Wireless Multicast: Theory and Approaches

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***Abstract*— We design transmission strategies for MAC layer multicast that maximize the utilization of available bandwidth. Bandwidth efficiency of wireless multicast can be improved substantially by exploiting the feature that a single transmission can be intercepted by several receivers at the MAC layer. The multicast nature of transmissions, however, changes the fundamental relations between the QoS parameters, throughput, stability and loss, e.g., a strategy that maximizes the throughput does not necessarily maximize the stability region or minimize the packet loss. We explore the trade-offs between the QoS parameters, and provide optimal transmission strategies that maximize the throughput subject to stability and loss constraints. The numerical performance evaluations demonstrate that the optimal strategies significantly outperform the existing approaches.**

***Index Terms*— Multicast, optimization, scheduling, stability, stochastic control, throughput, wireless**

I. Introduction

SEveral wireless applications need one to many (*multicast*) communication, e.g., conference meetings, sensor networks, rescue and disaster recovery and military operations. The existing research in wireless multicast have predominantly lead to the development of end-to-end error recovery and routing protocols [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13]. End-to-end error recovery protocols retrieve lost packets with minimum information exchange among nodes, e.g., [1], [2]. Protocols for energy efficient multicast routing have been proposed in [11], [12], [13]. Though the overall performance of the network depends on the efficiency of the underlying scheduling strategy used at the Medium Access Control (MAC) layer, MAC layer multicast has not been adequately explored. Our research is directed towards filling this void.

Wireless communication is inherently broadcast in nature, i.e., a packet can be intercepted by all nodes in the transmission range of the sender (e.g., Figure 1). Hence, it suffices to transmit each packet once in order to reach all the intended receivers; this substantially reduces bandwidth and power consumption in wireless multicast. A multicast specific challenge in exploiting broadcast nature of wireless medium is that *some* but *not all* the receivers may be ready to receive. For example, in Figure 1, when sender S_2 is transmitting to receiver R_5, receiver R_2 cannot receive the transmission from sender S_1 as both the transmissions will collide at R_2; though receivers R_1, R_3 and R_4 can still receive the transmission.

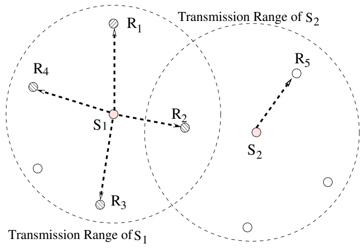

Fig. 1. Figure shows an example to demonstrate the advantages and the challenges associated with wireless multicast. There are two senders S_1, S_2 and 5 receivers R_1 to R_5. Dashed circle indicates the communication range of a sender. A single transmission from S_1 can reach all its receivers, $R_1, \ldots, R_4$.

The policy decision is whether the sender S_1 should transmit or it should wait until all the receivers are ready. A policy that does not transmit until a sufficient number of receivers are ready may render the system unstable (i.e., the queue length at the sender becomes unbounded). On the other hand, if the sender transmits irrespective of the readiness states of the receivers, then the transmitted packet may be lost at several receivers that were not ready. The resulting packet loss at the receivers may be unacceptably high. The throughput, which is the total number of packets received by all the receivers per unit time, may be low at both extremes and maximum somewhere in between. This is because the transmission rate is low in the first case, and packets do not reach most receivers in the latter case. Thus, there is a multicast specific trade-off between throughput, stability and packet loss.

We show that the fundamental relations between QoS parameters such as throughput, loss and stability change due to the multicast nature of transmissions, e.g., a strategy that maximizes the throughput does not necessarily maximize the stability region or minimize the packet loss. We propose a policy that decides when a sender should transmit a packet so as to maximize the throughput subject to (a) system stability and (b) packet loss constraints at the receivers. We prove using the large deviation theory that a sender can attain the above optimality objective by transmitting only when the number of ready receivers is above a certain threshold. This threshold based policy is simple to implement once the optimal threshold is known, as the sender need not know the individual readiness states of the receivers. The optimal value of the threshold depends on the statistics of the arrival and the receiver readiness processes. We present an adaptive approach that computes the threshold based on the estimates of the statistics obtained from system observations, and prove that the computation converges to the optimal value.

The threshold based scheme is a generalization of a *Threshold*-1 protocol proposed by Tang *et al.*, where a sender transmits whenever at least one receiver is ready [14], [15].

Manuscript received Mon XX, XXXX; revised Mon XX, XXXX. This work was supported by the National Science Foundation under grants ANI-0106984, NCR-0238340 and CNS-0435306. Parts of this paper have been presented at IEEE International Symposium on Information Theory, Yokohama, Japan, Jul 2003 and IEEE Conference on Decision and Control, Maui, Hawaii, Dec 2003.

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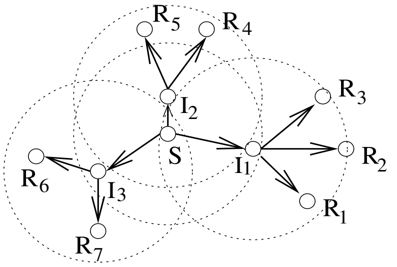


Fig. 2. Figure shows a multicast tree in a multi-hop wireless network from a source S to destinations R_1 to R_7. At the MAC layer, sender S multicasts to intermediate nodes I_1 to I_3, I_1 multicasts to the receivers R_1 to R_3, etc. Thus, there are four separate MAC sessions: S to $[I_1, I_2, I_3]$, I_1 to $[R_1, R_2, R_3]$, I_2 to $[R_4, R_5]$ and I_3 to $[R_6, R_7]$.

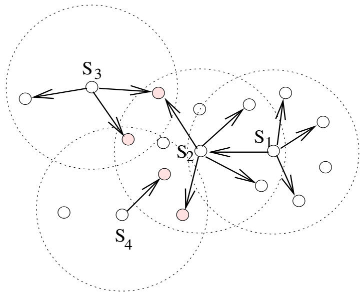


Fig. 3. Figure shows four MAC layer multicast sessions. Nodes S_1 to S_4 are the senders for multicast groups 1 to 4 respectively. Arrows from a sender point to its designated receivers. Note that the node S_2 is a sender for multicast group 2, and a receiver for multicast group 1.

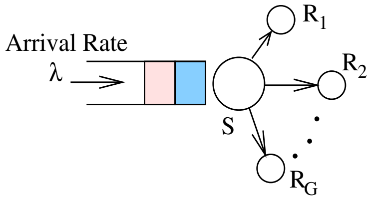


Fig. 4. Figure shows an isolated MAC layer multicast session. The packets arrive at the rate λ at the sender S. The receivers are R_1 to R_G.

Other existing multiple access strategies for wireless multicast are *Threshold*-0, which is used in IEEE 802.11, and *unicast based multicast* [16]. The former transmits a packet irrespective of the existing transmissions and the readiness states of the receivers. This causes packet loss at the receivers because of collision due to second hop interference. The latter attains multicast by transmitting a packet separately to each receiver in round robin manner [16], and thus does not exploit the broadcast nature of wireless medium. We analyze the existing approaches, and show using numerical performance evaluation that the proposed optimal policy provides significantly more efficient usage of bandwidth.

Now, we briefly discuss research contributions in other areas of wireless multicast. Zhou *et al.* have investigated content based multicast in adhoc networks [17]. Nagy *et al.* have investigated multicast in cellular networks [18]. Singh *et al.* have proposed a protocol for power aware broadcast [11]. Kuri *et al.* have proposed a contention resolution protocol for multicast in wireless local area networks [19]. This protocol can be used only when all the nodes are in the transmission ranges of each other, which does not hold in a multi-hop wireless adhoc network.

The paper is organized as follows. In Section II, we define our system model. In Section III, we investigate the trade off between the different QoS parameters for multicast transmission. In Section IV, we obtain threshold-based transmission policies that maximize the throughput subject to stability and loss constraints, and propose adaptive schemes for computing the optimal thresholds. We compare the performance of the optimal policy with other existing multicast policies in Section V. For simplicity, we assume that the wireless channel to a receiver can have only two states (*ready* and *not ready*) in most of the paper. We relax this assumption to consider three or more states for the transmission channel to each receiver in Section VI. We discuss several open problems in Section VII. We present all proofs in the appendix.

II. System Model

The objective is to design efficient transmission strategies for a wireless network with several MAC layer multicast sessions, e.g., Figure 3. Each multicast session has a sender and a set of receivers (multicast group). At the MAC layer, all the receivers are within the transmission range of the sender. The scenario described above is motivated by multicast communication in a multi-hop wireless network (Figure 2).

In this paper, we consider a single multicast session in isolation with G receivers (Figure 4). The impact of the network and the channel errors on the multicast session is that the receivers are not always ready to receive. This may happen because of transmission in the neighborhood of a receiver, bursty channel error, or power saving operation of a receiver. Thus, the receiver readiness states are correlated in the same time slot, and across the time slots. We model the readiness process of all the receivers as a Markov chain (MC) with an arbitrary Transition Probability Matrix (TPM) $\widehat{P}$. We discuss the implications of the Markovian assumption in Sections VI and VII. A state of the MC is a G-dimensional vector $\vec{j} = [j_1 \ j_2 \ \ldots \ j_G]$, where the component j_l is 1 if the l^{th} receiver is ready and 0 otherwise. Let $\mathcal{S}$ be the state space of the receiver readiness process. We assume that the $2^G \times 2^G$ TPM $\widehat{P}$ is irreducible, aperiodic and time-homogeneous. Thus, $\widehat{P}$ has a unique stationary distribution $\vec{B} = \{B_{\vec{j}} : \vec{j} \in \mathcal{S}\}$, which depends on the network load, channel characteristics, and power saving scheme. Let b_u be the steady state probability that u receivers are ready to receive, $b_u = \sum_{\vec{j}:\sum_{l=1}^{G} j_l = u} B_{\vec{j}}$, $b_u > 0$, for each u. We refer to the probability distribution $\vec{b} = [b_0 \ b_1 \ \ldots \ b_G]$ as the aggregate stationary distribution of the receiver readiness process. In Section VI, we consider the more general case where a receiver is ready with a probability that depends on the state of the receiver readiness process. We have summarized the notations in Table I.

A sender queries the readiness states of the receivers by transmitting control packets, and decides whether or not to transmit a packet depending on the transmission strategy, availability of packet and result of the query. Every receiver maintains its readiness state throughout the transmission. This

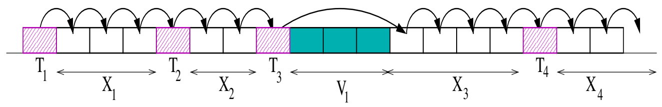


Fig. 5. Figure shows typical transitions of the receiver readiness process. A box indicates a time slot. The T_i's, V_i's and X_i's denote the sample points, duration of transmission and duration of back-off, respectively. Solid arcs indicate transitions in the receiver readiness process.

assumption is justified because the time scale of a change of transmission quality is large as compared to the packet sizes. Also, the level of interference does not change during a packet transmission, since in several MAC protocols (e.g., IEEE 802.11), the exchange of control messages prevents a new transmission during an ongoing transmission in the reception range of the receiver. The sender backs off for a random duration before querying the system again, irrespective of the transmission decision, so as to allow other senders to use the shared medium. The structure of the multiple access protocol described above is similar to IEEE 802.11. Note that the receiver readiness process is Markovian only when restricted to the slots in which the sender queries or backs off, e.g., in duration $[T_1, T_3] \cup X_3 \cup T_4 \cup X_4$ in Figure 5.

