in the matching.

In the node-exclusive spectrum sharing model, maximal scheduling always selects sessions that constitute a maximal matching in G'. This follows from the definition of maximal scheduling and because of the pair-wise interference relations in the node-exclusive spectrum sharing model. Lin *et. al.* [2] has proved that maximal matching attains at least $1/2$ the maximum throughput region in the node-exclusive spectrum sharing model. Thus, maximal scheduling also attains at least $1/2$ the maximum throughput region. We would like to remark that in this model in any network the interference set of any session consists of at most 2 sessions; therefore, in this case too, the throughput guarantee seems to be related to this quantity.

We next prove that there exists networks where maximal scheduling attains at most $1/2$ the maximum throughput region.

Lemma 4: Consider an arbitrary positive constant Z such that $Z < 2$. For the node-exclusive spectrum sharing model, there exists a network and an arrival rate vector $\vec{\lambda}$, such that $\vec{\lambda} \in \Lambda$ in $\mathcal{N}$, but $\vec{\lambda}/Z \notin \Lambda^{MS}$ in $\mathcal{N}$.

Proof: Proof is presented in Appendix IV-C. ■

We present the intuition behind this result. We construct a network with 3 sessions where one session (session 1) interferes with all other sessions and none of the other sessions interfere with each other (Fig. 3). Like for Lemma 2, we can prove that in such a network there exists an arrival rate vector $\vec{\lambda}$, such that $\vec{\lambda} \in \Lambda$, but $\vec{\lambda}/Z \notin \Lambda^{\text{MS}}$.

The throughput regions for the maximal scheduling in the node-exclusive spectrum sharing model are again significantly different from those in the bidirectional and unidirectional equal power models. We conclude that these guarantees will critically depend on the interference relations. Furthermore, the differences between the characterizations obtained for the bidirectional and the unidirectional interference models demonstrate that slight changes in interference conditions can significantly alter the guarantees. We can not therefore draw conclusions about the performance under different models from the results in a few representative scenarios. Also, given that large number of interference relations exist, case by case investigations may not be feasible. We therefore need a framework for characterizing the throughput region of maximal scheduling in arbitrary wireless networks.

V. Performance guarantees of maximal scheduling in arbitrary networks

We design a framework for characterizing the throughput region of maximal scheduling Λ^{MS} for an arbitrary wireless network.

We first introduce the notion of "interference degree" for sessions.

Definition 13: The *interference degree of a session* i is (i) the maximum number of sessions in its interference set S_i that can simultaneously transmit, if S_i is non-empty and (ii) 1 if S_i is empty.

The interference degrees depend on the links traversed by the sessions and the topology $G = (V, E)$ which in turn depends on the node locations, propagation conditions and interference models. For example, in Fig. 1(b), $S_{S1} = \{S2, S3, S4, S5, S6\}$, and the largest set of sessions in S_{S1} that can simultaneously transmit is $\{S3, S4, S6\}$. Thus, the interference degree of $S1$ is 3.

The characterizations of Λ^{MS} obtained so far for specific interference models are closely related to the maximum interference degrees of sessions under these models. For example, for both the bidirectional equal power and the node-exclusive spectrum sharing models maximal scheduling attains exactly $1/P$ fraction of the maximum throughput region Λ, where P is the maximum interference degree of any session. In the unidirectional equal power model, we observed that maximal scheduling can not attain a constant fraction of Λ and also that a session can have arbitrarily large interference degree. We now prove that this relation is not a coincidence but reflects a general phenomenon that holds for arbitrary networks.

We first define the interference degree of a network.

Definition 14: The *interference degree of a network* $\mathcal{N}$, $K(\mathcal{N})$, is the maximum interference degree of sessions in the network.

In Fig. 1(b) and (c), the interference degrees of the network are 3 and 2 respectively. Session $S1$ has these interference degrees in both cases.

Theorem 1: In any wireless network $\mathcal{N}$, if $\vec{\lambda} \in \Lambda$ in $\mathcal{N}$, $\vec{\lambda}/K(\mathcal{N}) \in \Lambda^{MS}$ in $\mathcal{N}$.

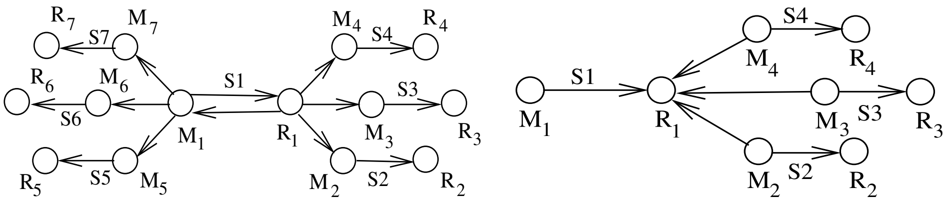
Proof: Proof is presented in Appendix I. ■

Theorem 2: Consider an arbitrary wireless network $\mathcal{N}$ and a constant Z such that $Z < K(\mathcal{N})$. There exists an arrival rate vector $\vec{\lambda}$ such that $\vec{\lambda} \in \Lambda$ in $\mathcal{N}$, but $\vec{\lambda}/Z \notin \Lambda^{MS}$ in $\mathcal{N}$.

Proof: Proof is presented in Appendix II. ■

The intuition behind Theorems 1 and 2 are similar to that for Lemmas 1 and 2 respectively. The generalization here is that we obtain the characterizations in terms of $K(\mathcal{N})$ because for any session i at most $K(\mathcal{N})$ sessions in S_i can transmit simultaneously.

Theorems 1 and 2 allow us to characterize Λ^{MS} under arbitrary interference relations, node locations, edge sets, session configurations and propagation models, provided we can compute $K(\mathcal{N})$ in these cases. We now obtain upper bounds for $K(\mathcal{N})$ for arbitrary bidirectional and unidirectional communications models, in terms of the maximum link degrees δ_G and Δ_G in the underlying topology G. We also prove that the bounds are tight as there exists networks $\mathcal{N}$ where $K(\mathcal{N})$ equals these bounds. These bounds and



(a) Network $\mathcal{N}_1'$ (b) Network $\mathcal{N}_2'$

Fig. 5. Fig. (a) shows a network $\mathcal{N}_1'$ with bidirectional communication model and 7 sessions: $(S1, M_1, R_1), \ldots, (S7, M_7, R_7)$. Session $S1$ interferes with all the remaining sessions, and none of the remaining sessions interferes with each other. Thus, $K(\mathcal{N}_1') = 6$. The degree of (M_1, R_1) is 10, which is also equal to δ_G. Thus, $K(\mathcal{N}_1') = \delta_G - 4 = \max(\delta_G - 4, 1)$.
Fig. (b) shows a network $\mathcal{N}_2'$ with unidirectional communication model and four sessions: $(S1, M_1, R_1), \ldots, (S4, M_4, R_4)$. Sessions $S2, S3$ and $S4$ interfere with $S1$, but not with each other. Thus, $K(\mathcal{N}_2') = 3$. The directed degree of (M_1, R_1) is 5, which is also equal to Δ_G. Thus, $K(\mathcal{N}_2') = \Delta_G - 2 = \max(\Delta_G - 2, 1)$. In both figures, arrows indicate directed links between the nodes.

the resulting characterizations of Λ^{MS} hold even when different nodes use different transmission powers and propagation conditions in different directions are different.

Lemma 5: In a wireless network $\mathcal{N}$ with bidirectional communication and underlying topology $G = (V, E)$, $K(\mathcal{N}) \leq \max(\delta_G - 4, 1)$. Moreover, there exists a wireless network $\mathcal{N}_1$ with bidirectional communication and underlying topology $G = (V, E)$, such that $K(\mathcal{N}_1) = \max(\delta_G - 4, 1)$.

Proof: Proof is presented in Appendix IV-D. ■

The upper-bound in Lemma 5 follows because for bidirectional communication the interference-degree of each session (i, u, v) is at most the degree of link (u, v) minus 4, and the degree of any link in G is at most δ_G. Fig. 5(a) shows an example network $\mathcal{N}_1'$ where $K(\mathcal{N}_1') = \delta_G - 4$; the bound is therefore tight.

Lemma 6: In a wireless network $\mathcal{N}$ with unidirectional communication and underlying topology $G = (V, E)$, $K(\mathcal{N}) \leq \max(\Delta_G - 2, 1)$. Moreover, there exists a wireless network $\mathcal{N}_1$ with unidirectional communication and underlying topology $G = (V, E)$, such that $K(\mathcal{N}_1) = \max(\Delta_G - 2, 1)$.

Proof: Proof is presented in Appendix IV-E. ■

Lemmas 5 and 6 are similar. Due to unidirectional communication, the bound in Lemma 6 however depends on Δ_G instead of δ_G. Fig. 5(b) provides an example to illustrate the tightness.

Theorems 1 and 2 explain the characterizations of Λ^{MS} for all the specific interference models considered so far. For the bidirectional equal power model, for any network $\mathcal{N}$, $K(\mathcal{N}) \leq 8$ (refer to Appendix III), and there exists a network $\mathcal{N}_1$ where $K(\mathcal{N}_1) = 8$ (Fig. 2(a)). Thus, Lemmas 1 and 2 follow as special cases of Theorems 1 and 2. For the unidirectional equal power model, given any Z a network can be constructed so as to attain the interference degree $Z + 1$ (Fig. 2(b)). Theorem 2 now explains Lemma 3. Theorem 1 also explains the throughput characterization for the maximal matching policy in the node-exclusive spectrum sharing model obtained by Lin *et. al* [2]. In this model, for any network $\mathcal{N}$, $K(\mathcal{N}) \leq 2$. Also, there exists a network $\mathcal{N}_1$ with $K(\mathcal{N}_1) = 2$ (Fig. 3). Thus, the throughput guarantee of $1/2$ obtained in this case follows as a special case of Theorem 1, and Lemma 4 follows as a special case of Theorem 2.