We assume that time is slotted. The number of packets arriving in a slot constitutes an irreducible, aperiodic Markov chain with a finite number of states. The expected number of arrivals in a slot under the stationary distribution is denoted as λ. The packet transmission times and back-off durations are independent and identically distributed (i.i.d.) random variables with arbitrary probability distributions and the expected values $\mathbf{E}[V]$ and $\mathbf{E}[X]$ respectively. We assume that $\mathbf{E}[V]$ and $\mathbf{E}[X]$ are finite. Let $\mathbf{E}[A]$ and $\mathbf{E}[\bar{A}]$ be the expected number of arrivals in the duration of a back-off and a packet transmission respectively, under the stationary distribution of the arrival process. Then, $\mathbf{E}[A] = \lambda\mathbf{E}[X]$, and $\mathbf{E}[\bar{A}] = \lambda\mathbf{E}[V]$.

We consider data traffic and assume FIFO selection of packets for transmission. We consider three Quality of Service (QoS) measures: (a) *throughput* (b) *packet loss* and (c) *system stability*.

Definition 1: A *reward* for a transmission is the number of receivers that receive the packet successfully.

Definition 2: *Throughput* is the expected reward per unit time.

Definition 3: The *loss at a receiver* is the fraction of transmitted packets that are either not received or received in error at the receiver. The *system loss*, or simply the *loss*, is the sum of the losses at all the receivers in the multicast group. A *loss constraint* specifies an upper bound (L) on the system loss.

Definition 4: The *sample points* are the epochs at which the sender samples (queries) the receiver readiness states.

Definition 5: A *transmission policy* is an algorithm that decides whether or not to transmit a packet at a sample.

Definition 6: A system is *stable* if the mean queue length at the sender is bounded. A *stable transmission policy* is one that stabilizes the system.

Definition 7: The *stability region of a transmission policy* is the maximum value of λ for which the policy stabilizes the system. The *stability region of the system* is the maximum value of λ for which some transmission policy stabilizes the system.

TABLE I
NOTATIONS USED IN THE ENTIRE PAPER

Notation	Definition
G	Size of the multicast group
L	Upper bound on loss
λ	The expected number of packets arriving at the sender in each slot under the steady state distribution of the arrival process
$\mathbf{E}[X]$	The expected back-off duration
$\mathbf{E}[V]$	The expected packet transmission time
$\mathcal{S}$	The state space for the receiver readiness process
$\vec{B} = \{B_{\vec{j}} : \vec{j} \in \mathcal{S}\}$	The steady state distribution of $\widehat{P}$
$\vec{b} = [b_1 \ \ldots \ b_G]$	The aggregate steady state distribution of $\widehat{P}$
$\mathbf{E}[A] = \lambda\mathbf{E}[X]$	The expected number of arrivals in the back-off duration under the steady state distribution of the arrival process
$\mathbf{E}[\bar{A}] = \lambda\mathbf{E}[V]$	The expected number of arrivals in the duration of a packet transmission under the steady state distribution of the arrival process
Ω^{Δ}	The throughput of the policy Δ
R^{Δ}	The expected reward per packet of the policy Δ
$\Delta_{(T,q)}$	The two-threshold policy(T, q)

A transmission policy can be either offline or online. An offline strategy uses a prior knowledge of packet arrivals at all (including future) slots and the readiness states at all (including future) samples in its decision process. Thus, an offline strategy knows the readiness states at all slots apriori in the special case that the sender samples the system every slot, i.e., when every packet has length 1 slot and there is no back-off. An online strategy takes the transmission decision based on the current packet availability, and the number of ready receivers at the current sample. We show that online strategies maximize the throughput subject to stability and loss constraints.

We will demonstrate that a small loss tolerance significantly increases the throughput and the stability region of the system in wireless multicast. The lost information can be recovered by using coding redundancy, or a reliable protocol at a higher layer. We impose a constraint on the sum of the loss at the receivers, as a receiver can often retrieve lost packets from other receivers who have received the packet. A sender may satisfy the loss constraint by transmitting a packet several times until a sufficient number of receivers receive the packet, e.g., in Figure 1, S_1 may transmit a packet to the R_1, R_3 and R_4 even when the R_2 is not ready, and then retransmit the packet when R_2 becomes ready. But each additional transmission increases the power consumption. Therefore, we assume that a packet can be transmitted only once at the MAC layer.

III. Relation between Throughput, Stability and Loss

We first investigate the relation between the throughput and stability for multicast transmission. In the unicast case, a throughput optimal strategy is one that attains the stability region of the system (Definition 7) [20], [21]. Let us exclude the policies that transmit even when no receiver is ready. Then, in Figure 4 if $G = 1$, a policy that transmits whenever the sender has a packet and the receiver is ready maximizes the throughput, and attains the stability region of the system. This relation between throughput optimality and system stability does not hold in the multicast case. Let $G = 2$ in Figure 4. A policy that transmits when at least one receiver is ready attains the stability region of the system. However, the policy that transmits only when both receivers are ready has a smaller stability region, but it can provide a higher throughput for appropriate choice of the system parameters. Assume that each receiver is ready with a probability of $p = 0.1$ in each slot independent of the other receiver and the readiness states in the other slots. Let $\mathbf{E}[X] = 1, \mathbf{E}[V] = 1000, \lambda = 1/1050$. Then the throughputs of the two policies are 1.058×10^{-3} and 1.818×10^{-3}, respectively. The first (second) policy renders the system stable (unstable). If the arrival rate is such that both the policies stabilize the system, then throughputs are 1.1111λ and 2λ, respectively. Thus, in the multicast case, a policy that maximizes the throughput need not attain the stability region of the system, and vice-versa. Hence, Lyapunov function based approaches cannot be directly used to prove the throughput optimality of a transmission strategy in the multicast case. We note that the throughput is the product of the transmission rate and the expected reward per transmission. The stability is guaranteed for both unicast and multicast if the transmission rate equals the arrival rate. Now, the equivalence between the maximization of throughput and attaining the stability region in the unicast case is because a transmitted packet always fetches a reward of 1 unit. The reward obtained by a transmitted packet can, however, be any integer between 1 and G in the multicast case depending on the readiness states of the receivers. Therefore, the equivalence does not exist in the multicast case.

We investigate the relation between throughput and loss now. First, consider a stable system. The throughput of a transmission policy is λR, where R is the average reward received by the policy per transmission. Further, the loss is $G - R$. Thus, a throughput optimal policy minimizes the loss for stable systems. This relation, however, does not hold for unstable systems. An unstable system is saturated in the sense that the sender always has a packet to transmit. For example, let $G = 2$ in Figure 4. Now, let one receiver be ready with probability p in each slot; the other is always ready. Let $\mathbf{E}[X] = \mathbf{E}[V] = 1$. We consider a policy that transmits only when both the receivers are ready, and another that transmits with probability q if only one receiver is ready and with probability 1 if both the receivers are ready. Let $\lambda > p+q(1-p)$. The transmission rates are p and $p+q(1-p)$ respectively. Thus, neither policy is stable. The throughputs are $2p$ and $2p+q(1-p)$ respectively. The losses are 0 and $\frac{q(1-p)}{p+q(1-p)}$ respectively. Thus, for $p, q \in (0, 1)$, the second policy has both higher throughput and higher loss. Hence, the maximization of the throughput is not equivalent to the minimization of loss.

We now discuss whether the saturated region is relevant in practice. If a policy that maximizes the throughput subject to stability does not satisfy the loss constraint, then no stable policy can do so. Thus, a system must operate in the saturated region, if satisfying the loss constraint is more important than attaining the stability. We note that it is always possible to satisfy the loss constraint if the stability requirement is relaxed. For example, a policy that transmits only when all G receivers are ready has zero loss, but can render the system unstable.

IV. Throughput Optimal Transmission Policy

In Subsection IV-A, we obtain a transmission policy that maximizes the throughput subject to attaining system stability. Next, in Subsection IV-B, we obtain a transmission policy that maximizes the throughput subject to satisfying the loss constraint. In each subsection we provide algorithms that decide the parameters of the optimum strategies without using any information about the system statistics. We first present some definitions.

Definition 8: The *busy samples* are the sample points at which the sender's queue is non-empty. Sample points that are not busy are called *idle samples*.

Definition 9: A *single-threshold transmission policy(T)* is a policy that transmits a packet at every busy sample with T or more ready receivers. The parameter T is the *threshold*.

Definition 10: A *two-threshold transmission policy(T,q)* is a policy that sets threshold T for a given sample w.p. q, or a threshold $T+1$ w.p. $1-q$, and transmits in accordance with the threshold.

Definition 11: A stable transmission policy Δ is ϵ-*throughput optimal* if no other stable transmission policy can achieve throughput more than ϵ plus that achieved by Δ.

A. Throughput Optimality subject to Stability

We first describe the stability region of the system. The service time of a packet is the difference between the times at which the packet finishes transmission and reaches the head of line position in the queue. For the system to be stable, the expected service time must be less than the expected inter-arrival time of packets. The sum of the transmission time plus one back-off duration is the lower bound on the service time of a packet for any transmission policy. Hence, for stability we need $\lambda(\mathbf{E}[X] + \mathbf{E}[V]) < 1$, i.e.,

$$\mathbf{E}[A] + \mathbf{E}[\bar{A}] < 1. \tag{1}$$

We show that if (1) is satisfied, we can choose a threshold T and a probability q such that the corresponding two-threshold policy(T, q) is ϵ-throughput optimal.

Theorem 1: Let the stability condition (1) hold. For every $\epsilon > 0$, there exists a choice of parameters T and q such that the corresponding two-threshold policy(T, q) is ϵ-throughput optimal with probability 1. The optimal values of the parameters T and q are

$$T^* = \arg\max_{0 \le i \le G} \left\{ \frac{\mathbf{E}[A] + \hat{\epsilon}\mathbf{E}[X]}{1 - \mathbf{E}[\bar{A}]} \le \sum_{u=i}^{G} b_u \right\} \tag{2}$$

$$q^* = \frac{1}{b_{T^*}} \left[\frac{\mathbf{E}[A] + \hat{\epsilon}\mathbf{E}[X]}{1 - \mathbf{E}[\bar{A}]} - \sum_{u=T^*+1}^{G} b_u \right], \tag{3}$$

where

$$\widehat{\epsilon} = \min\left\{\frac{\epsilon}{G}, \frac{1-(\mathbf{E}[A]+\mathbf{E}[\bar{A}])}{\mathbf{E}[X]}\right\}. \quad (4)$$

Let Δ^* denote the two-threshold policy(T^*, q^*). Then, the throughput of Δ^* can be lower bounded as

$$\Omega^{\Delta^*} \geq \frac{(T^* q^* b_{T^*} + \sum_{u=T^*+1}^{G} u b_u)(1-\mathbf{E}[\bar{A}])}{\mathbf{E}[X]} - \epsilon \quad \text{w.p. 1.} \quad (5)$$

We now motivate the above result in a special case. Let the sender sample the system in every slot, i.e., every packet has length 1, and there is no back-off. The number of packets served per unit time under any stable policy is equal to the arrival rate λ. A stable policy Δ_1 can achieve throughput higher than that of another stable policy Δ_2 only by attaining a higher reward for infinitely many packets. Now, for a two-threshold policy(T, q), $\Delta_{(T,q)}$, the sender transmits a packet for every busy sample that has $T+1$ or more ready receivers. Each of the remaining packets achieves reward T. Thus, some other policy can achieve a higher reward infinitely often only by sending packets in the idle samples of $\Delta_{(T,q)}$. The choice of parameters T^* and q^* in Theorem 1 ensures that the ratio of idle samples and the total samples is less than or equal to $\frac{\epsilon}{G}$. Now, even if all the idle samples of $\Delta_{(T,q)}$ have G ready receivers and if all of these samples are used by some other policy, then the improvement in the throughput is not more than ϵ. Thus, $\Delta_{(T,q)}$ is ϵ-throughput optimal.

The computation of the optimal parameters provided in equations (2) and (3) of Theorem 1 depends on $\vec{b}, \mathbf{E}[A], \mathbf{E}[\bar{A}]$ and $\mathbf{E}[X]$. We assume that the sender knows the values of $\mathbf{E}[A], \mathbf{E}[\bar{A}]$ and $\mathbf{E}[X]$. Next, we design an adaptive approach $\hat{\Delta}(t)$ that computes T^* and q^* accurately without prior knowledge of $\vec{b}$.

Let $n_r(t)$ be the number of samples with r ready receivers and $n(t)$ be the number of samples until time t. Let $\vec{\hat{b}}(t) = [\frac{n_0(t)}{n(t)} \; \frac{n_1(t)}{n(t)} \; \dots \; \frac{n_G(t)}{n(t)}]$. Now, estimates $\hat{T}(t)$ and $\hat{q}(t)$ for T^*, q^* are computed by substituting $\vec{b}(t)$ with its estimate $\vec{\hat{b}}(t)$ in equations (2) and (3). Since the MC $\hat{P}$ is ergodic,

$$\lim_{t\to\infty} \frac{n_r(t)}{n(t)} = b_r \qquad \text{w.p. 1 for every } r \in \{0, 1, \dots, G\}$$

The above discussion motivates the following result.

Theorem 2: Let there exist a T such that $\sum_{u=T+1}^{G} b_u < \frac{\mathbf{E}[A]}{1-\mathbf{E}[A]} < \sum_{u=T}^{G} b_u$.
Let $0 < \epsilon < \frac{1-\mathbf{E}[\bar{A}]}{\mathbf{E}[X]}\left(\sum_{u=T}^{G} b_u - \frac{\mathbf{E}[A]}{1-\mathbf{E}[A]}\right)$.

$$\text{Then, } \lim_{t\to\infty} \hat{T}(t) = T^* \quad \text{w.p. 1,} \quad (6)$$

$$\text{and, } \lim_{t\to\infty} \hat{q}(t) = q^* \quad \text{w.p. 1.} \quad (7)$$

Since $\lambda > 0, EX > 0$ and $EV > 0$, $0 < \frac{\mathbf{E}[A]}{1-\mathbf{E}[A]} < 1$ from (1). In addition, $b_u > 0$, for each u, $\sum_{u=T+1}^{G} b_u = 0$ for $T = G$ and $\sum_{u=T}^{G} b_u = 1$ for $T = 0$. Thus, there always exists a T such that $\sum_{u=T+1}^{G} b_u < \frac{\mathbf{E}[A]}{1-\mathbf{E}[A]} \leq \sum_{u=T}^{G} b_u$. We assume a strict inequality in the theorem.

The outputs $\hat{T}(t)$ and $\hat{q}(t)$ converge to T^* and q^*, respectively, even when $\mathbf{E}[X], \mathbf{E}[A]$ and $\mathbf{E}[\bar{A}]$ are substituted with their estimates in equations (2) and (3).

B. Throughput Optimality subject to Loss Constraint

For stable systems throughput maximization is equivalent to loss minimization. Thus, we will assume a saturated system throughout this subsection. We show that for appropriate choice of parameters T and q, a two-threshold policy(T, q), Δ_S^*, maximizes the throughput subject to any given loss constraint. First, we quantify the throughput for a two-threshold policy(T, q).

Proposition 1: For a saturated system, the throughput $(\Omega^{\Delta_{(T,q)}})$ and the expected reward achieved per transmission $(R^{\Delta_{(T,q)}})$ by a two threshold policy(T, q), $\Delta_{(T,q)}$, are as follows.

$$\Omega^{\Delta_{(T,q)}} = \frac{qTb_T + \sum_{r=T+1}^{G} r b_r}{\mathbf{E}[X] + \mathbf{E}[V](qb_T + \sum_{u=T+1}^{G} b_u)} \text{ w.p. 1, and}$$

$$R^{\Delta_{(T,q)}} = \frac{qTb_T + \sum_{r=T+1}^{G} r b_r}{qb_T + \sum_{u=T+1}^{G} b_u} \qquad \text{w.p. 1.}$$

We next show that a single-threshold policy maximizes the throughput in a saturated system.

Theorem 3: A single-threshold policy(T_S) maximizes the throughput in a saturated system, if

$$T_S = \arg\max_{0 \leq T \leq G} \left\{\Omega^{\Delta_{(T,1)}}\right\}.$$

The optimum threshold T_S can now be computed from Proposition 1.

Under any policy Δ, the loss is $G - R^{\Delta}$ like in stable systems. The difference with stable systems is that the throughput is not a monotonic increasing function of the expected reward. This explains the observation that a throughput optimal transmission policy need not minimize the loss in a saturated system unlike that in a stable systems. Thus, the policy proposed in Theorem 3 may not satisfy the loss constraint. We present a two-threshold transmission policy that maximizes the throughput subject to satisfying the loss constraint.

Theorem 4: In a saturated system, the two-threshold policy(T_S^*, q_S^*) Δ_S^*, presented in Figure 6), maximizes the throughput subject to satisfying the loss constraint L. The throughput attained by Δ_S^* is

$$\Omega^{\Delta_S^*} = \frac{q_S^* T_S^* b_{T_S^*} + \sum_{r=T_S^*+1}^{G} r b_r}{\mathbf{E}[X] + \mathbf{E}[V](q_S^* b_{T_S^*} + \sum_{u=T_S^*+1}^{G} b_u)} \qquad \text{w.p. 1.}$$

The expressions for the throughput and reward per packet of a two-threshold policy, which are obtained in Proposition 1, can be used in the computations in Figure 6. We motivate Theorem 4 now. We show that it is sufficient to consider only the two threshold policies(T, q) that satisfy the loss constraint. The loss constraint is satisfied if (a) $T \geq T_M$, or (b) $T = T_M$ and $q \leq q_2$. It can be shown that Δ_1 maximizes the throughput in the first case, and Δ_2 maximizes the throughput in the second case.

Adaptive policies can be designed for saturated systems like in Section IV-A. Let $\hat{T}_S(t)$, $\hat{T}_S^*(t)$ and $\hat{q}_S^*(t)$ be the values of the parameters obtained in Theorem 3 and Figure 6, if $\vec{b}$ is replaced by its estimate $\vec{\hat{b}}(t)$. If $R^{\Delta_{(0,1)}} > G - L$, or there exists a T such that $R^{\Delta_{(T,1)}} < G - L < R^{\Delta_{(T+1,1)}}$, then $\lim_{t\to\infty} \hat{T}_S(t) = T_S$, $\lim_{t\to\infty} \hat{T}_S^*(t) = T_S^*$, and $\lim_{t\to\infty} \hat{q}_S^*(t) = q_S^*$ w.p. 1.

A two-threshold(T_S^*, q_S^*) policy Δ_S^* for saturated systems
begin
if $(R^{\Delta(0,1)} \geq G - L)$ **then**
$T_S^* = \arg\max_{0 \leq T \leq G} \left\{ \Omega^{\Delta(T,1)} \right\}$ and $q_S^* = 1$,
else
Let $T_M = \arg\max_{0 \leq u \leq G} \{R^{\Delta(u,1)} < G - L\}$.
Let $T_1 = \arg\max_{T_M + 1 \leq u \leq G} \{\Omega^{\Delta(u,1)}\}$.
Let Δ_1 be a single-threshold policy with parameter T_1,
Let Δ_2 be a two-threshold policy with parameters (T_M, q_2), where
$q_2 = \frac{\sum_{r=T_M+1}^{G} (r+L-G) b_r}{b_{T_M}(G-L-T_M)}$.
if $(\Omega^{\Delta_1} \geq \Omega^{\Delta_2})$ **then**
$T_S^* = T_1$ and $q_S^* = 1$,
else
$T_S^* = T_M$ and $q_S^* = q_2$.
end

Fig. 6. Figure shows the pseudo code of an algorithm that maximizes the throughput subject to loss constraints in a saturated system

V. Performance Analysis of the Existing Multiple Access Multicast Strategies

A. *Threshold*-0 *Multicast* (Δ_B)

In Threshold-0 multicast (Δ_B), the sender transmits a packet at every busy sample without querying the receiver about its readiness. It is thus a two-threshold policy(T, q) with $T = 0$ and $q = 1$. IEEE 802.11 implements Δ_B.

Theorem 5: If $\mathbf{E}[A] + \mathbf{E}[\bar{A}] < 1$, then Δ_B is stable, and w.p. 1, $R^{\Delta_B} = \sum_{u=0}^{G} u b_u$, and $\Omega^{\Delta_B} = \lambda \sum_{u=0}^{G} u b_u$.

B. *Threshold*-1 *Multicast* (Δ_C)

In Threshold-1 multicast (Δ_C) [16], the sender transmits a packet whenever at least one receiver is ready. It is thus a two-threshold policy(T, q) with $T = 1$ and $q = 1$.

Theorem 6: Let, $n_{u,I}^{\Delta_C}(t)$, $u > 0$, be the number of samples until time t such that u receivers are ready, and the sender's queue is empty. If $\frac{\mathbf{E}[A]}{1-\mathbf{E}[\bar{A}]} < \sum_{u=1}^{G} b_u$, then Δ_C is stable, and

$$\Omega^{\Delta_C} = \frac{1 - \mathbf{E}[\bar{A}]}{\mathbf{E}[X]} \sum_{u=1}^{G} u b_u - \lim_{t \to \infty} \frac{1}{t} \sum_{u=1}^{G} u n_{u,I}^{\Delta_C}(t) \text{ w.p. 1.}$$

C. *Unicast Based Multicast* (Δ_U)

In unicast based multicast (Δ_U), the sender transmits a packet separately to each receiver in round robin manner. A packet is delivered to a receiver only when it is ready. Hence, Δ_U has no loss. Thus, Δ_U has a high throughput (λG) in its stability region. A necessary condition for the system to be stable under Δ_U is that $\mathbf{E}[A] + \mathbf{E}[\bar{A}] < \frac{1}{G}$, since a lower bound on the mean service time is $G(\mathbf{E}[X] + \mathbf{E}[V])$. Thus, stability region of Δ_U is at most $\frac{1}{G}$ times that of Δ^*.

In its stability region, Δ_U can attain throughput higher than that of Δ^*. Note that Δ^* maximizes throughput among the policies that transmit a packet once in the MAC layer. Thus, our framework *does not* apply to Δ_U as it transmits a packet several times. Multiple transmissions of a packet result in high power consumption, low stability region, and high network load. The increase in network load decreases the throughput for other nodes.

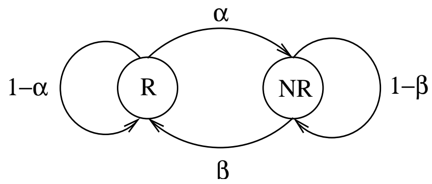


Fig. 7. Figure shows the readiness process for a receiver. The states R and NR denote ready and not ready states, respectively.