The characterizations of Λ^{MS} for specific interference models are often obtained for the worst network under the interference model. This observation applies to all results obtained in Section IV and also the guarantees obtained by Lin *et. al* [2]. Theorems 1 and 2 allow the guarantees to cater to specific networks, and therefore often provide better guarantees. For example, Lemma 3 states that for the unidirectional equal power model, given a constant, there exists topologies where the throughput region of maximal scheduling is less than that constant fraction of the maximum throughput region. But, Lemma 6[*] shows that even in this model maximal scheduling attains a guaranteed fraction of the maximum throughput region; Lemma 6 shows that the guarantee however depends on the degrees in the underlying topology G.

[*]Note that Lemma 6 holds for all unidirectional communication models and hence for the unidirectional equal power model.

Although in the worst case, these degrees can be arbitrarily large (and therefore the guaranteed fraction can not be lower bounded by a constant in the worst case), these degrees are usually small. Thus, for several topologies Lemma 6 guarantees an acceptable performance even for the unidirectional equal power model. Similarly, for the bidirectional equal power model, whenever $\delta_G < 12$, Lemma 5 guarantees that $K(\mathcal{N}) \leq 7$, and Theorem 1 provides a throughput guarantee that is better than the lower bound of $1/8$ in Lemma 1.

VI. Generalizations

We now generalize the analytical results and the framework so as to consider some additional scenarios of practical interest.

First, the characterizations of Λ^{MS} obtained so far demonstrate that maximal scheduling does not attain the maximum throughput region of a network. This is clearly expected as maximal scheduling uses only local information and the maximum throughput region has so far only been obtained by centralized scheduling policies [6], [5]. The contribution of these results is to characterize the penalty due to the use of such limited information, and provide tight "uniform" bounds on the penalty in the arbitrary networks. The bounds are "uniform" because they uniformly apply to all sessions. In subsection VI-A, we generalize Theorems 1 and 2 to obtain better throughput guarantees for specific sessions by allowing different bounds for different sessions (Lemma 7).

We next generalize the framework to characterize Λ^{MS} for some additional scenarios of practical interest. In subsection VI-B we consider a network with multicast sessions. In subsection VI-C, we consider a network with multiple (M) frequencies. Here, M may not be so large that every node can be allocated a frequency that is unique in its 2-hop neighborhood, and thus the node exclusive spectrum sharing model may not apply. We demonstrate that the overall framework may easily be extended to consider both cases, and the uniform bounds in Theorems 1 and 2 and the non-uniform bound in Lemma 7 hold.

A. Nonuniform Bounds

We now describe how we obtain different performance guarantees for different sessions. In Theorems 1 and 2, we have proved that in an arbitrary network $\mathcal{N}$, due to the use of maximal scheduling, the arrival rate that can be accommodated for each session reduces by at most $K(\mathcal{N})$, and the arrival rate that can be accommodated for at least one session reduces by at least $K(\mathcal{N})$. This uniform bound of a factor of $1/K(\mathcal{N})$ is obtained considering the worst session, and it is possible that for most sessions the penalty is less. The question now is whether it is possible to obtain better non-uniform bounds by considering the constraints of individual sessions. We need to introduce an additional concept to address this question.

The *interference component degree* of a session i in network $\mathcal{N}$, $K(\mathcal{N})_i$ is the maximum interference degree of a session in the component of the interference graph $I^{\mathcal{N}} = (V_I^{\mathcal{N}}, E_I^{\mathcal{N}})$ of $\mathcal{N}$ that contains i. For example, in Fig. 1(c), $K(\mathcal{N}')_{S1} = 2 = K(\mathcal{N}')$, and $K(\mathcal{N}')_{S3} = 1$. The following lemma now generalizes Theorems 1 and 2.

Lemma 7: In any wireless network $\mathcal{N}$, if $(\lambda_1, \ldots, \lambda_N) \in \Lambda$ in $\mathcal{N}$, $(\lambda_1/K(\mathcal{N})_1, \ldots, \lambda_N/K(\mathcal{N})_N) \in \Lambda^{MS}$ in $\mathcal{N}$.

Consider positive constants $Z_1, \ldots Z_N$ such that $Z_i < K(\mathcal{N})_i$ for some session i. There exists an arrival rate vector $(\lambda_1, \ldots, \lambda_N)$ such that $(\lambda_1, \ldots, \lambda_N) \in \Lambda$ in $\mathcal{N}$, but $(\lambda_1/K(\mathcal{N})_1, \ldots, \lambda_N/K(\mathcal{N})_N) \notin \Lambda^{MS}$ in $\mathcal{N}$.

Proof: Proof is presented in Appendix IV-F. ■

In an arbitrary network $\mathcal{N}$, Lemma 7 characterizes the penalties by considering the worst session (i.e., the session with the maximum interference degree) in each component of the interference graph $I^{\mathcal{N}}$. Thus, due to the use of maximal scheduling, the arrival rate that can be accommodated for each session reduces by at most its interference component degree $K(\mathcal{N})_i$. Also, the arrival rate that can be accommodated for at least one session i in each component of the interference graph $I^{\mathcal{N}}$ reduces by at least $K(\mathcal{N})_i$. Thus, the bounds are now uniform in each component of $I^{\mathcal{N}}$, but different for sessions in different components.

Note that when the interference graph has multiple components, then each of the components can be treated as a separate network. Thus, Lemma 7 is a straight forward extension of Theorems 1 and 2. We now investigate whether better non-uniform bounds can be obtained for each session in terms of its interference degree rather than its interference component degree. In the following lemma we show that it is indeed possible to obtain a better non-uniform bound by considering the interference degrees of a session and of its neighbors in the interference graph.

Lemma 8: Let $k_i(\mathcal{N})$ denote the interference degree of session i in the network $\mathcal{N}$. Let us define $\beta_i(\mathcal{N})$ as follows.

$$\beta_i(\mathcal{N}) \stackrel{\text{def}}{=} \max_{j \in S_i \cup \{i\}} \{k_j(\mathcal{N})\}. \tag{3}$$

Then, if $\vec{\lambda} = [\lambda_1 \ \cdots \ \lambda_N]$ belongs to Λ, then $\frac{\vec{\lambda}}{\vec{\beta}} = [\frac{\lambda_1}{\beta_1} \ \cdots \ \frac{\lambda_N}{\beta_N}]$ belongs to Λ^{MS}.

Proof: Proof is presented in Appendix V-A. ∎

The non-uniform bound obtained above takes into account the interference degree of a session i and the sessions that interfere with i. In the following lemma, we show that this non-uniform bound is tight in a sense that non-uniform bounds can not be obtained in terms of interference degrees of individual sessions alone.

Lemma 9: Let $k_i(\mathcal{N})$ denote the interference degree of session i in network $\mathcal{N}$. Then, there exists a network $\mathcal{N}$ and an arrival rate vector $\vec{\lambda} = (\lambda_1, \ldots, \lambda_N)$ such that $\vec{\lambda} \in \Lambda$, but $(\lambda_1/k_1(\mathcal{N}), \ldots, \lambda_N/k_N(\mathcal{N})) \notin \Lambda^{MS}$.

Proof: Proof is presented in Appendix V-B ∎

B. Multicast Networks

We now generalize the framework to support multicast (one-to-many) communications. Each multicast session has one sender and one or more receivers, and therefore has two or more end points. Thus, unicast sessions (which we considered so far) are special cases of multicast.

A session i has transmitter u, G_i receivers $(v_1, \ldots, v_{G_i})$ and is completely specified by $(i, u, v_1, \ldots, v_{G_i})$. For the bidirectional communication model, the description of the pairwise interference relations remain the same as in the unicast case. For the unidirectional communication models, the description must be generalized as follows: session i interferes with session j if i and j have a common end point, or one or more of j's receivers are neighbors of i's transmitter.

Given that the interference relations are still between two sessions, maximal scheduling can be used to schedule sessions. All the definitions introduced in context of arbitrary wireless networks again remain valid in this case. We now characterize Λ^{MS} in arbitrary wireless networks with multicast sessions. Theorems 1 and 2 and Lemma 7 also hold for multicast networks.

We now introduce some additional notations to generalize the results for specific interference models. The *multicast degree of a session* $(i, u, v_1, \ldots, v_{G_i})$ is the sum of the degrees of $u, v_1, \ldots, v_{G_i}$ and $-4G_i$. Let $\gamma(\mathcal{N})$ be the maximum multicast degree of a session in a network. The *multicast directional degree of a session* $(i, u, v_1, \ldots, v_{G_i})$ is the sum of the out-degree of u, and in-degrees of $v_1, \ldots, v_{G_i}$ and $-2G_i$. Let $\Gamma(\mathcal{N})$ be the maximum multicast directional degree of a session in a network. Let $G(\mathcal{N})$ be the maximum number of receivers in a multicast session in a network.

We first upper bound $K(\mathcal{N})$ for specific interference models, which would in turn provide lower bounds for Λ^{MS} using Theorem 1.

Lemma 10: Consider a wireless network $\mathcal{N}$ with multicast sessions.

1) *In the bidirectional communication model,* $K(\mathcal{N}) \leq \max(\gamma(\mathcal{N}), 1)$.
2) *In the unidirectional communication model,* $K(\mathcal{N}) \leq \max(\Gamma(\mathcal{N}), 1)$.
3) *In the bidirectional equal power model,* $K(\mathcal{N}) \leq 25$.
4) *In the node exclusive spectrum sharing model,* $K(\mathcal{N}) \leq G(\mathcal{N}) + 1$.

Proof: Proof is presented in Appendix VI-A. ∎

We now lower bound $K(\mathcal{N})$ for specific interference models, which would in turn provide upper bounds for Λ^{MS} using Theorem 2.

Lemma 11: 1) *In the bidirectional communication model, there exists a wireless network $\mathcal{N}$ such that $K(\mathcal{N}) = \max(\gamma(\mathcal{N}), 1)$.*

2) *In the unidirectional communication model, there exists a wireless network $\mathcal{N}$ such that $K(\mathcal{N}) = \max(\Gamma(\mathcal{N}), 1)$.*
3) *In the bidirectional equal power model, there exists a wireless network $\mathcal{N}$ such that $K(\mathcal{N}) \geq 19$.*
4) *In the unidirectional equal power model, given any constant Z there exists a wireless network $\mathcal{N}$ such that $K(\mathcal{N}) > Z$.*
5) *In the node exclusive spectrum sharing model, there exists a wireless network $\mathcal{N}$ such that $K(\mathcal{N}) = G(\mathcal{N}) + 1$.*

Proof: Proof is presented in Appendix VI-B. ∎

Using Lemmas 10 and 11 and Theorems 1 and 2, Λ^{MS} can now be characterized for specific interference models.

The generalizations in Lemmas 10 and 11 for the bidirectional and unidirectional communication models have been obtained by substituting $\max(\delta_G - 2, 1)$ and $\max(\Delta_G - 2, 1)$ in Lemmas 5 and 6 with $\max(\gamma(\mathcal{N}), 1)$ and $\max(\Gamma(\mathcal{N}), 1)$ in Lemmas 10 and 11 respectively. Note that when all sessions are unicast, $\gamma(\mathcal{N}) = \delta_G - 4$ and $\Gamma(\mathcal{N}) = \Delta_G - 2$. Thus, Lemma 5 can be obtained as a special case of Lemmas 10 and 11.

Lemma 11 shows that in the unidirectional equal power model, maximal scheduling may not in general attain a constant fraction of the maximum throughput region. This is expected as a similar negative result holds for unicast networks (Lemma 3) and unicast is a special case of multicast.

When all sessions are unicast, $G(\mathcal{N}) = 1$. Then, Lemmas 10 and 11 and Theorems 1 and 2 guarantee that in the node exclusive spectrum sharing model, maximal scheduling attains at least $1/2$, and in some topologies no more than $1/2$ the maximum throughput region. This is consistent with the result obtained by Lin *et. al.* [2] and Lemma 4.

C. Multichannel Wireless Networks

We consider a wireless network with M channels. We assume that the transmissions from a session always use the same frequency which is pre-determined. We characterize Λ^{MS} for arbitrary frequency allocation strategies, but do not investigate the design of such strategies. A session i that traverses link (u, v) and transmits in channel k is now completely specified by the $4-$tuple (i, u, v, k). We first describe the transmission constraints. Now, node j can not receive any packet successfully in channel k if more than one of its neighbors are transmitting simultaneously in channel k. Thus, a transmission on edge $(i, j) \in E$ using channel k is successful in a slot if and only if no neighbor of j other than i transmits in channel k in the slot.

We now obtain the pairwise interference relations for both the bidirectional and unidirectional communication models using the above constraints. In the bidirectional communication model, a session i interferes with session j if they have a common end point (transmitter or receiver), or if they have the same frequency and one end point of j is a neighbor of an end point of i. In the unidirectional communication model, session i interferes with session j if they have a common end point, or if they have the same frequency and j's receiver is a neighbor of i's transmitter.

Given the above pairwise interference relations, all the definitions introduced in context of arbitrary wireless networks remain valid in this case. Theorems 1 and 2 and Lemma 7 also hold for arbitrary multichannel wireless networks. Both Lemmas 5 and 6 can be generalized to obtain specific results for bidirectional and unidirectional communication models.

VII. Concluding Remarks

In this paper, we have addressed the long-standing open question of attaining throughput guarantees with distributed scheduling in wireless networks. We have studied the performance of a simple distributed scheduling policy, maximal scheduling, which had earlier been investigated in context of node-exclusive spectrum sharing model and input-queued switches. We have obtained tight performance guarantees for maximal scheduling under arbitrary interference models and topologies, and have characterized the throughput region attained by maximal scheduling in terms of the interference degree of the network. The characterizations demonstrate that the performance bounds depend heavily on the nature of communication and interference models. We prove that maximal scheduling is guaranteed to attain a constant fraction of the maximum throughput region for certain communication and interference models, while it is also guaranteed to not attain a constant fraction in the worst case for some other models. Our results can be generalized to networks with multicast communication and an arbitrary number of frequencies.

The class of maximal scheduling policies is quite broad, and our performance bounds apply to all policies in this class. However, it remains to be seen whether certain policies in this class can attain better performance bounds, while still being amenable to low-complexity distributed implementation. Similar questions remain open for distributed scheduling policies outside this class as well.

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Appendix I: Proof of Theorem 1

Select an arbitrary network $\mathcal{N}$. Let $Q_i^{\pi}(n)$ denote the queue length of session i in the beginning of n^{th} slot under π. Then, the dynamic queue length equation is

$$Q_i^{\pi}(n+1) = Q_i^{\pi}(0) + A_i(n) - D_i^{\pi}(n) \; \forall\, n \geq 1 \text{ and } i = 1, \ldots, N. \tag{4}$$

Now, we fornally define fluid limits. The definitions are similar to that in [1] by Dai and Prabhakar.

Appendix I-A: Fluid Limits

We present the definitions for a simpler case in which the number of arrivals in a slot for any session i is bounded above by $A_{\max}$. Throughout this subsection, we fix a scheduling policy π, and omit superscript π for brevity. Also, we denote by $\mathbb{N}$ and $\mathbb{R}$ the set of non-negative integers and reals respectively. For a random process $\{f(t)\}_{t\geq 0}$, we denote its value at time t along a sample path ω by $f(t,\omega)$.

Note that the domain of the functions $A(\cdot)$, $D(\cdot)$ and $Q(\cdot)$ is $\mathbb{N}$. Now, we define these functions for arbitrary $t \in \mathbb{R}$ by using a piecewise linear interpolation. The piecewise linear interpolation of a function $f : \mathbb{N} \rightarrow \mathbb{R}$ is defined as follows. For $t \in (n, n+1]$

$$f(t) = f(n) + (t-n)\left(f(n+1) - f(n)\right).$$

Note that $f(t)$ defined as above is a continuous function.

From any sender i, at most one packet can be served in a slot. Also, the maximum number of packets arriving in a slot at i is bounded by $A_{\max}$. Thus, for every i, ω, $t \geq 0$ and $\delta > 0$

$$A_i(t+\delta,\omega) - A_i(t,\omega) \;\leq\; \delta A_{\max}, \tag{5}$$

$$D_i(t+\delta,\omega) - D_i(t,\omega) \;\leq\; \delta, \tag{6}$$

$$Q_i(t+\delta,\omega) - Q_i(t,\omega) \;\leq\; \delta A_{\max}. \tag{7}$$

Now, let us define a family of functions for any given function $f(\cdot)$ as follows.

$$f^r(t,\omega) \stackrel{\text{def}}{=} \frac{f(rt,\omega)}{r} \text{ for every } \ r > 0.$$

It follows from (5), (6) and (7), that for every $r > 0$,

$$\begin{aligned} A_i^r(t+\delta,\omega) - A_i^r(t,\omega) &\leq \delta A_{\max}, &(8)\\ D_i^r(t+\delta,\omega) - D_i^r(t,\omega) &\leq \delta, &(9)\\ Q_i^r(t+\delta,\omega) - Q_i^r(t,\omega) &\leq \delta A_{\max}. &(10)\end{aligned}$$

Thus, all the above functions are Lipschitz continuous, and hence uniformly continuous on any compact interval. Clearly, the above functions are also bounded on any compact interval. Fix a compact interval $[0,t]$. Now, consider any sequence r_n such that $r_n \to \infty$ as $n \to \infty$. Then, by Arzela-Ascoli Theorem, there exists a subsequence r_{n_k} and continuous functions $\overline{A}_i(\cdot), \overline{D}_i(\cdot)$ and $\overline{Q}_i(\cdot)$ such that for every i

$$\begin{aligned} \lim_{k\to\infty} \sup_{\widehat{t}\in[0,t]} \left| A_i^{r_{n_k}}(\widehat{t},\omega) - \overline{A}_i(\widehat{t},\omega)\right| &= 0, &(11)\\ \lim_{k\to\infty} \sup_{\widehat{t}\in[0,t]} \left| D_i^{r_{n_k}}(\widehat{t},\omega) - \overline{D}_i(\widehat{t},\omega)\right| &= 0, &(12)\\ \lim_{k\to\infty} \sup_{\widehat{t}\in[0,t]} \left| Q_i^{r_{n_k}}(\widehat{t},\omega) - \overline{Q}_i(\widehat{t},\omega)\right| &= 0. &(13)\end{aligned}$$

Now, we formally define fluid limits.

Definition 15: Any $(\overline{A}_i, \overline{D}_i, \overline{Q}_i)$ is called a fluid limit for $\mathcal{N}$ if there exists r_{n_k} such that all the relations (11) to (13) are satisfied.

Appendix I-B: Properties of Fluid Limits

Now, we state some important properties of the fluid limits which we use to prove Theorem 1.

Lemma 12: Every fluid limit satisfies, $\overline{A}_i(t) = \lambda_i t$ w.p. 1 for every i and $t \geq 0$.