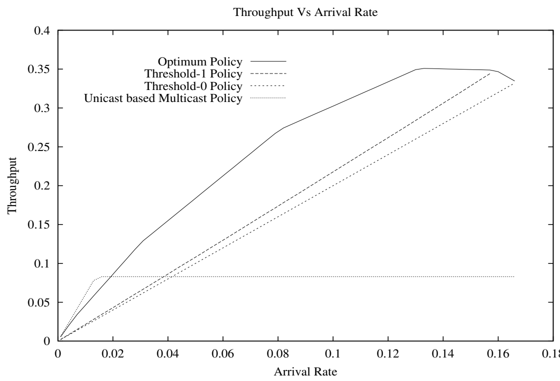
D. *Performance Comparison of the Policies*

We compare the performances in the special case that the readiness process for each receiver is Markovian, and independent of the readiness process of any other receiver. For each receiver, let α (resp., β) denote the transition probability from *ready* (resp., *not ready*) to *not ready* (resp., *ready*) state (Figure 7). We select $G = 6$, $\alpha = 0.2$, $\beta = 0.1$ and $\mathbf{E}[V] = \mathbf{E}[X] = 3$. The throughput of Δ_U is $\min\{\lambda G, \frac{1}{EV + EX(1 + \frac{\alpha}{(1-\beta)(\beta+\alpha)})}\}$. For every $u \in \{0, 1, \ldots, G\}$, $b_u = \binom{G}{u}(\frac{\beta}{\alpha+\beta})^u(\frac{\alpha}{\alpha+\beta})^{G-u}$. We numerically compute the throughput and expected reward per packet under Δ^* and Δ_B using Theorems 1 and 5. We simulate the performance of Δ_C as we have not been able to obtain closed form expressions for $\lim_{t \to \infty} \frac{1}{t} \sum_{u=1}^{G} u n_{u,I}^{\Delta_C}(t)$ for an arbitrary TPM, $\hat{P}$ (Theorem 6).

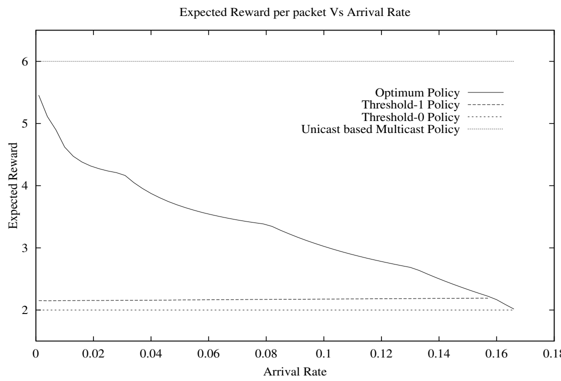
We plot in Figure 8 the throughput and the expected reward per packet of the policies Δ^* (Two Threshold policy), Δ_B (Threshold-0 policy), Δ_C (Threshold-1 policy) and Δ_U (Unicast based multicast policy) as a function of the arrival rate λ. We consider only the stability region of the system. Note that both the throughput and the expected reward per packet are much higher for Δ^* than that for Δ_B and Δ_C. Since the expected loss is the group size minus the expected reward, the loss under Δ^* is significantly lower than that under Δ_B and Δ_C.

Recall that the throughput for a stable policy is a product of the arrival rate and the expected reward per packet. Figure 8(b) illustrates that the expected reward for Δ^* decreases with increase in the arrival rate. This happens because the threshold decreases as the arrival rate increases so as to ensure stability. From Figure 8(a), the throughput increases until a certain value of the arrival rate, i.e., for $\lambda < 0.125$. In this region, the increase of the arrival rate compensates for the decrease of the expected reward. The transmission decision and hence the expected reward per packet of Δ_B and Δ_C does not depend on the arrival rate. Hence, the throughputs of Δ_B and Δ_C increase linearly with the increase in the arrival rate. The policy Δ_C attains a higher expected reward per packet and a higher throughput than that attained by Δ_B, since unlike Δ_B, Δ_C transmits only when at least one receiver is ready. However, Δ_B has a stability region larger than that of Δ_C; Δ^* and Δ_B attain the stability region of the system (Theorems 1,5,6).

The policy Δ_U incurs zero loss; therefore, in its stability region attains a throughput slightly higher than that of Δ^*. However, Δ_U has considerably small stability region and its throughput saturates outside its stability region, i.e., for



(a) Throughput Vs. λ



(b) Expected Reward per Packet Vs. λ

Fig. 8. Figures show the throughput and the expected reward per packet of various multiple access multicast strategies as a function of the arrival rate. Here, optimum policy refers to the two-threshold policy Δ^* that maximizes the throughput subject to stability.

TABLE II
A TABLE SUMMARIZING THE PERFORMANCES OF DIFFERENT POLICIES

Policy	Throughput	Loss	Stability Region
Δ^*	High	Low (can be controlled)	Attains stability region of the system
Δ_B	Low	High	Attains stability region of the system
Δ_C	Low	High	Medium
Δ_U	High in its stability region, and low otherwise	Zero	Low

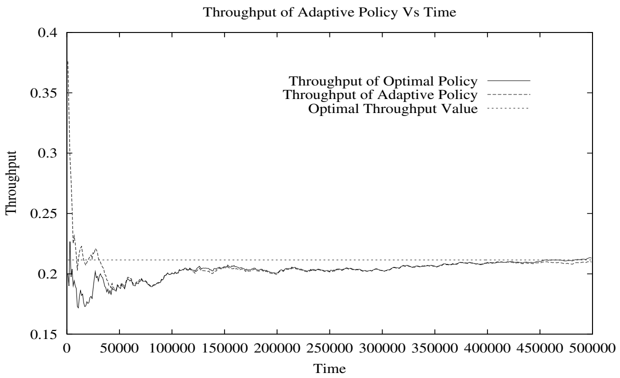


Fig. 9. Figure shows the throughputs of the optimal and adaptive policies as a function of time. Here, $\lambda = 0.055$.

$\lambda > 0.017$ in Figure 8(a). The policy Δ^* incurs some loss (Figure 8(b)), but achieves substantially larger stability region ($\lambda < 0.167$) and a much higher throughput. Thus, the loss tolerance of the system can be exploited to provide a significant gain in throughput. We summarize the performance comparisons in Table II.

Finally, Figure 9 shows the convergence of the throughputs of the optimal and adaptive policies. The figure illustrates that both the policies have similar convergence times.

We do not compare the performance of the various policies outside the stability region of the system, since the performance objective there is to maximize the throughput subject to loss constraints, and Δ_B and Δ_C suffer high loss in this region.

VI. Optimal Transmission Strategies for Multiple Readiness States

We generalize the analytical framework to allow three or more states of the channel to a receiver. The readiness process is an irreducible, aperiodic, time-homogeneous discrete time Markov chain with L states where the lth state is $[j_{l,1}, \ldots, j_{l,G}]$, $j_{l,i}$ is the probability of error-free reception of a packet at the ith receiver in the lth state. Recall that earlier $j_{l,i} \in \{0, 1\}$. The expected reward associated with a state l, r_l is the expected number of receivers that receive a packet without any error in state l, $r_l = \sum_{i=1}^{G} j_{l,i}$. Let $r_0 < r_1 < \ldots < r_K$ be the distinct values of the rewards for different states ($K \le L$). Let b_u be the steady state probability that the readiness process is in a state where the reward is r_u, $u \in \{0, \ldots, K\}$.

A single-threshold transmission policy(T) transmits a packet only when the expected reward is greater than or equal to r_T, $0 \le T \le K$. The other definitions can be generalized similarly. We now generalize the analytical results presented earlier.

Theorem 7 (Genaralization of Theorem 1): Let the stability condition (1) hold. For every $\epsilon > 0$, there exists a choice of parameters T and q such that the corresponding two-threshold policy(T, q) is ϵ-throughput optimal with probability 1. The optimal values of parameters T and q are

$$T_g^* = \arg \max_{0 \le i \le K} \left\{ \frac{\mathbf{E}[A] + \widehat{\epsilon}\mathbf{E}[X]}{1 - \mathbf{E}[\bar{A}]} \le \sum_{u=i}^{K} b_u \right\},$$

$$q_g^* = \frac{1}{b_{T_g^*}} \left[\frac{\mathbf{E}[A] + \widehat{\epsilon}\mathbf{E}[X]}{1 - \mathbf{E}[\bar{A}]} - \sum_{u=T_g^*+1}^{K} b_u \right],$$

where

$$\widehat{\epsilon} = \min \left\{ \frac{\epsilon}{G}, \frac{1 - (\mathbf{E}[A] + \mathbf{E}[\bar{A}])}{\mathbf{E}[X]} \right\}.$$

Let Δ_g^* denote the two-threshold policy(T_g^*, q_g^*). Then, the throughput of Δ_g^* can be lower bounded as

$$\Omega^{\Delta_g^*} \geq \frac{(r_{T_g^*} q^* b_{T_g^*} + \sum_{u=T_g^*+1}^{K} r_u b_u)(1 - \mathbf{E}[\bar{A}])}{\mathbf{E}[X]} - \epsilon \text{ w.p. } 1.$$

Proposition 2 (Genaralization of Proposition 1): For a saturated system, the throughput $(\Omega^{\Delta_{(T,q)}})$ and the mean reward achieved per transmission $(R^{\Delta_{(T,q)}})$ by a two threshold policy(T, q), $\Delta_{(T,q)}$, are

$$\Omega^{\Delta_{(T,q)}} = \frac{q r_T b_T + \sum_{u=T+1}^{K} r_u b_u}{\mathbf{E}[X] + \mathbf{E}[V](q b_T + \sum_{u=T+1}^{K} b_u)} \text{ w.p. 1, and}$$

$$R^{\Delta_{(T,q)}} = \frac{q r_T b_T + \sum_{u=T+1}^{K} r_u b_u}{q b_T + \sum_{u=T+1}^{K} b_u} \text{ w.p. } 1.$$

Theorem 8 (Genaralization of Theorem 3): A single-threshold policy(T_S^g) attains the maximum possible throughput in a saturated system, if $T_S^g = \arg\max_{0 \leq T \leq K} \left\{ \Omega^{\Delta_{(T,1)}} \right\}$.

The algorithm presented in Figure 6 needs to be modified as follows. The maximizations should be for thresholds less than or equal to K instead of G. Now, $q_2 = \frac{\sum_{u=T_M+1}^{K} (r_u + L - G) b_u}{b_{T_M}(G - L - T_M)}$. The modified algorithm maximizes the throughput subject to satisfying the required loss constraint L in a saturated system. The maximum throughput is

$$\Omega^{\Delta_S^*} = \frac{q_S^* r_{T_S^*} b_{T_S^*} + \sum_{u=T_S^*+1}^{K} r_u b_u}{\mathbf{E}[X] + \mathbf{E}[V](q_S^* b_{T_S^*} + \sum_{u=T_S^*+1}^{K} b_u)} \quad \text{w.p. } 1.$$

The estimates of $\vec{b}, \mathbf{E}[X], \mathbf{E}[V]$ and $\mathbf{E}[A], \mathbf{E}[\bar{A}]$ can be used to compute the parameters of the optimal strategies in an adaptive manner as discussed before.

Theorem 9 (Genaralization of Theorem 5): If $\mathbf{E}[A] + \mathbf{E}[\bar{A}] < 1$, then Δ_B is stable, and w.p. 1, $R^{\Delta_B} = \sum_{u=0}^{K} r_u b_u$, and $\Omega^{\Delta_B} = \lambda \sum_{u=0}^{K} r_u b_u$.

Theorem 10 (Genaralization of Theorem 6): Let $n_{u,I}^{\Delta_C}(t)$, $u > 0$, be the number of samples until time t such that u receivers are ready and the sender's queue is empty. If $\frac{\mathbf{E}[A]}{1 - \mathbf{E}[\bar{A}]} < \sum_{u=1}^{K} b_u$,, then the policy Δ_C is stable, and

$$\Omega^{\Delta_C} = \frac{1 - \mathbf{E}[\bar{A}]}{\mathbf{E}[X]} \sum_{u=1}^{K} r_u b_u - \lim_{t \to \infty} \frac{1}{t} \sum_{u=1}^{K} r_u n_{u,I}^{\Delta_C}(t) \text{ w.p. } 1.$$

The proofs for the Proposition and Theorems presented in this section are similar to those for the special case of on-off readiness states.