Proof: Since $\overline{A}_i(t)$ is a fluid limit, by Definition 15, there exists a sequence r_{n_k} such that $\lim_{k\to\infty} r_{n_k} = \infty$ and

$$\begin{aligned} \overline{A}_i(t) &= \lim_{k\to\infty} A_i^{r_{n_k}}(t) \quad \text{(from (11))}\\ &= \lim_{k\to\infty} \frac{A_i(r_{n_k}t)}{r_{n_k}}\\ &= \lim_{k\to\infty} \frac{A_i(r_{n_k}t)}{r_{n_k}t}t\\ &= \lambda_i t \quad \text{w.p. 1 (since } A_i(\cdot) \text{ satisfy SLLN).}\end{aligned}$$

This proves the required. ■

Lemma 13: Any fluid limit $(\overline{A}_i, \overline{D}_i, \overline{Q}_i)$ for $\mathcal{N}$ satisfies the following equality for every i and $t \geq 0$ with probability (w.p.) 1.

$$\overline{Q}_i(t) = \overline{Q}_i(0) + \lambda_i t - \overline{D}_i(t). \tag{14}$$

Proof: Since $\overline{Q}_i(\cdot)$, $\overline{A}_i(\cdot)$ and $\overline{D}_i(\cdot)$ are fluid limits, there exists a sequence r_{n_k} such that $\lim_{k\to\infty r_{n_k}} = \infty$ and they are obtained as a uniform limits of functions $Q_i^{r_{n_k}}(\cdot)$, $A_i^{r_{n_k}}(\cdot)$ and $D_i^{r_{n_k}}(\cdot)$ respectively. Now, from (4) it follows that for every r_{n_k} and $t \geq 0$,

$$Q_i^{r_{n_k}}(t) = Q_i^{r_{n_k}}(0) + A_i^{r_{n_k}}(t) - D_i^{r_{n_k}}(t).$$

The result follows from Lemma 12 after taking the limit $k \to \infty$ on both sides of the above equality. ■

Lemma 14: Let $\overline{Q}_i^{MS}(0) = 0$ for every i. Also, let $\sum_{j\in S_i\cup\{i\}} \lambda_j \leq 1$ for every i. Then, under Maximal Scheduling (MS) policy every fluid limit satisfies that $\overline{Q}_i^{MS}(t) = 0$ for every $t \geq 0$ w.p. 1 for every i.

Proof: Here, we omit superscript MS for brevity. We prove the required by contradiction. Let $\overline{Q}_i(t) \neq 0$ for every t and i. Then, there exists a session i, $\widehat{t}$, $y_1 > 0$ and $x_1 > 0$ such that

$$\sum_{j\in S_i\cup\{i\}} \overline{Q}_j(\widehat{t}) \;=\; y_1, \tag{15}$$

$$\sum_{j\in S_i\cup\{i\}} \overline{Q}_j(t) \;<\; y_1 \text{ for every } t \in [0, \widehat{t}), \tag{16}$$

$$\overline{Q}_i(\widehat{t}) \;=\; x_1. \tag{17}$$

Clearly, $x_1 \leq y_1$ as $\overline{Q}_j(\cdot) \geq 0$ for every j. Relation (16) holds because its left hand side is the sum of continuous functions $\overline{Q}_j(\cdot)$ whose values at 0 are 0. In other words, (16) implies that $\widehat{t}$ is the first time instance such that $\sum_{j\in S_i\cup\{i\}} \overline{Q}_j(t)$ reaches level y_1. Since $\overline{Q}_i(\cdot)$ is a continuos function, there exists $t' \in [0, \widehat{t})$ such that

$$\overline{Q}_i(t) \geq \frac{x_1}{2} \text{ for every } t \in [t', \widehat{t}]. \tag{18}$$

Now, since $\overline{Q}_j(\cdot)$ is a fluid limit, by Definition 15, there exists a sequence r_{n_k} such that $\lim_{k\to\infty} r_{n_k} = \infty$ and $\lim_{k\to\infty} Q_j^{r_{n_k}}(t) = \overline{Q}_j(t)$ for every j and t in an interval $[0, \widehat{t}]$. Thus, we can draw two conclusions. First, for sufficiently large r_{n_k}, $Q_i^{r_{n_k}}(t) > x_1/4$ for every $t \in [t', \widehat{t}]$. Thus, $Q_i(r_{n_k}t) > r_{n_k}x_1/4$. This implies that for every $r_{n_k} > 4/x_1$,

$$Q_i(r_{n_k}t) > 1 \text{ for every } t \in [t', \widehat{t}]. \tag{19}$$

The second conclusion is that for every sufficiently large r_{n_k}, there exists $\epsilon > 0$ such that

$$\begin{aligned} \sum_{j\in S_i\cup\{i\}} Q_j^{r_{n_k}}(\widehat{t}) - \sum_{j\in S_i\cup\{i\}} Q_j^{r_{n_k}}(t') \;&>\; \epsilon, \text{ (from (16), since } t' < \widehat{t}) \\ \Rightarrow \lim_{k\to\infty} \left[\sum_{j\in S_i\cup\{i\}} Q_j^{r_{n_k}}(\widehat{t}) - \sum_{j\in S_i\cup\{i\}} Q_j^{r_{n_k}}(t') \right] \;&\geq\; \epsilon. \end{aligned} \tag{20}$$

Relation (20) follows from (15), (16), $t' < \widehat{t}$ and the definition of fluid limits. Select $\widehat{r}_{n_k}$ large enough such that (19) hold. For every $r_{n_k} > \widehat{r}_{n_k}$,

$$\begin{aligned} &\sum_{j\in S_i\cup\{i\}} Q_j^{r_{n_k}}(\widehat{t}) - \sum_{j\in S_i\cup\{i\}} Q_j^{r_{n_k}}(t') \\ &= \sum_{j\in S_i\cup\{i\}} \left[A_j^{r_{n_k}}(\widehat{t}) - A_j^{r_{n_k}}(t')\right] - \left[\sum_{j\in S_i\cup\{i\}} D_j^{r_{n_k}}(\widehat{t}) - \sum_{j\in S_i\cup\{i\}} D_j^{r_{n_k}}(t') \right] \text{ (from (4)).} \end{aligned} \tag{21}$$

Since the scheduling policy is MS and (19) holds, at least one packet from some session in $S_i \cup \{i\}$ departs in every slot. Thus, $\sum_{j \in S_i \cup \{i\}} D_j^{r_{n_k}}(\widehat{t}) - \sum_{j \in S_i \cup \{i\}} D_j^{r_{n_k}}(t') \geq (\widehat{t} - t')$. Now, from (21),

$$\begin{aligned}
\sum_{j \in S_i \cup \{i\}} Q_j^{r_{n_k}}(\widehat{t}) - \sum_{j \in S_i \cup \{i\}} Q_j^{r_{n_k}}(t') &\leq \sum_{j \in S_i \cup \{i\}} \left[A_j^{r_{n_k}}(\widehat{t}) - A_j^{r_{n_k}}(t')\right] - (\widehat{t} - t') \\
\Rightarrow \lim_{k \to \infty} \left[\sum_{j \in S_i \cup \{i\}} Q_j^{r_{n_k}}(\widehat{t}) - \sum_{j \in S_i \cup \{i\}} Q_j^{r_{n_k}}(t')\right] &\leq \lim_{k \to \infty} \sum_{j \in S_i \cup \{i\}} \left[A_j^{r_{n_k}}(\widehat{t}) - A_j^{r_{n_k}}(t')\right] - (\widehat{t} - t') \\
&= \left(\sum_{j \in S_i \cup \{i\}} \lambda_j - 1\right)(\widehat{t} - t') \quad \text{w.p. 1 (from Lemma 12)} \\
&\leq 0. \qquad (22)
\end{aligned}$$

Note that (22) contradicts (20). Thus, the result follows. ■

Appendix I-C: Proof of Theorem 1

Proof: We prove the required in two steps. First, we show that under MS scheduling, if $\sum_{j \in S_i \cup \{i\}} \lambda_j \leq 1$, then $\mathcal{N}$ is rate stable. Second, we show that a necessary condition for stability of $\mathcal{N}$ under any scheduling policy is that $\sum_{j \in S_i \cup \{i\}} \lambda_j \leq K(\mathcal{N})$. Thus, the result follows.

STEP 1: First, we show that $\lim_{r \to \infty} D_i^r(t) = \lambda_i t$ w.p. 1 for every t. Then, the result follows by choosing $t = 1$.

Under MS scheduling, if $\overline{Q}_i(0) = 0$ and $\sum_{j \in S_i \cup \{i\}} \lambda_i \leq 1$ for every i, then $\overline{Q}_i(t) = 0$ w.p. 1 for every i and $t \geq 0$ (Lemma 14). Thus, by Lemma 13, $\overline{D}_i(t) = \lambda_i t$ w.p. 1 for every $t \geq 0$. Since $\overline{D}_i(\cdot)$ is a fluid limit, there exists a subsequence r_{n_k} such that $\lim_{k \to \infty} r_{n_k} = \infty$ and $\lim_{k \to \infty} D_i^{r_{n_k}}(t) = \overline{D}_i(t) = \lambda_i t$ w.p. 1 (Section I-A). Thus, $\liminf_{r \to \infty} D_i^r(t) \leq \lambda_i t$ w.p. 1. Now, we argue that $\liminf_{r \to \infty} D_i^r(t) = \lambda_i t$ w.p. 1.

Suppose, $\liminf_{r \to \infty} D_i^r(t) < \lambda_i t$ w.p. 1. Then, there exists a subsequence $\widehat{r}_{n_k}$ such that $\lim_{k \to \infty} \widehat{r}_{n_k} = \infty$ and $\lim_{k \to \infty} D_i^{\widehat{r}_{n_k}}(t) = \lambda_i t - \epsilon$ w.p. 1 for some $\epsilon > 0$. Now, note that

$$Q^{\widehat{r}_{n_k}}(t) = Q^{\widehat{r}_{n_k}}(0) + A^{\widehat{r}_{n_k}}(t) - D^{\widehat{r}_{n_k}}(t) \text{ (from (4))}.$$

Now, by taking limit as $k \to \infty$ on both sides of the above equation we obtain

$$\begin{aligned}
\overline{Q}_i^1(t) &= \overline{Q}_i^1(0) + \lambda_i t - \overline{D}_i^1(t) \quad \text{w.p. 1 (from Lemma 12)} \\
&= \epsilon, \ (\text{since } \overline{D}_i^1(t) = \lim\nolimits_{k \to \infty} D_i^{\widehat{r}_{n_k}}(t) = \lambda_i t - \epsilon).
\end{aligned}$$

Since, $\overline{Q}_i^1(t)$ is also a fluid limit under MS, the above equation contradicts Lemma 14. Thus,

$$\liminf_{r \to \infty} D_i^r(t) = \lambda_i t \quad \text{w.p. 1.}$$

Now, for every $r > 0$, $D_i^r(t) \leq A_i^r(t)$ as the number of departures from i can at most be equal to the arrivals for i till time rt. Thus, clearly,

$$\limsup_{r \to \infty} D_i^r(t) \leq \lambda_i t \quad \text{w.p. 1.}$$

This shows that

$$\lim_{r \to \infty} D_i^r(t) = \lambda_i t \quad \text{w.p. 1.}$$

Now, select $t = 1$, and consider subsequence r_n such that $r_n = n$. Here, for every i

$$\begin{aligned}
\lim_{n \to \infty} D_i^{r_n}(1) &= \lambda_i \quad \text{w.p. 1} \\
\lim_{n \to \infty} \frac{D_i(n)}{n} &= \lambda_i \quad \text{w.p. 1.}
\end{aligned}$$

This proves STEP 1.

STEP 2: We show that if $\sum_{j\in S_i\cup\{i\}}\lambda_j > K(\mathcal{N})$ for some i, then no policy can stabilize the network. Let us select i such that $\sum_{j\in S_i\cup\{i\}}\lambda_j > K(\mathcal{N})$. Now, note that under any scheduling policy π, $\sum_{j\in S_i\cup\{i\}} D_i^{\pi}(n) \le nK(\mathcal{N})$ for every $n \ge 0$ as at most $K(\mathcal{N})$ nodes among $S_i \cup \{i\}$ can be scheduled concurrently. Thus,

$$\begin{aligned}
\liminf_{n\to\infty} \sum_{j\in S_i\cup\{i\}} \frac{D_j^{\pi}(n)}{n} &\le K(\mathcal{N}) \\
\Rightarrow \quad \sum_{j\in S_i\cup\{i\}} \liminf_{n\to\infty} \frac{D_j^{\pi}(n)}{n} &\le K(\mathcal{N}) \\
&< \sum_{j\in S_i\cup\{i\}} \lambda_j. \\
\Rightarrow \quad \liminf_{n\to\infty} \frac{D_j^{\pi}(n)}{n} &< \lambda_j \text{ for some } j \in S_i\cup\{i\}.
\end{aligned}$$

Now, if $\lim_{n\to\infty}\frac{D_j^{\pi}(n)}{n}$ exists, then its value is less than λ_j. Thus, $\mathcal{N}$ is not stable. Alternatively, if the limit does not exists, then also the $\mathcal{N}$ is not stable.

From STEPs 1 and 2, it follows that if $\vec{\lambda} \in \Lambda$, then $\vec{\lambda}/K(\mathcal{N}) \in \Lambda^{MS}$. ■

APPENDIX II: PROOF OF THEOREM 2

Proof: Consider an arbitrary network $\mathcal{N}$ with interference degree $K(\mathcal{N})$. By Definition 13, there exists i such that the interference degree of session i is $K(\mathcal{N})$. Consider sessions $j_1,\ldots,j_{K(\mathcal{N})} \in S_i$ such that they are pair-wise non-interfering. Now, consider the following arrival rate vector $\vec{\lambda}$: $\lambda_j = Z/K(\mathcal{N})$ if $j \in \{j_1,\ldots,j_{K(\mathcal{N})}\}$, and $\lambda_j = (K(\mathcal{N})-Z)/K(\mathcal{N})$ if $j = i$, and $\lambda_j = 0$ otherwise. Thus, effectively the network consists only of sessions i and $j_1,\ldots,j_{K(\mathcal{N})}$. Note that since $1 \le Z < K(\mathcal{N})$, $\lambda_j > 0$ for every $j \in \{i, j_1,\ldots,j_{K(\mathcal{N})}\}$. Now, consider a scheduling policy π that schedules i w.p. $(K(\mathcal{N})-Z)/K(\mathcal{N})$ and sessions $j_1,\ldots,j_{K(\mathcal{N})}$ concurrently in the remaining slots. Clearly, π is rate stable. Thus, $\vec{\lambda} \in \Lambda$.

Now, consider arrival rate vector $\vec{\lambda}/Z$ and the following arrival pattern. A packet corresponding to session j_u arrives in slots t if $u = t \bmod K(\mathcal{N})+1$, where "mod" is the modulo operator. In every slot a packet arrives w.p. $(K(\mathcal{N})-Z)/K(\mathcal{N})$. Clearly, the arrivals are in accordance with $\vec{\lambda}/Z$. Let MS policy schedule i only when none of the sessions in S_i have a packet to transmit. Note that under MS policy and the described arrival pattern, j_u is scheduled in slot t such that $u = t \bmod K(\mathcal{N})+1$, and thus i is never sheduled. Since $\lambda_i/Z > 0$, i is not stable. Thus, $\vec{\lambda}/Z \notin \Lambda^{MS}$. ■

APPENDIX III: PROOF OF LEMMA 1

We prove Lemma 1 by considering an arbitrary session (S_0, T_0, R_0) and showing that K_0, the maximum number of sessions that interfere with S_0 but do not interfere with each other, must satisfy $K_0 \le 8$. Thus, Lemma 1 follows from Theorem 1.

We assume that the nodes are deployed on a two-dimensional Euclidean plane. Let the distance between the transmitting node T_0 and receiving node R_0 be $\rho \le r$, where r is the transmission range of any node.

Without loss of generality let us assume that the line joining T_0 and R_0 is aligned along the x-axis. Let D_{T_0} and D_{R_0} represent disks of radius r around T_0 and R_0, respectively. Then the interference area of session S_0 is $D_{T_0} \cup D_{R_0}$.

In the following, a node is said to be the *transceiver node* of a session if it is either the transmitting node or the receiving node of that session; thus each session has two transceiver nodes. Note that if a session interferes with S_0, at least one of its transceiver nodes must lie in $D_{T_0} \cup D_{R_0}$. Now for each of the sessions that interfere with S_0 but do not interfere with each other, choose any one transceiver node of that session that lies in $D_{T_0} \cup D_{R_0}$; let $\mathcal{U}_0$ denote the set of the transceiver nodes thus chosen. We will show $K_0 \leq 8$ by showing $U_0 = |\mathcal{U}_0| \leq 8$.

The proof of $K_0 \leq 8$ is quite involved; therefore, we will first show that $K_0 \leq 9$, the proof of which is considerably simpler. We will then extend our arguments to show that $K_0 \leq 8$.

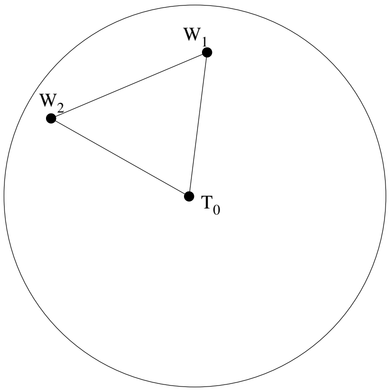

Fig. 6. Diagram used in proof of Lemma 15

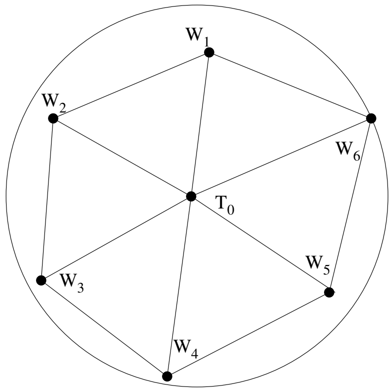

Fig. 7. Diagram used in proof of Lemma 16

Lemma 15: Let $W_1, W_2 \in \mathcal{U}_0$. If $W_1, W_2 \in D_{T_0}$ ($W_1, W_2 \in D_{R_0}$), and none of them coincide with T_0 (R_0), then the line segment joining W_1 and W_2 subtends an angle greater than $\frac{\pi}{3}$ at T_0 (R_0).

Proof: We only consider the case of $W_1, W_2 \in D_{T_0}$ (the $W_1, W_2 \in D_{R_0}$ case is similar). Let $|w_1 w_2|$ denote the length of the line segment joining w_1 and w_2, for any two points w_1, w_2. Refer to Fig. 6. Since $W_1, W_2 \in D_{T_0}$, we have $|W_1 T_0| \leq r$, $|W_2 T_0| \leq r$. Also, since sessions of transceiver nodes W_1 and W_2 do not interfere with each other, we have $|W_1 W_2| > r$. Thus, in triangle $W_1 W_2 T_0$, $W_1 W_2$ is longer than each of the other sides, and its opposite angle $\angle W_1 T_0 W_2$ must be greater than $\frac{\pi}{3}$, from elementary geometry. ■

Fig. 8 shows the area $D_{T_0} \cup D_{R_0}$. Note that $|T_0 R_0| = \rho \leq r$. Let A_1 (A_2) be a point on the circumference of D_{T_0} (D_{R_0}), such that $\angle A_1 T_0 R_0$ $\frac{\pi}{3}$ ($\angle A_2 R_0 T_0$ $\frac{\pi}{3}$). Similarly, let B_1 (B_2) be a point on the circumference of D_{T_0} (D_{R_0}), such that $\angle B_1 T_0 R_0$ $\frac{\pi}{3}$ ($\angle B_2 R_0 T_0$ $\frac{\pi}{3}$). Let line segments $T_0 A_1$ and $R_0 A_2$ intersect at A, and line segments $T_0 B_1$ and $R_0 B_2$ intersect at B.

Recall that $T_0 R_0$ is aligned along the x-axis. Let points C, D, E be points on the circumference of D_{T_0} such that CT_0, DT_0 and ET_0 subtend angles of $\frac{2\pi}{3}$, π and $\frac{4\pi}{3}$ with the x-axis, respectively. Also, let points F, G, H be points on the circumference of D_{R_0} such that FR_0, GR_0 and HR_0 subtend angles of $\frac{\pi}{3}$, 0 and $\frac{5\pi}{3}$ with the x-axis, respectively. Let P_1 (P_2) denote the points at which the line $T_0 R_0$ extended ($R_0 T_0$ extended) intersects the circumference of D_{T_0} (D_{R_0}). Thus, line segments $A_1 T_0$, CT_0, DT_0, ET_0, $B_1 T_0$ and $P_1 T_0$ divide D_{T_0} into six $\frac{\pi}{3}$ sectors. Similarly, line segments $A_2 R_0$, FR_0, GR_0, HR_0, $B_2 R_0$ and $P_2 R_0$ divide D_{R_0} into six $\frac{\pi}{3}$ sectors. From Lemma 15, it follows that each of these sectors can contain at most one node in $\mathcal{U}_0$.