Furthermore, Proposition 1, Theorems 3, 4, and their generalizations hold for any stationary ergodic readiness process. The Theorems 1, 5, 6 and their generalizations hold for any stationary ergodic process that satisfies the following additional condition. Let $n_u(t)$ denote the number of sample points with u ready receivers, and let b_u be the steady state probability of u receivers being ready. Then the additional condition is that the empirical distribution $(\frac{n_u(t)}{t})$ converges to the stationary distribution (b_u) at rate $o(\frac{1}{t})$, i.e., there exists a $\delta > 0$ such that for every $\epsilon > 0$ there exists time $\widehat{t}$ such that for every $t > \widehat{t}$

$$\mathbf{P} \left\{ \left| \frac{n_u(t)}{t} - b_u \right| > \epsilon \right\} < \frac{1}{t^{1+\delta}}.$$

VII. Conclusion and Discussion

We design transmission strategies for MAC layer multicast that maximize the utilization of available bandwidth. We establish that the relation between QoS parameters like throughput, loss and stability changes due to the multicast nature of transmissions. The maximization of the throughput is no longer equivalent to attaining the stability region of the system or the minimization of loss. We show that threshold based transmission policies maximize the throughput subject to stability and loss constraints, and present an adaptive approach to compute the parameters of the optimum policies without any knowledge of system statistics. The sender only needs to know the number of ready receivers in each slot, and not the individual readiness states of the receivers so as to implement the threshold based policies. We analyze other existing policies, and show using numerical performance evaluations that the optimal policies provide significantly more efficient usage of bandwidth. Our investigation provides the first step towards understanding MAC layer multicast. We however considered somewhat restricted systems and made some simplifying assumptions, which we elaborate on next. We believe that our results will provide foundation for addressing more general versions of this problem.

Our simplifying assumption was that the receiver readiness process does not depend on the sender's transmission policy. In practice it may however be possible to design a transmission policy that generates favorable readiness states and thereby improve throughput. But designing such a policy is likely to involve coordination among the senders. This may be difficult to attain in ad-hoc networks that do not support centralized control. Our initial research suggests that designing such a policy is NP-hard, but efficient approximation algorithms may exist. This intellectually challenging problem remains open.

The restriction we considered was that each packet can be transmitted only once at the MAC layer. Now we discuss the open problems that arise when this restriction is removed. When a sender can transmit a packet multiple times, its throughput may increase. But retransmissions also increase the network load, and thereby adversely affect the overall readiness process, which in turn reduces the throughput. Multiple transmissions also increase power consumption of each sender. The major challenges in designing optimal retransmission schemes are to determine (a) the number of transmissions for each packet and (b) when to retransmit the packets. Next, we discuss possible approaches for these problems.

Suppose the maximum rate at which the sender can transmit is $\widehat{\lambda}$ which is determined by the network load and power constraints. Then for a stable system the expected number of transmissions (K) allowed per packet is $\frac{\widehat{\lambda}}{\lambda}$. It is however not clear how $\widehat{\lambda}$ can be determined. We now discuss how to formulate the problem of computing the optimal retransmission strategy that maximizes throughput subject to stability assuming that $\widehat{\lambda}$ and hence K is known. Let $\mathcal{A}$ denote a power set of set $\{1, 2, \ldots, G\}$ minus the set itself. The sender maintains $2^G - 1$ queues, where each queue corresponds to a member of $\mathcal{A}$. A queue indexed by a set $A \in \mathcal{A}$ contains packets that have already been received by the receivers in A. At every sample point, a transmission policy decides whether to transmit, and which queue to serve if it transmits. The decisions should maximize the throughput subject to (a) maintaining the transmission rate below $\widehat{\lambda}$ and (b) attaining bounded expected queue length in every queue. Our initial investigation indicates that this problem is *NP-hard*, which is intuitive as the number of queues is exponential in the number of receivers. It may be worthwhile to investigate approximation algorithms.

We have studied a simpler version of the problem, where the packets are served in a first come first serve (FCFS) order, i.e., the sender has a single queue and can serve only the head-of-line (HoL) packet [22], [23]. We assume that each packet can be transmitted at most K times and must be delivered to at least Z receivers, where K, Z are given constants. Using a Markov decision process (MDP) based formulation, we prove that a threshold type policy minimizes the total time for delivering a packet to at least Z receivers. For each retransmission a new threshold is selected, depending on the number of previous transmissions of the packet and the reward received in those transmissions.

ACKNOWLEDGMENT

The authors wish to thank the anonymous reviewers and Associate Editor Galen Sasaki for suggestions that improved the paper.

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TABLE III
NOTATIONS USED IN THE PROOFS

Notation	Definition
V_k	The length of the kth packet
X_k	The length of the kth back-off duration
$z^{\Delta}(t)$	Total number of packets departed until time t under policy Δ
$n^{\Delta}(t)$	Total number of sample points until time t under policy Δ
$n_u^{\Delta}(t)$	Total number of sample points until time t with u ready receivers under policy Δ

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APPENDIX

NOTATIONS AND GENERAL PROPERTIES

We present some general properties of the receiver readiness process and various transmission strategies which we will use in the proofs. We summarize frequently used notations in Table III.

In any transmission policy, the sender samples the number of ready receivers and subsequently may or may not transmit based on the readiness state, packet availability and the transmission rule. If the sender decides to transmit, then the receiver readiness states do not change until the transmission is over. Irrespective of the transmission decision, the sender backs off for a random time interval before sampling the receiver readiness process again. The receiver readiness process observed at

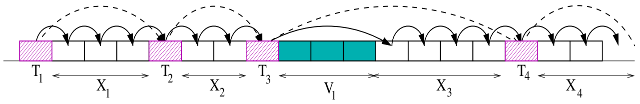

Fig. 10. Figure shows typical transitions of the actual and the sampled receiver readiness processes. A box indicates a time slot. The T_i's, V_i's and X_i's denote the sample points, duration of transmission and duration of back-off, respectively. Solid (dashed) arcs indicate transitions in the actual (sampled) receiver readiness process.

the sampling points is the *sampled receiver readiness process* (Figure 10).

Property 1: The sampled receiver readiness process is a finite state, irreducible and aperiodic DTMC. The unique stationary distribution of the sampled process is equal to that of the original receiver readiness process ($\vec{B}$).

Proof: The above property follows since the receiver readiness process does not change during a packet transmission, the back-off intervals are i.i.d., and the original receiver readiness process is a finite state, irreducible and aperiodic DTMC. ∎

Property 2: For any transmission policy Δ, $n^{\Delta}(t) \to \infty$ as $t \to \infty$ w.p. 1.

Proof: We observe from Figure 10 that

$$\sum_{k=1}^{z^{\Delta}(t)} V_k + \sum_{k=1}^{n^{\Delta}(t)} X_k \leq t \leq \sum_{k=1}^{z^{\Delta}(t)+1} V_k + \sum_{k=1}^{n^{\Delta}(t)+1} X_k. \quad (8)$$

Since the sender backs-off after every transmission, $z^{\Delta}(t) \leq n^{\Delta}(t)$. From the right inequality in (8), $t \leq \sum_{k=1}^{n^{\Delta}(t)+1}(V_k + X_k)$. The result follows since $\mathbf{E}[V+X] < \infty$, i.e., $\mathbf{P}\{V+X = \infty\} = 0$. ∎

Property 3: For any transmission policy Δ, if $\lim_{t\to\infty} \frac{z^{\Delta}(t)}{t}$ exists w.p. 1, then

$$\lim_{t\to\infty} \frac{n^{\Delta}(t)}{t}\mathbf{E}[X] + \lim_{t\to\infty} \frac{z^{\Delta}(t)}{t}\mathbf{E}[V] = 1 \qquad \text{w.p. } 1.$$

Proof: Divide all sides by t in (8) and take limit as $t \to \infty$. Since V_k's and X_k's are i.i.d. with finite mean, the result follows from Kolmogorov's Strong Law of Large Numbers (KSLLN). ∎

Property 4: Let $\lim_{t\to\infty} \frac{z^{\Delta_1}(t)}{t} = \lim_{t\to\infty} \frac{z^{\Delta_2}(t)}{t}$ w.p. 1, for two transmission policies Δ_1 and Δ_2. Then, $\lim_{t\to\infty} \frac{n^{\Delta_1}(t)}{t} = \lim_{t\to\infty} \frac{n^{\Delta_2}(t)}{t}$ w.p. 1.

Proof: Follows from Property 3. ∎

Property 5: For every stable policy Δ,

$$\lim_{t\to\infty} \frac{z^{\Delta}(t)}{t} = \lambda \qquad \text{w.p. } 1 \quad (9)$$

$$\lim_{t\to\infty} \frac{n^{\Delta}(t)}{t} = \frac{1-\mathbf{E}[\bar{A}]}{\mathbf{E}[X]} \qquad \text{w.p. } 1. \quad (10)$$

Proof: Clearly, (9) holds. Equation (10) follows from Property 3 and (9).

Property 6: For any policy Δ, and $u \in \{0,1,\dots,G\}$, $\lim_{t\to\infty} \frac{n_u^{\Delta}(t)}{n^{\Delta}(t)} = b_u > 0$ w.p. 1.

Proof: Follows from Property 1 and the ergodicity of the sampled receiver readiness process. ∎

Property 7: Consider a saturated system where the sender always has a packet to transmit. Let Δ be a two-threshold policy(T,q). Then,

$$\lim_{t\to\infty} \frac{t}{n^{\Delta}(t)} = \mathbf{E}[X] + \mathbf{E}[V]\left(qb_T + \sum_{r=T+1}^{G} b_r\right) \quad (11)$$

$$\lim_{t\to\infty} \frac{z^{\Delta}(t)}{t} = \frac{qb_T + \sum_{r=T+1}^{G} b_r}{\mathbf{E}[X] + \mathbf{E}[V]\left(qb_T + \sum_{r=T+1}^{G} b_r\right)} \quad (12)$$

Proof: Every sample with T ready receivers corresponds to a packet transmission with probability q. Let $S_T^{\Delta}(t)$ be the number of such samples before time t. Every sample with $T+1$ or more ready receivers corresponds to a packet transmission. There is a random back-off before every new sample. Hence, for every t, the following relations hold.

$$t \geq \sum_{k=1}^{n^{\Delta}(t)} X_k + \sum_{k=1}^{S_T^{\Delta}(t)} V_k + \sum_{r=T+1}^{G} \sum_{k=1}^{n_r^{\Delta}(t)} V_k \quad (13)$$

$$t \leq \sum_{k=1}^{n^{\Delta}(t)+1} X_k + \sum_{k=1}^{S_T^{\Delta}(t)+1} V_k + \sum_{r=T+1}^{G} \sum_{k=1}^{n_r^{\Delta}(t)+1} V_k. \quad (14)$$

Here, X_k and V_k are i.i.d. sequences. Inequality (11) follows by dividing both sides of (13) and (14) by t, taking the limit as $t \to \infty$, using property 6, $\lim_{t\to\infty} n_r^{\Delta}(t) = \infty$ w.p. 1 for every $r \in \{0\dots,G\}$ and KSLLN in (13) and (14). From Properties 1, 2 and 6, $\lim_{t\to\infty} n_r^{\Delta}(t) = \infty$ w.p. 1 for every $r \in \{0\dots,G\}$. Next,

$$z^{\Delta}(t) = qn_T^{\Delta}(t) + \sum_{r=T+1}^{G} n_r^{\Delta}(t)$$

$$\frac{z^{\Delta}(t)}{t} = \frac{qn_T^{\Delta}(t) + \sum_{r=T+1}^{G} n_r^{\Delta}(t)}{n^{\Delta}(t)} \frac{n^{\Delta}(t)}{t}$$

$$= \frac{qb_T + \sum_{u=T+1}^{G} b_u}{\mathbf{E}[X] + \mathbf{E}[V]\left(qb_T + \sum_{u=T+1}^{G} b_u\right)} \text{ w. p. } 1$$

(from Property 6 and (11)).