Lemma 16: The number of nodes in $\mathcal{U}_0$ that lie in D_{T_0} (D_{R_0}) can be no greater than 5.

Proof: We only consider the case of D_{T_0} (the case of D_{R_0} is similar). Let $\hat{U}$ denote the number of nodes in $\mathcal{U}_0$ that lie in D_{T_0}. Since D_{T_0} is contained in six $\frac{\pi}{3}$ sectors, $\hat{U} \leq 6$.

For the sake of contradiction let us assume that $\hat{U} = 6$, and let $W_i, i = 1, \ldots, 6$ denote the six nodes in $\mathcal{U}_0$ that lie in D_{T_0}, as shown in Fig. 7. Note that none of these nodes can lie at the center of D_{T_0}, i.e., at T_0. Then, from Lemma 15, the angle subtended at T_0 by each of the line segments $W_i W_j$, $j = (i+1) \bmod 6$, is greater than $\frac{\pi}{3}$. Since the total angle subtended at T_0 cannot exceed 2π, we have a contradiction, thereby proving the lemma. ■

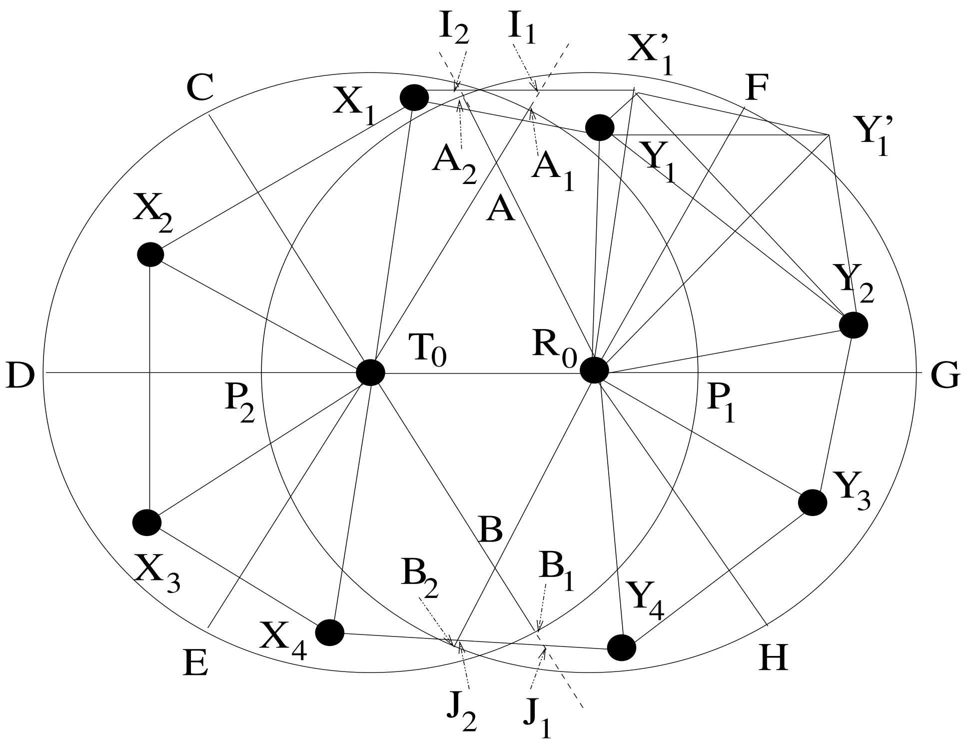

Fig. 8. Diagram used in proof of Lemma 1

From Fig. 8, note that $D_{R_0} \setminus D_{T_0}$ is contained in four $\frac{\pi}{3}$ sectors. Therefore, at most 4 nodes in $\mathcal{U}_0$ can lie in $D_{R_0} \setminus D_{T_0}$. Since at most 5 nodes in $\mathcal{U}_0$ can lie in D_{T_0}, we have the following result:

Corollary 1: U_0, the number of nodes in $\mathcal{U}_0$ can be no greater than 9.

The above result implies that $K_0 \leq 9$. Now we proceed to tighten this upper bound by showing $K_0 \leq 8$.

Now let us assume, for the sake of contradiction, that $K_0 = 9$; this implies $U_0 = 9$.

Corollary 2: If $U_0 = 9$, then the number of nodes in $\mathcal{U}_0$ that lie in $D_{T_0} \setminus D_{R_0}$, $D_{T_0} \cap D_{R_0}$, and $D_{R_0} \setminus D_{T_0}$ are 4, 1 and 4, respectively.

Proof: Let U_1, U_2 and U_3 respectively denote the nodes in $\mathcal{U}_0$ that lie in $D_{T_0} \setminus D_{R_0}$, $D_{T_0} \cap D_{R_0}$, and $D_{R_0} \setminus D_{T_0}$. Then, $U_1 + U_2 + U_3 = 9$. Without loss of generality, assume $U_1 \geq U_3$.

We first argue that $U_2 \neq 0$. Note that if $U_2 = 0$, then $U_1 + U_3 = 9$, implying $U_1 \geq 5$, which is impossible since $D_{T_0} \setminus D_{R_0}$ is contained in four $\frac{\pi}{3}$ sectors. This implies that $U_2 > 0$.

Now we argue that $U_2 \leq 1$. Let us assume, for the sake of contradiction, that $U_2 \geq 2$. Then, $U_1 + U_3 = 9 - U_2 \leq 7$. Thus $U_3 \leq 3$. Therefore, $U_1 + U_2 = 9 - U_3 \geq 6$, which is impossible (from Lemma 16). Therefore, $U_2 \leq 1$. Since $U_2 > 0$ (as shown previously), we have $U_2 = 1$.

Therefore, $U_1 + U_3 = 8$. Since $U_1 \leq 4, U_3 \leq 4$ (each of $D_{T_0} \setminus D_{R_0}$ and $D_{R_0} \setminus D_{T_0}$ are contained in four $\frac{\pi}{3}$ sectors), we must have $U_1 = U_3 = 4$. ■

From the above lemma, we see that if $K_0 = 9$, then $D_{T_0} \setminus D_{R_0}$ and $D_{R_0} \setminus D_{T_0}$ must each contain 4 nodes in $\mathcal{U}_0$. For the sake of contradiction, let us assume that this is true. Note that none of these 8 nodes can lie at the centers of the two disks, i.e., at T_0 or R_0. Also, exactly one of these 8 points must lie in each of the $\frac{\pi}{3}$ sectors of $D_{T_0} \setminus D_{R_0}$ and $D_{R_0} \setminus D_{T_0}$. Let X_1, X_2, X_3 and X_4 respectively denote the nodes in $\mathcal{U}_0$ that lie in sectors A_1T_0C, CT_0D, DT_0E and ET_0B_1. Let Y_1, Y_2, Y_3 and Y_4 respectively denote the nodes in $\mathcal{U}_0$ that lie in sectors A_2R_0F, FR_0G, GR_0H and HR_0B_2. Join X_1, X_2, X_3, X_4 with T_0, and Y_1, Y_2, Y_3, Y_4 with R_0 (refer to Fig. 8). Now, construct the octagon by joining $X_1X_2, X_2X_3, X_3X_4, Y_1Y_2, Y_2Y_3, Y_3Y_4$, and X_1Y_1, X_4Y_4. Note that the length of each side of this octagon must be greater than r. Let line segment

X_1Y_1 intersect line segments T_0A and R_0A (possibly extended) at points I_1 and I_2, respectively. Let line segment X_4Y_4 intersect line segments T_0B and R_0B (possibly extended) at points J_1 and J_2, respectively.

Note that the angle subtended at T_0 by $I_1X_1X_2X_3X_4J_1$ (which is a collection of the line segments $I_1X_1, X_1X_2, \ldots, X_4J_1$), is equal to $\frac{4\pi}{3}$. Similarly, the angle subtended at R_0 by $I_2Y_1Y_2Y_3Y_4J_2$ (which is a collection of the line segments $I_2Y_1, Y_1Y_2, \ldots, Y_4J_2$), is equal to $\frac{4\pi}{3}$. In the following, we show however that the angle subtended at T_0 by $I_1X_1X_2X_3X_4J_1$ plus the angle subtended at R_0 by $I_2Y_1Y_2Y_3Y_4J_2$ must be greater than $\frac{8\pi}{3}$, thus arriving at a contradiction.

We will show that the angle subtended by $X_2X_1I_1$ at T_0 plus the angle subtended by $I_2Y_1Y_2$ at R_0 is greater than π. Without loss of generality, assume that X_1 has a higher y-coordinate than Y_1 (recall that T_0R_0 is aligned along the x-axis). As shown in Fig. 8, choose $X_1^{'}$ such that $X_1T_0R_0X_1^{'}$ is a parallelogram. Join $X_1^{'}$ with Y_1 and Y_2. Note, $\angle X_1T_0I_1 + \angle I_2R_0X_1^{'} = \pi - \angle I_1T_0R_0 - \angle I_2R_0T_0 = \pi - \frac{\pi}{3} - \frac{\pi}{3} = \frac{\pi}{3}$.

We consider the following two cases separately: (i) Y_1 lies within parallelogram $X_1T_0R_0X_1^{'}$, and (ii) Y_1 lies outside parallelogram $X_1T_0R_0X_1^{'}$. Let us consider case (i) first (Fig. 8 shows this case). In this case, we claim that $\angle X_1^{'}R_0Y_2 > \frac{\pi}{3}$. To see this, choose $Y_1^{'}$ such that $X_1Y_1Y_1^{'}X_1^{'}$ is a parallelogram. Join $Y_1^{'}$ with R_0 and Y_2. Note, $|X_1^{'}Y_1^{'}| = |X_1Y_1| > r$. Note that Y_2 must lie "below" $Y_1Y_1^{'}$, since it is easy to see that there is no point in sector FR_0G that is "above" $Y_1Y_1^{'}$ and whose distance from Y_1 is greater than r.

Note that $|Y_1Y_1^{'}| = |X_1X_1^{'}| = |T_0R_0| = \rho$ (by construction). Therefore, it is easy to see that $Y_1^{'}$ must lie outside D_{R_0}. Thus, line segment $X_1^{'}Y_2$ must intersect line segment $Y_1Y_1^{'}$. In triangle $Y_1Y_2Y_1^{'}$, $|Y_1Y_2| > r$ and $|Y_1Y_1^{'}| = \rho \le r$. Therefore, $\angle Y_1Y_1^{'}Y_2 > \angle Y_1Y_2Y_1^{'}$. Thus, $\angle X_1^{'}Y_1^{'}Y_2 \ge Y_1Y_1^{'}Y_2 > \angle Y_1Y_2Y_1^{'} \ge \angle X_1^{'}Y_2Y_1^{'}$. Thus, comparing angles in triangle $X_1^{'}Y_2Y_1^{'}$, we get $|X_1^{'}Y_2| > |X_1^{'}Y_1^{'}| > r$.

Note that since X_1 lies in sector CT_0A_1, it follows that $X_1^{'}$ must lie in sector A_2R_0F. Therefore, $X_1^{'}$ lies in D_{R_0}. In triangle $X_1^{'}R_0Y_2$, therefore, we have $|X_1^{'}R_0| \le r, |Y_2R_0| \le r$, and $|X_1^{'}Y_2| > r$. Therefore, $\angle X_1^{'}R_0Y_2 > \frac{\pi}{3}$.

Thus, if Y_1 lies in the parallelogram $X_1T_0R_0X_1^{'}$, we have $\angle X_1T_0I_1 + \angle I_2R_0X_1^{'} + \angle X_1^{'}R_0Y_2 > \frac{\pi}{3} + \frac{\pi}{3} = \frac{2\pi}{3}$. Moreover, since $\angle I_2R_0X_1^{'} + \angle X_1^{'}R_0Y_2 = \angle I_2R_0Y_1 + \angle Y_1R_0Y_2$, we have $\angle X_1T_0I_1 + \angle I_2R_0Y_1 + \angle Y_1R_0Y_2 > \frac{2\pi}{3}$. From Lemma 15, $\angle X_2T_0X_1 > \frac{\pi}{3}$. Therefore, $\angle X_2T_0X_1 + \angle X_1T_0I_1 + \angle I_2R_0Y_1 + \angle Y_1R_0Y_2 > \pi$. In other words, the angle subtended by $X_2X_1I_1$ at T_0 plus the angle subtended by $I_2Y_1Y_2$ at R_0 is greater than π.

Now let us consider the case where Y_1 does not lie inside parallelogram $X_1T_0R_0X_1^{'}$. Since Y_1 has a lower y-coordinate than X_1, it follows that Y_1 must lie below the line $X_1X_1^{'}$. Thus Y_1 must lie to the "right" of line $R_0X_1^{'}$. Thus, $\angle X_1T_0I_1 + \angle I_2R_0Y_1 > \angle X_1T_0I_1 + I_2R_0X_1^{'} = \frac{\pi}{3}$. From Lemma 15, we get $\angle X_2T_0X_1 > \frac{\pi}{3}, \angle Y_1R_0Y_2 > \frac{\pi}{3}$. Therefore, we obtain $\angle X_2T_0X_1 + \angle X_1T_0I_1 + \angle I_2R_0Y_1 + \angle Y_1R_0Y_2 > \pi$, implying that the angle subtended by $X_2X_1I_1$ at T_0 plus the angle subtended by $I_2Y_1Y_2$ at R_0 is greater than π.

Using similar arguments as above, it follows that the angle subtended by $X_3X_4J_1$ at T_0 plus the angle subtended by $J_2Y_4Y_3$ is greater than π. From Lemma 15, we obtain $\angle X_2T_0X_3 \ge \frac{\pi}{3}, \angle Y_2R_0Y_3 \ge \frac{\pi}{3}$. Combining all of the above results, we see that the angle subtended at T_0 by $I_1X_1X_2X_3X_4J_1$ plus the angle subtended at R_0 by $I_2Y_1Y_2Y_3Y_4J_2$ must be greater than $\pi + \pi + \frac{\pi}{3} + \frac{\pi}{3} = \frac{8\pi}{3}$. Thus we arrive at a contradiction showing that our assumption that $K_0 = 9$ was incorrect. Therefore $K_0 \le 8$. ■

Appendix IV: Proofs of Lemmas 2, 3 and 4

Appendix IV-A: Proof of Lemma 2

Proof: Figure 2(a) shows a network $\mathcal{N}$ with bidirectional equal power model such that $K(\mathcal{N}) = 8$. Thus, the lemma follows immediately from Theorem 2. ■

Appendix IV-B: Proof of Lemma 3

Proof: Fix any Z and note that in the network $\mathcal{N}$ of Fig. 2(b), for $\theta < 2\pi/(Z+2)$, $K(\mathcal{N}) > Z$ under unidirectional equal power model. Thus, the lemma follows immediately from Theorem 2. ■

Appendix IV-C: Proof of Lemma 4

Proof: Fig. 3 shows an example of a network $\mathcal{N}$ under node exclusive spectrum sharing model with $K(\mathcal{N}) = 2$. Thus, the lemma follows immediately from Theorem 2. ■

PROOFS OF LEMMAS 5, 6 AND 7

Appendix IV-D: Proof of Lemma 5

Proof: Consider a network $\mathcal{N}$ that has bidirectional communication and underlying topology $G = (V, E)$. Select a session i from u to v. Since we are considering bidirectional communication, $(u, v) \in E$ and $(v, u) \in E$. Note that at most one session along every link from u and v, and every link to u and v can be scheduled concurrently in the interference region of i without interfering with each other. Let $d_{(u,v)}$ denote the degree of link (u, v). Now, i's interference degree $k_i(\mathcal{N})$ satisfies the following inequality.

$$\begin{aligned}
k_i(\mathcal{N}) &\leq \sum_{\substack{j\in V\\ j\neq v}} \left[\mathbf{1}_{\{(j,u)\in E\}} + \mathbf{1}_{\{(u,j)\in E\}}\right] + \sum_{\substack{j\in V\\ j\neq u}} \left[\mathbf{1}_{\{(j,v)\in E\}} + \mathbf{1}_{\{(v,j)\in E\}}\right] \\
&= \sum_{j\in V} \left[\mathbf{1}_{\{(j,u)\in E\}} + \mathbf{1}_{\{(u,j)\in E\}}\right] + \sum_{j\in V} \left[\mathbf{1}_{\{(j,v)\in E\}} + \mathbf{1}_{\{(v,j)\in E\}}\right] - 4 \\
&= d_{(u,v)} - 4 \\
\Rightarrow \quad \max_i\{k_i(\mathcal{N})\} &\leq \max_{(u,v)\in E}\{d_{(u,v)}\} - 4 \\
\Rightarrow \quad K(\mathcal{N}) &\leq \delta_G - 4. \qquad (23)
\end{aligned}$$

Now, Fig. 5(a) shows an example of a network that achieves the equality in (23). ■

Appendix IV-E: Proof of Lemma 6

Proof: Consider a network $\mathcal{N}$ and with unidirectional communication on underlying topology $G = (V, E)$. Fix a session i from u to v. Since we are considering unidirectional communication, $(u, v) \in E$. Let $\widehat{d}_{(u,v)}$ denote the directional degree of link (u, v). Now, i's interference degree $k_i(\mathcal{N})$ satisfies the following inequality.

$$\begin{aligned}
k_i(\mathcal{N}) &\leq \sum_{\substack{j\in V\\ j\neq v}} \mathbf{1}_{\{(u,j)\in E\}} + \sum_{\substack{j\in V\\ j\neq u}} \mathbf{1}_{\{(j,v)\in E\}} \\
&= \sum_{j\in V} \mathbf{1}_{\{(u,j)\in E\}} + \sum_{j\in V} \mathbf{1}_{\{(j,v)\in E\}} - 2 \\
&= \widehat{d}_{(u,v)} - 2 \\
\Rightarrow \quad \max_i\{k_i(\mathcal{N})\} &\leq \max_{(u,v)\in E}\{\widehat{d}_{(u,v)}\} - 2 \\
\Rightarrow \quad K(\mathcal{N}) &\leq \Delta_G - 2. \qquad (24)
\end{aligned}$$

Now, Fig. 5(b) shows an example of a network that achieves the equality in (24). ■

Appendix IV-F: Proof of Lemma 7

Proof: Note that if the interference graph has more than one component, then the transmission in one component do not interfere with that in other components. Thus, under MS policy, each of these components can be treated as independent networks. Now, the result follows by applying Theorems 1 and 2 to each of these sub-networks. ■

Appendix V: Proofs of Lemmas 8 and 9

Appendix V-A: Proof of Lemma 8

Proof: First, we note that $k_i(\mathcal{N}) \leq \beta_j(\mathcal{N})$ for every $j \in S_i \cup \{i\}$. This is because if $j \in S_i$, then $i \in S_j$. Thus, maximization in computation of $\beta_j(\mathcal{N})$ (refer to (3)) involves term $k_i(\mathcal{N})$ for every $j \in S_i \cup \{i\}$.