Thus (12) follows. ∎

Proof for ϵ-Throughput Optimality of the Policy Δ^* (Theorem 1)

We prove the ϵ-throughput optimality of the two-threshold policy Δ^* in the following four steps. (a) In Lemma 1, we obtain a sufficient condition for the stability of a two-threshold policy(T,q). (b) In Lemma 2, we obtain a lower bound on the throughput of a stable two-threshold policy(T,q). (c) In Lemma 3, we obtain an upper bound on the throughput of any stable policy. (d) We use results obtained in Lemmas 1, 2 and 3 to show that for every $\epsilon > 0$, Δ^* is a stable policy that provides throughput more than the highest throughput possible for any stable policy minus ϵ. We first state and prove the supporting Lemmas 1, 2 and 3 in Subsection A. We prove Theorem 1 in Subsection B.

A. Proof of supporting Lemmas

Lemma 1: A two-threshold policy(T,q) (Δ) is stable if

$$\frac{\mathbf{E}[A]}{(1-\mathbf{E}[\bar{A}])} < qb_T + \sum_{u=T+1}^{G} b_u. \quad (15)$$

Proof: Let (15) hold. Let $\mathcal{B}$ denote a random variable indicating the length of an arbitrary busy period under Δ. We show that $\mathbf{E}[\mathcal{B}] < \infty$. The lemma follows.

The number of arrivals in time slot t is A_t. The number of departures until time t is $z^{\Delta}(t)$. Without loss of generality, we assume that the busy period under consideration starts in slot 1, i.e., $A_1 > 0$. We first consider a fictitious system in which the sender's queue is never empty. Let $D^{\Delta}(t)$ denote the number of departures until time t under two-threshold policy Δ in the fictitious system. We assume that both the actual and the fictitious systems start with the same receiver readiness state. For any sample path ω, if $\mathcal{B}(\omega) \geq t$,

$$z^{\Delta}(\omega, t) = D^{\Delta}(\omega, t). \tag{16}$$

We note that

$$\begin{aligned}\lim_{t\to\infty} \frac{D^{\Delta}(t)}{t} &= \frac{qb_T + \sum_{u=T+1}^{G} b_u}{\mathbf{E}[X] + (qb_T + \sum_{u=T+1}^{G} b_u)\mathbf{E}[V]} \\ &\qquad \text{(from (12) in Property 7)} \\ &> \lambda. \end{aligned}\tag{17}$$

Inequality (17) follows from (15) since $\mathbf{E}[A] = \lambda \mathbf{E}[X]$ and $\mathbf{E}[\bar{A}] = \lambda \mathbf{E}[V]$. Hence, there exists a $\delta > 0$, such that

$$\lim_{t\to\infty} \frac{D^{\Delta}(t)}{t} = \lambda + \delta \quad \text{w.p. 1.} \tag{18}$$

We use (18) for the fictitious system, to show that the expected length of a busy period is bounded in the actual system. Consider an event where the busy period under consideration is larger than t.

$$\begin{aligned}&\{\omega : \mathcal{B}(\omega) \geq t\} \\ &= \left\{ \bigcap_{\widehat{t}=1}^{t} \{\omega : \sum_{l=1}^{\widehat{t}} A_l(\omega) - z^{\Delta}(\omega, \widehat{t}) > 0\} \right\} \\ &= \left\{ \bigcap_{\widehat{t}=1}^{t} \{\omega : \sum_{l=1}^{\widehat{t}} A_l(\omega) - D^{\Delta}(\omega, \widehat{t}) > 0\} \right\}. \end{aligned}\tag{19}$$

The last equality follows from (16). Thus,

$$\begin{aligned}&\mathbf{P}\{\mathcal{B} \geq t\} \\ &= \mathbf{P}\left\{ \bigcap_{\widehat{t}=1}^{t} \{\sum_{l=1}^{\widehat{t}} A_l - z^{\Delta}(\widehat{t}) > 0\} \right\} \\ &= \mathbf{P}\left\{ \bigcap_{\widehat{t}=1}^{t} \{\sum_{l=1}^{\widehat{t}} A_l - D^{\Delta}(\widehat{t}) > 0\} \right\} \quad \text{(from (19))} \\ &\leq \mathbf{P}\left\{ \sum_{l=1}^{t} A_l - D^{\Delta}(t) > 0 \right\} \\ &= \mathbf{P}\left\{ \frac{1}{t}\sum_{l=1}^{t} A_l - \frac{D^{\Delta}(t)}{t} > 0 \right\} \\ &= 1 - \mathbf{P}\left\{ \frac{1}{t}\sum_{l=1}^{t} A_l - \frac{D^{\Delta}(t)}{t} \leq 0 \right\} \\ &\leq 1 - \mathbf{P}\left\{ \{\frac{1}{t}\sum_{l=1}^{t} A_l \leq \lambda + \frac{\delta}{2}\} \cap \{\frac{D^{\Delta}(t)}{t} \geq \lambda + \frac{\delta}{2}\} \right\} \\ &= \mathbf{P}\left\{ \{\frac{1}{t}\sum_{l=1}^{t} A_l > \lambda + \frac{\delta}{2}\} \cup \{\frac{D^{\Delta}(t)}{t} < \lambda + \frac{\delta}{2}\} \right\} \\ &\leq \mathbf{P}\left\{ \frac{1}{t}\sum_{l=1}^{t} A_l > \lambda + \frac{\delta}{2} \right\} + \mathbf{P}\left\{ \frac{D^{\Delta}(t)}{t} < \lambda + \frac{\delta}{2} \right\}. \end{aligned}\tag{20}$$

Using exponentially fast convergence of empirical distribution to the unique stationary distribution for ergodic MC's [24],

$$\sum_{t=1}^{\infty} \mathbf{P}\left\{ \frac{1}{t}\sum_{l=1}^{t} A_l > \lambda + \frac{\delta}{2} \right\} < \infty, \tag{21}$$

$$\sum_{t=1}^{\infty} \mathbf{P}\left\{ \frac{D^{\Delta}(t)}{t} < \lambda + \frac{\delta}{2} \right\} < \infty \quad \text{(from (18)).} \tag{22}$$

From (20), (21) and (22),

$$\mathbf{E}[\mathcal{B}] = \sum_{t=1}^{\infty} \mathbf{P}\{\mathcal{B} \geq t\} < \infty.$$

This proves the Lemma. ∎

In the next Lemma, we obtain a lower bound on the throughput of a stable two-threshold policy(T, q).

Lemma 2: Let Δ denote a two-threshold policy(T, q) that satisfies (15). Then,

$$\Omega^{\Delta} \geq \frac{(Tqb_T + \sum_{u=T+1}^{G} ub_u)(1 - \mathbf{E}[\bar{A}])}{\mathbf{E}[X]} - G\widehat{\delta} \quad \text{w.p. 1,} \tag{23}$$

where $\widehat{\delta} = \delta\left[1 + \frac{(qb_T + \sum_{u=T+1}^{G} b_u)\mathbf{E}[V]}{\mathbf{E}[X]}\right]$, and $\delta = \frac{qb_T + \sum_{u=T+1}^{G} b_u}{\mathbf{E}[X] + (qb_T + \sum_{u=T+1}^{G} b_u)\mathbf{E}[V]} - \lambda$.

Proof: Let $\tilde{n}^{\Delta}(t)$ be the total number of samples where the sender decides to transmit. This happens with probability 1 (q) if the number of ready receivers is more than (equal to) T. Even if a sender decides to transmit, it will not transmit if a packet is not available. From Property 6,

$$\lim_{t\to\infty} \frac{\tilde{n}^{\Delta}(t)}{n^{\Delta}(t)} = qb_T + \sum_{u=T+1}^{G} b_u. \tag{24}$$

Let $\tilde{n}_u^{\Delta}(t)$ be the total number of samples with u ready receivers and the sender decides to transmit. Then, from Property 6,

$$\lim_{t\to\infty} \frac{\tilde{n}_u^{\Delta}(t)}{n^{\Delta}(t)} = \begin{cases} b_u & \text{if } u > T \\ qb_T & \text{if } u = T. \end{cases} \tag{25}$$

Furthermore,

$$\begin{aligned}\lim_{t\to\infty} \frac{\tilde{n}^{\Delta}(t)}{t} &= \lim_{t\to\infty} \frac{\tilde{n}^{\Delta}(t)}{n^{\Delta}(t)} \lim_{t\to\infty} \frac{n^{\Delta}(t)}{t} \\ &= \frac{(qb_T + \sum_{u=T+1}^{G} b_u)(1 - \mathbf{E}[\bar{A}])}{\mathbf{E}[X]} \quad \text{w.p. 1} \\ &\qquad \text{(from (10) and (24)).}\end{aligned}\tag{26}$$

From (17) in Lemma 1, the stability condition (15) implies that there exists $\delta > 0$ such that

$$\frac{qb_T + \sum_{u=T+1}^{G} b_u}{\mathbf{E}[X] + (qb_T + \sum_{u=T+1}^{G} b_u)\mathbf{E}[V]} = \lambda + \delta. \tag{27}$$

Then,

$$\begin{aligned}
&\lim_{t\to\infty}\frac{\tilde{n}^{\Delta}(t)}{t}\\
&= \lambda+\delta\left[1+\frac{(qb_T+\sum_{u=T+1}^{G}b_u)\mathbf{E}[V]}{\mathbf{E}[X]}\right] \quad \text{w.p. 1}\\
&\qquad\qquad \text{(from (26) and (27))}\\
&= \lambda+\widehat{\delta} \qquad \text{w.p. 1.} \qquad (28)
\end{aligned}$$

Let $\tilde{n}^{\Delta}_{u,\mathcal{B}}(t)$ be the total number of samples until time t such that u receivers are ready, sender decides to transmit and the sender's queue is non-empty. Here, $\tilde{n}^{\Delta}_{u,\mathcal{B}}(t)=0$ for every $u<T$. Let $\tilde{n}^{\Delta}_{u,I}(t)$ be the total number of samples until time t such that u receivers are ready, sender decides to transmit but the sender's queue is empty. Then, $\tilde{n}^{\Delta}_{u}(t)=\tilde{n}^{\Delta}_{u,\mathcal{B}}(t)+\tilde{n}^{\Delta}_{u,I}(t)$. In addition, let $\tilde{n}^{\Delta}_{\mathcal{B}}(t)=\sum_{u=T}^{G}\tilde{n}^{\Delta}_{u,\mathcal{B}}(t)$, and $\tilde{n}^{\Delta}_{I}(t)=\sum_{u=T}^{G}\tilde{n}^{\Delta}_{u,I}(t)$. Then,

$$\tilde{n}^{\Delta}(t)=\tilde{n}^{\Delta}_{\mathcal{B}}(t)+\tilde{n}^{\Delta}_{I}(t). \qquad (29)$$

We note that $\tilde{n}^{\Delta}_{\mathcal{B}}(t)=z^{\Delta}(t)$. From (9),

$$\lim_{t\to\infty}\frac{\tilde{n}^{\Delta}_{\mathcal{B}}(t)}{t}=\lim_{t\to\infty}\frac{z^{\Delta}(t)}{t}=\lambda \quad \text{w.p. 1.} \qquad (30)$$

In addition,

$$\lim_{t\to\infty}\frac{\tilde{n}^{\Delta}_{I}(t)}{t}=\widehat{\delta} \ \text{ w.p. 1 (from (28), (29) and (30)).} \qquad (31)$$

Throughput of the policy Δ is given as follows.

$$\begin{aligned}
\Omega^{\Delta} &= \lim_{t\to\infty}\frac{1}{t}\sum_{u=T}^{G}u\tilde{n}^{\Delta}_{u,\mathcal{B}}(t)\\
&= \lim_{t\to\infty}\frac{1}{t}\sum_{u=T}^{G}u(\tilde{n}^{\Delta}_{u}(t)-\tilde{n}^{\Delta}_{u,I}(t))\\
&= \lim_{t\to\infty}\frac{1}{t}\sum_{u=T}^{G}u\tilde{n}^{\Delta}_{u}(t)-\lim_{t\to\infty}\frac{1}{t}\sum_{u=T}^{G}u\tilde{n}^{\Delta}_{u,I}(t). \qquad (32)
\end{aligned}$$

Now,

$$\begin{aligned}
\lim_{t\to\infty}\frac{\tilde{n}^{\Delta}_{u}(t)}{t} &= \lim_{t\to\infty}\frac{\tilde{n}^{\Delta}_{u}(t)}{n^{\Delta}(t)}\lim_{t\to\infty}\frac{n^{\Delta}(t)}{t}\\
&= b_u\frac{1-\mathbf{E}[\bar{A}]}{\mathbf{E}[X]} \qquad \text{if } u>T \qquad (33)\\
&\qquad\qquad \text{(from (10) and (25)),}\\
\lim_{t\to\infty}\frac{\tilde{n}^{\Delta}_{T}(t)}{t} &= qb_T\frac{1-\mathbf{E}[\bar{A}]}{\mathbf{E}[X]} \qquad (34)\\
&\qquad\qquad \text{(from (10) and (25)).}
\end{aligned}$$

It follows that

$$\begin{aligned}
\Omega^{\Delta} &\geq \lim_{t\to\infty}\frac{1}{t}\sum_{u=T}^{G}u\tilde{n}^{\Delta}_{u}(t)-G\lim_{t\to\infty}\frac{1}{t}\sum_{u=T}^{G}\tilde{n}^{\Delta}_{u,I}(t) \qquad (35)\\
&\qquad\qquad \text{(from (32)),}\\
&= \lim_{t\to\infty}\frac{1}{t}\sum_{u=T}^{G}u\tilde{n}^{\Delta}_{u}(t)-G\lim_{t\to\infty}\frac{\tilde{n}^{\Delta}_{I}(t)}{t}\\
&= \frac{(Tqb_T+\sum_{u=T+1}^{G}ub_u)(1-\mathbf{E}[\bar{A}])}{\mathbf{E}[X]}-G\widehat{\delta} \quad \text{w.p. 1}\\
&\qquad\qquad \text{(from (31), (33) and (34)).}
\end{aligned}$$

The lemma follows. ■

In the following lemma, we prove an upper bound on the throughput of any stable policy.

Lemma 3: Let Δ_1 be an arbitrary stable transmission policy. Then,

$$\Omega^{\Delta_1}\leq\frac{(\widehat{T}\widehat{q}b_{\widehat{T}}+\sum_{u=\widehat{T}+1}^{G}ub_u)(1-\mathbf{E}[\bar{A}])}{\mathbf{E}[X]}, \qquad (36)$$

where $\widehat{T}\in\{0,\ldots,G\}$ and $\widehat{q}\in[0,1]$ are chosen such that

$$\widehat{q}b_{\widehat{T}}+\sum_{u=\widehat{T}+1}^{G}b_u=\frac{\mathbf{E}[A]}{1-\mathbf{E}[\bar{A}]}. \qquad (37)$$

Proof: From (1), for any policy to be stable we need $\frac{\mathbf{E}[A]}{1-\mathbf{E}[\bar{A}]}<1$. For any number $\sigma\in[0,1]$, we can find a threshold T and probability q such that $qb_T+\sum_{u=T+1}^{G}b_u=\sigma$ for any valid distribution $\vec{b}$. Hence, there exist $\widehat{T}$ and $\widehat{q}$ that satisfy (37).

Let $\mathbf{1}^{\Delta_1}_u(t)$ be an indicator such that $\mathbf{1}^{\Delta_1}_u(t)=1$ if queue is served and the reward achieved is u in time slot t; it is zero otherwise. Then,

$$z^{\Delta_1}(t)=\sum_{u=0}^{G}\sum_{l=0}^{t}\mathbf{1}^{\Delta_1}_u(l) \ \text{ and } \ n^{\Delta_1}_u(t)\geq\sum_{l=0}^{t}\mathbf{1}^{\Delta_1}_u(l).$$

Let us define

$$\gamma^{\Delta_1}_u=\lim_{t\to\infty}\frac{\sum_{l=0}^{t}\mathbf{1}^{\Delta_1}_u(l)}{t}.$$

Since Δ_1 is a stable policy, from (9) of Property 5,

$$\sum_{u=0}^{G}\gamma^{\Delta_1}_u=\lim_{t\to\infty}\frac{z^{\Delta_1}(t)}{t}=\lambda \quad \text{w.p. 1.} \qquad (38)$$

In addition,

$$\begin{aligned}
\gamma^{\Delta_1}_u &\leq \lim_{t\to\infty}\frac{n^{\Delta_1}_u(t)}{t}\\
&= \frac{b_u(1-\mathbf{E}[\bar{A}])}{\mathbf{E}[X]} \ \text{(from Property 6 and (10)).} \qquad (39)
\end{aligned}$$

In addition, on any sample path,

$$\Omega^{\Delta_1}=\sum_{u=0}^{G}u\gamma^{\Delta_1}_u. \qquad (40)$$

Now, the following Linear Program (LP) provides an upper bound on the throughput of Δ_1.

Maximize : $\sum_{u=0}^{G}u\gamma_u$

Subject to :

1) $\sum_{u=0}^{G}\gamma_u=\lambda$
2) $\gamma_u\leq\frac{b_u(1-\mathbf{E}[\bar{A}])}{\mathbf{E}[X]}$
3) $\gamma_u\geq 0$

The objective function follows from (40) and the constraints follow from (38) and (39). The linear program is a fractional knapsack problem [25] with knapsack volume λ units and $G+1$ items. The volume and the value per unit volume of the uth item are $\frac{b_u(1-\mathbf{E}[\bar{A}])}{\mathbf{E}[X]}$ and u respectively. The variables γ_u indicate the volume of item u put in the knapsack. The goal is to maximize the total value of items put in the knapsack ($\sum_{u=0}^{G} u\gamma_u$) without exceeding its volume λ (first constraint) and the volume of any item (second constraint). The optimum strategy is to put the items in the knapsack in descending order of their value per unit volume, e.g., first put the item G entirely, then $G+1$ etc., until the first constraint is violated [25]. The last item may be partially added in the knapsack. Hence, the optimal value of the objective function is $\frac{(\widehat{T}\widehat{q}b_{\widehat{T}}+\sum_{u=\widehat{T}+1}^{G} ub_u)(1-\mathbf{E}[\bar{A}])}{\mathbf{E}[X]}$, where $\widehat{T}$ and $\widehat{q}$ satisfy,

$$\begin{aligned}\frac{(\widehat{q}b_{\widehat{T}}+\sum_{u=\widehat{T}+1}^{G} b_u)(1-\mathbf{E}[\bar{A}])}{\mathbf{E}[X]} &= \lambda.\\ \Rightarrow \quad \widehat{q}b_{\widehat{T}}+\sum_{u=\widehat{T}+1}^{G} b_u &= \frac{\mathbf{E}[A]}{1-\mathbf{E}[\bar{A}]}.\end{aligned}$$

This proves the desired result. ■

B. Proof of Theorem 1

Proof: First, we show that Δ^* is a valid two-threshold policy, i.e., $T^* \in \{0,1,\ldots,G\}$ and $q^* \in [0,1]$. Since $\widehat{\epsilon} \le \frac{1-(\mathbf{E}[A]+\mathbf{E}[\bar{A}])}{\mathbf{E}[X]}$ from (4), $\frac{\mathbf{E}[A]+\widehat{\epsilon}\mathbf{E}[X]}{1-\mathbf{E}[\bar{A}]} \le 1$. Thus, $\sum_{u=0}^{G} b_u = 1 \ge \frac{\mathbf{E}[A]+\widehat{\epsilon}\mathbf{E}[X]}{1-\mathbf{E}[\bar{A}]}$. Hence, (2) results in a valid choice for T^*. Now, we show that $q^* \in [0,1]$. From (2),

$$\frac{\mathbf{E}[A]+\widehat{\epsilon}\mathbf{E}[X]}{1-\mathbf{E}[\bar{A}]} > \sum_{u=T^*+1}^{G} b_u \text{ and } b_{T^*} > 0.$$

Hence, from (3), $q^* > 0$. Now, we show that $q^* \le 1$ using contradiction. Let $q^* > 1$. From (3),

$$\begin{aligned}\frac{1}{b_{T^*}}\left[\frac{\mathbf{E}[A]+\widehat{\epsilon}\mathbf{E}[X]}{1-\mathbf{E}[\bar{A}]} - \sum_{u=T^*+1}^{G} b_u\right] &> 1.\\ \Rightarrow \quad \frac{\mathbf{E}[A]+\widehat{\epsilon}\mathbf{E}[X]}{1-\mathbf{E}[\bar{A}]} &> \sum_{u=T^*}^{G} b_u.\end{aligned}$$

This contradicts the choice of T^* in (2). Thus, Δ^* is a valid two-threshold policy.

Now, we show that Δ^* is a stable policy. From Lemma 1, it suffices to show (15), i.e.,

$$\frac{\mathbf{E}[A]}{1-\mathbf{E}[\bar{A}]} < q^* b_{T^*} + \sum_{u=T^*+1}^{G} b_u. \tag{41}$$

The proof is by contradiction. Let

$$\begin{aligned}\frac{\mathbf{E}[A]}{1-\mathbf{E}[\bar{A}]} &\ge q^* b_{T^*} + \sum_{u=T^*+1}^{G} b_u\\ &= \frac{\mathbf{E}[A]+\widehat{\epsilon}\mathbf{E}[X]}{1-\mathbf{E}[\bar{A}]} \text{ (from (3)).} \qquad (42)\\ \Rightarrow \quad \mathbf{E}[A] &\ge \mathbf{E}[A]+\widehat{\epsilon}\mathbf{E}[X].\end{aligned}$$

This contradicts the fact that $\mathbf{E}[X] \ge 1$ and $\widehat{\epsilon} > 0$. Now, $\widehat{\epsilon} > 0$ follows from (4) since $\epsilon > 0$ and $\mathbf{E}[A]+\mathbf{E}[\bar{A}] < 1$. Thus, (41) holds, and hence Δ^* is a stable policy.

Now, we show that Δ^* is ϵ-throughput optimal. From Lemma 2,

$$\Omega^{\Delta^*} \ge \frac{(1-\mathbf{E}[\bar{A}])P_{T^*}}{\mathbf{E}[X]} - G\widehat{\delta}, \tag{43}$$

where $P_{T^*} = T^* q^* b_{T^*} + \sum_{u=T^*+1}^{G} ub_u$ and $\widehat{\delta}$ is as defined in (23).