Recall that we have shown in the proof of Theorem 1 that if $\sum_{j \in S_i \cup \{i\}} \lambda_j \leq 1$ for every i, then the MS policy is stable. Thus, if $\frac{\vec{\lambda}}{\vec{\beta}} \notin \Lambda^{MS}$, then there exists i such that

$$\begin{aligned}
\sum_{j \in S_i \cup \{i\}} \frac{\lambda_j}{\beta_j} &> 1, \\
\Rightarrow \sum_{j \in S_i \cup \{i\}} \frac{\lambda_j}{k_i(\mathcal{N})} &> 1 \quad (\text{since } k_i(\mathcal{N}) \leq \beta_j(\mathcal{N})), \\
\Rightarrow \sum_{j \in S_i \cup \{i\}} \lambda_j &> k_i(\mathcal{N}).
\end{aligned}$$

Now, we show that if $\sum_{j \in S_i \cup \{i\}} \lambda_j > k_i(\mathcal{N})$ for some i, then no policy can stabilize $\vec{\lambda}$. Let us select i such that $\sum_{j \in S_i \cup \{i\}} \lambda_j > k_i(\mathcal{N})$. Now, note that under any scheduling policy π, $\sum_{j \in S_i \cup \{i\}} D_i^{\pi}(n) \leq nk_i(\mathcal{N})$ for every $n \geq 0$ as at most $k_i(\mathcal{N})$ nodes among $S_i \cup \{i\}$ can be scheduled concurrently. Thus,

$$\begin{aligned}
\liminf_{n \to \infty} \sum_{j \in S_i \cup \{i\}} \frac{D_j^{\pi}(n)}{n} &\leq k_i(\mathcal{N}) \\
\Rightarrow \sum_{j \in S_i \cup \{i\}} \liminf_{n \to \infty} \frac{D_j^{\pi}(n)}{n} &\leq k_i(\mathcal{N}) \\
&< \sum_{j \in S_i \cup \{i\}} \lambda_j. \\
\Rightarrow \liminf_{n \to \infty} \frac{D_j^{\pi}(n)}{n} &< \lambda_j \text{ for some } j \in S_i \cup \{i\}.
\end{aligned}$$

Now, if $\lim_{n \to \infty} \frac{D_j^{\pi}(n)}{n}$ exists, then its value is less than λ_j. Thus, $\mathcal{N}$ is not stable under π. Alternatively, if the limit does not exists, then also the $\mathcal{N}$ is not stable under π. Thus, the result follows. ■

Appendix V-B: Proof of Lemma 9

Consider a network $\mathcal{N}$ with three sessions i_1, i_2 and i_3 such that $S_{i_1} = \{i_2, i_3\}$ and $S_{i_2} = S_{i_3} = \{i_1\}$. Thus, $k_{i_1}(\mathcal{N}) = 2$ and $k_{i_2}(\mathcal{N}) = k_{i_3}(\mathcal{N}) = 1$. Note that such a network can be constructed under all the interference models considered in this paper. Let $\lambda_{i_1} = \lambda_{i_2} = \lambda_{i_3} = 1/2$. Note that a policy that schedules session i_1 in odd slots and i_2 and i_3 in the even slots stabilizes the system. Hence, $(1/2, 1/2, 1/2) \in \Lambda$.

Now, consider the rate vector $(\lambda_{i_1}/k_{i_1}(\mathcal{N}), \lambda_{i_2}/k_{i_2}(\mathcal{N}), \lambda_{i_3}/k_{i_3}(\mathcal{N})) = (1/4, 1/2, 1/2)$. In addition, consider the following arrival pattern. A packer arrives at i_2 (i_3, resp.) in every even (odd, resp.) slot, and a packet arrives at i_1 in every slot w.p. 1/4. Clearly, this arrival pattern is in accordance with the rate vector $(1/4, 1/2, 1/2)$. Note that a MS policy that schedules i_1 only when i_2 and i_3 do not have a packet to transmit is unstable. Thus, the result follows. ■

Appendix VI: Proofs for Lemmas 10 and 11

Let $\mathcal{G}_i$ denote the set of receivers for session i and let u denote the receiver for session i. Also, let $k_i(\mathcal{N})$ denote the interference degree of session i.

Appendix VI-A: Proof for Lemma 10

Proof: 1) Consider network $\mathcal{N}$ with multicast sessions and bidirectional communication model. Since only one session can be scheduled along any link in a slot, the multicast degree of any session i (denoted by M_i) satisfies the following relation.

$$\begin{aligned} k_i(\mathcal{N}) &\leq M_i \\ \Rightarrow \quad \max_i\{k_i(\mathcal{N})\} &\leq \max_i\{M_i\} \\ \Rightarrow \quad K(\mathcal{N}) &\leq \gamma(\mathcal{N}). \end{aligned}$$

2) Consider network $\mathcal{N}$ with multicast sessions and unidirectional communication model. Note that i can interfere only with the sessions whose receiver is neighbor of u or whose sender has at least one $j \in \mathcal{G}_i$ as its neighbor. Moreover, each node can be involved in only one transmission in a slot. Thus, the directional multicast degree (denoted by $\tilde{M}_i$) satisfies the following relation.

$$\begin{aligned} k_i(\mathcal{N}) &\leq \tilde{M}_i \\ \Rightarrow \quad \max_i\{k_i(\mathcal{N})\} &\leq \max_i\{\tilde{M}_i\} \\ \Rightarrow \quad K(\mathcal{N}) &\leq \Gamma(\mathcal{N}). \end{aligned}$$

3) Let interference area for session i ($\mathcal{A}_i$) denote the area such that if an end-point of session j lies in $\mathcal{A}_i$ then session i and j interfere with each other. Note that $\mathcal{A}_i$ for any i is a subset of the area covered by a disk of radius $2d$ centered at the sender of i under bidirectional equal power model with the transmission radius of a node being d. Now, for two sessions j and k to belong to a interference set of i at least one of their end-points should belong to $\mathcal{A}_i$. Moreover, for these sessions to be mutually non-interfering, the distance between their end-points should be greater than d. Thus, if we place a disk of radius $d/2$ around one end-point of the interfering sessions j and k, then these disks do not intersect. Thus, the maximum number interfering sessions that are pair-wise non-interfering for any session i is less than or equal to the maximum number of non-overlapping disks of radius $d/2$ whose center lies in $\mathcal{A}_i$. Thus, $K(\mathcal{N})$ is less than or equal to the maximum number of non-overlapping disks of radius $d/2$ such that the disks lie completely in the area covered by a disk of radius $5d/2$. Thus, $K(\mathcal{N}) \leq 25$.

4) Note that under node exclusive spectrum sharing model, a session j interferes with session i only if i and j has common end-point. Since the number of end-points for the session i is $G_i + 1$. Thus,

$$\begin{aligned} k_i(\mathcal{N}) &\leq G_i + 1 \\ \Rightarrow \quad \max_i\{k_i(\mathcal{N})\} &\leq \max_i\{G_i + 1\} \\ \Rightarrow \quad K(\mathcal{N}) &\leq G(\mathcal{N}) + 1. \end{aligned}$$

■

Appendix VI-B: Proof of Lemma 11

Proof: Note that unicast is a special case multicast and hence 1) and 2) follows immediately from Figures 5(a) and 5(b) respectively. Also, 4) and 5) follows from Figures 2(b) and 3 respectively. Now, we prove 3) by constructing a network $\mathcal{N}$ with a multicast session such that $K(\mathcal{N}) = 19$. We show such an network in Figure 9. Hence, the result follows. ■

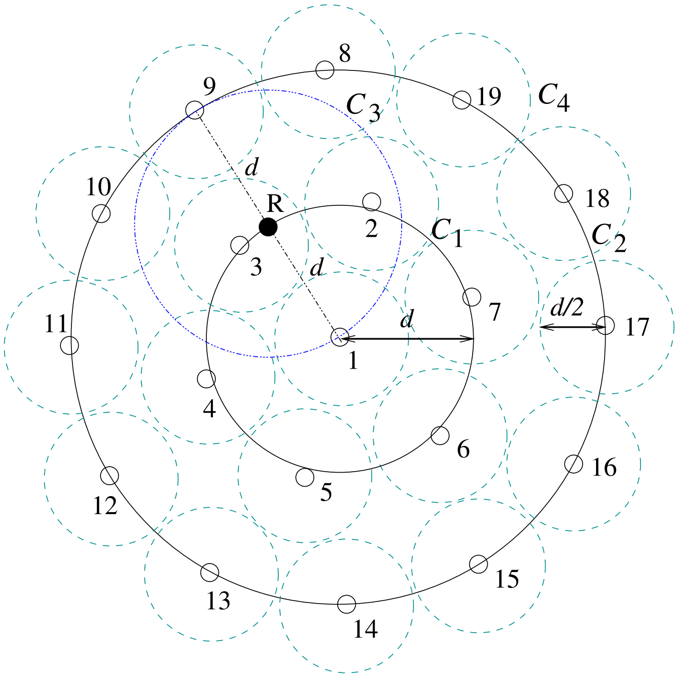

Fig. 9. Figure shows a construction to prove that a network $\mathcal{N}$ with a multicast session can be constructed such that $K(\mathcal{N}) = 19$. Now, we describe the construction. Figure shows 19 disks of radius $d/2$ whose center lies in a disk C_2 of radius $2d$. C_1 is another disk of radius d such that C_1 and C_2 are concentric. Now, consider a multicast session with 12 receivers such that the sender of the multicast session is at the center of C_1. Each of the receivers lie at the intersection of the line segment from the center of C_1 to nodes 8 to 19 and the boundary of C_1. One such receiver R that lies at the intersection of the line from the center of C_1 to node 9 and the boundary of C_1 is shown with the filled circle. This completes the construction of the multicast session. Now, we construct 19 unicast sessions that interfere with the multicast session but are pair-wise non-interfering. We place the senders of these unicast sessions at the center of each of the disks with radius $d/2$, i.e., at the location shown by the small circles numbered from 1 to 19. Note that since the disks with radius $d/2$ do not intersect, the distance between any $i \in \{1, \ldots, 19\}$ to any $j \in \{1, \ldots, 19\}$ is greater than $d + \epsilon$ for some $\epsilon > 0$. Now, we place receivers for each unicast session at the distance $\epsilon/4$ from its respective sender. Thus, note that the distance between any end-point of session i and any end-point of session j is strictly greater than d for every $i, j \in \{1, \ldots, 19\}$. In other words, the 19 unicast sessions are pair-wise non-interfering. But, clearly, each of the unicast session interferes with the multicast session. Thus, $K(\mathcal{N}) = 19$.