Let Δ_1 be an arbitrary stable policy. From Lemma 3 and (43),

$$\Omega^{\Delta_1} - \Omega^{\Delta^*} \le \frac{(1-\mathbf{E}[\bar{A}])(P_{\widehat{T}} - P_{T^*})}{\mathbf{E}[X]} + G\widehat{\delta}, \tag{44}$$

where $P_{\widehat{T}} = \widehat{q}b_{\widehat{T}}\widehat{T} + \sum_{u=\widehat{T}+1}^{G} ub_u$. Next, we show that

$$P_{\widehat{T}} - P_{T^*} \le 0.$$

From (37),

$$\sum_{u=\widehat{T}+1}^{G} b_u \le \frac{\mathbf{E}[A]}{1-\mathbf{E}[\bar{A}]} < \frac{\mathbf{E}[A]+\widehat{\epsilon}\mathbf{E}[X]}{1-\mathbf{E}[\bar{A}]} \le \sum_{u=T^*}^{G} b_u.$$

Thus, $\widehat{T}+1 > T^*$. First let $\widehat{T} = T^*$. Then,

$$\begin{aligned}&P_{\widehat{T}} - P_{T^*}\\ &= \left(\widehat{q}b_{\widehat{T}} - q^* b_{T^*}\right)T^*\\ &= T^*\left(\frac{\mathbf{E}[A]}{1-\mathbf{E}[\bar{A}]} - \frac{\mathbf{E}[A]+\widehat{\epsilon}\mathbf{E}[X]}{1-\mathbf{E}[\bar{A}]}\right)\\ &\quad +T^*\left(\sum_{u=T^*+1}^{G} b_u - \sum_{u=\widehat{T}+1}^{G} b_u\right)\\ &= -T^*\frac{\widehat{\epsilon}\mathbf{E}[X]}{1-\mathbf{E}[\bar{A}]} \quad (\text{since } \widehat{T} = T^*)\\ &\le 0.\end{aligned}$$

Now let $\widehat{T} > T^*$. Then,

$$\begin{aligned}P_{\widehat{T}} - P_{T^*} &= -q^* b_{T^*}T^* + \widehat{q}b_{\widehat{T}}\widehat{T} - \sum_{u=T^*+1}^{\widehat{T}} ub_u\\ &\le \widehat{q}b_{\widehat{T}}\widehat{T} - b_{\widehat{T}}\widehat{T} \quad \text{since } q^* \ge 0, \widehat{T} > T^*\\ &\le 0 \quad (\text{since } \widehat{q} \le 1).\end{aligned}$$

From (44) it follows that

$$\Omega^{\Delta_1} - \Omega^{\Delta^*} \le G\widehat{\delta}. \tag{45}$$

Now, we show that $\widehat{\delta} \le \frac{\epsilon}{G}$. From Lemma 2,

$$\widehat{\delta} = \delta\left[\frac{\mathbf{E}[X] + (q^* b_{T^*} + \sum_{u=T^*+1}^{G} b_u)\mathbf{E}[V]}{\mathbf{E}[X]}\right], \tag{46}$$

where $\delta = \frac{q^* b_{T^*} + \sum_{u=T^*+1}^{G} b_u}{\mathbf{E}[X] + (q^* b_{T^*} + \sum_{u=T^*+1}^{G} b_u)\mathbf{E}[V]} - \lambda$. Thus,

$$\begin{aligned}\widehat{\delta} &= \frac{(q^* b_{T^*} + \sum_{u=T^*+1}^{G} b_u)(1-\mathbf{E}[\bar{A}]) - \mathbf{E}[A]}{\mathbf{E}[X]}\\ &= \widehat{\epsilon} \qquad \text{(from (42))}\\ &\le \frac{\epsilon}{G} \qquad \text{(from (4)).}\end{aligned}$$

This proves that $\Omega^{\Delta_1} - \Omega^{\Delta^*} \le \epsilon$. The result follows since Δ_1 is an arbitrary stable policy.

The expression for the throughput of Δ^* follows from the lower bound in Lemma 2 and the fact that $\widehat{\delta} \le \epsilon/G$. ■

PROOF FOR CONVERGENCE OF THE PARAMETERS OF THE ADAPTIVE POLICY, $\hat{\Delta}(t)$ TO THOSE OF Δ^* (THEOREM 2)

Proof: We can show that $\hat{\Delta}(t)$ is a valid two-threshold policy for every t, using the fact that $\hat{\vec{b}}(t)$ is a probability distribution. The arguments are similar to those used for proving that Δ^* is a valid two-threshold policy in Theorem 1.

By assumption, there exists $T \in \{0, \ldots, G\}$ such that $\sum_{u=T+1}^{G} b_u < \frac{\mathbf{E}[A]}{1-\mathbf{E}[A]} < \sum_{u=T}^{G} b_u$. Clearly, there can be only one such T. Since $0 < \epsilon < \frac{1-\mathbf{E}[\bar{A}]}{\mathbf{E}[X]} \left(\sum_{u=T}^{G} b_u - \frac{\mathbf{E}[A]}{1-\mathbf{E}[A]} \right)$, and $0 < \hat{\epsilon} < \epsilon$, $\sum_{u=T+1}^{G} b_u < \frac{\mathbf{E}[A]+\hat{\epsilon}\mathbf{E}[X]}{1-\mathbf{E}[A]} < \sum_{u=T}^{G} b_u$. Let $\delta_1 = \sum_{u=T}^{G} b_u - \frac{\mathbf{E}[A]+\hat{\epsilon}\mathbf{E}[X]}{1-\mathbf{E}[A]}$, $\delta_2 = \frac{\mathbf{E}[A]}{1-\mathbf{E}[A]} - \sum_{u=T+1}^{G} b_u$, and $\delta = \min\{\delta_1, \delta_2\}$.

Since $\hat{b}_u(t)$ converges to b_u w.p. 1 for every u, there exists t' such that for every $t \geq t'$, $|b_u - \hat{b}_u(t)| < \frac{\delta}{2G}$, for every u, $0 \leq u \leq G$,

It follows that $\hat{T}(t) = T$ for all $t \geq t'$. Note that $T^* = T$. Thus (6) follows. Next,

$$\hat{q}(t) = \frac{1}{\hat{b}_{\hat{T}(t)}(t)} \left[\frac{\mathbf{E}[A]}{(1-\frac{\epsilon}{G})(1-\mathbf{E}[\bar{A}])} - \sum_{u=\hat{T}(t)+1}^{G} \hat{b}_u(t) \right]$$

(from equation (3))

$$\lim_{t\to\infty} \hat{q}(t) = \frac{1}{b_{T^*}} \left[\frac{\mathbf{E}[A]}{(1-\frac{\epsilon}{G})(1-\mathbf{E}[\bar{A}])} - \sum_{u=T^*+1}^{G} b_u \right] \text{ w.p. 1}$$
$$= q^*.$$

Thus, (7) follows. ■

PROOF OF THE ANALYTICAL RESULTS FOR A SATURATED SYSTEM (PROPOSITION 1, THEOREMS 3 AND 4)

We prove Proposition 1 in Subsection C, Theorems 3 and 4 in Subsections E and F respectively, and the supporting lemmas used in the proofs of Theorems 3 and 4 in Subsection D.

C. Proof of Proposition 1

Proof: Let Δ be a two-threshold policy(T, q). A sender always has a packet to transmit in a saturated system. The policy Δ transmits with probability q (1) for every sample with T ($T+1$ or more) ready receivers. Thus, the throughput of the policy is given as follows.

$$\begin{aligned}
\Omega^{\Delta} &= \lim_{t\to\infty} \frac{qTn_T^{\Delta}(t) + \sum_{r=T+1}^{G} rn_r^{\Delta}(t)}{t} \quad \text{w. p. 1} \\
&= \lim_{t\to\infty} \frac{qTn_T^{\Delta}(t) + \sum_{r=T+1}^{G} rn_r^{\Delta}(t)}{n^{\Delta}(t)} \lim_{t\to\infty} \frac{n^{\Delta}(t)}{t} \\
&= \frac{qTb_T + \sum_{r=T+1}^{G} rb_r}{\mathbf{E}[X] + \mathbf{E}[V](qb_T + \sum_{r=T+1}^{G} b_r)}
\end{aligned}$$

(from Property 6 and (11)).

The number of packets departed until time t satisfies $\lim_{t\to\infty} \frac{z^{\Delta}(t)}{t} = \lim_{t\to\infty} \frac{1}{t} \left[qn_T^{\Delta}(t) + \sum_{r=T+1}^{G} n_r^{\Delta}(t) \right]$, w.p. 1. Thus, the average reward per transmission is

$$\begin{aligned}
R^{\Delta} &= \lim_{t\to\infty} \frac{qTn_T^{\Delta}(t) + \sum_{r=T+1}^{G} rn_r^{\Delta}(t)}{z^{\Delta}(t)} \\
&= \lim_{t\to\infty} \frac{qTn_T^{\Delta}(t) + \sum_{r=T+1}^{G} rn_r^{\Delta}(t)}{qn_T^{\Delta}(t) + \sum_{r=T+1}^{G} n_r^{\Delta_T}(t)} \\
&= \lim_{t\to\infty} \frac{\left(qTn_T^{\Delta}(t) + \sum_{r=T+1}^{G} rn_r^{\Delta}(t) \right) / n^{\Delta}(t)}{\left(qn_T^{\Delta}(t) + \sum_{r=T+1}^{G} n_r^{\Delta}(t) \right) / n^{\Delta}(t)} \\
&= \frac{qTb_T + \sum_{r=T+1}^{G} rb_r}{qb_T + \sum_{r=T}^{G} b_r} \quad \text{w.p. 1 (from Property 6).}
\end{aligned}$$

■

D. Supporting Lemmas used in the proof of Theorems 3 and 4

Lemma 6 shows that a two-threshold policy maximizes throughput subject to a given loss constraint. Lemma 9 shows that a single-threshold policy maximizes throughput among all two-threshold policies with threshold greater than or equal to any given T. Theorem 3 follows from Lemmas 6 and 9.

Now we outline the proof of Theorem 4. Lemma 6 shows that there exists a two-threshold policy(T, q) that maximizes throughput subject to loss requirements. Lemma 7 shows that the reward of a two-threshold policy is a monotonic function of T and q. Refer to the algorithm in Figure 6. Lemma 9 shows that Δ_1 maximizes throughput among two-threshold policies with threshold greater than T_M. Lemma 8 shows that the throughput of either Δ_1 or Δ_2 is greater than or equal to that of any two-threshold policy(T_M, q) if $q \leq q_2$. Thus, Theorem 4 follows.

We now state and prove Lemmas 4 and 5 which we use in proving Lemma 6.

Lemma 4: The throughput of any transmission policy Δ_1 can be upper-bounded as follows.

$$\begin{aligned}
\Omega^{\Delta_1} \leq \min_{0 \leq T \leq G} \Bigg(&\lim_{t\to\infty} \frac{\sum_{u=T+1}^{G} un_u^{\Delta_1}(t)}{t} \\
&+ \lim_{t\to\infty} \frac{T \max\{0, z^{\Delta_1}(t) - \sum_{u=T+1}^{G} n_u^{\Delta_1}(t)\}}{t} \Bigg).
\end{aligned}$$

Proof: Consider an arbitrary T. The throughput of Δ_1 is upper bounded by that obtained if all the samples with $T+1$ or more ready receivers can be used for packet transmission, and the remaining packets (which may be zero) can be transmitted for a reward of T each. ■

The upper bound in the previous property may not be tight depending on the choice of T and the policy. For example, the remaining packets may receive reward less than T as the number of samples with T ready receivers may be less than the number of remaining packets. In addition, if the total number of samples before time t $\sum_{u=0}^{G} n_u(t)$ is high, then the number of packets transmitted before t may be upper bounded to a quantity less than $\sum_{u=T+1}^{G} n_u(t)$ depending on the packet lengths and the backoff intervals. Thus, the number of transmitted packets may be less than the total number of samples with $T+1$ or more ready receivers.

Lemma 5: The throughput of a two-threshold policy(T, q) Δ is given by

$$\Omega^{\Delta} = \lim_{t\to\infty} \frac{\sum_{u=T+1}^{G} u n_u^{\Delta}(t)}{t} + \lim_{t\to\infty} \frac{T\max\{0, z^{\Delta}(t) - \sum_{u=T+1}^{G} n_u^{\Delta}(t)\}}{t}.$$

Proof: The result follows since, for a saturated system, Δ transmits a packet for every sample with $T+1$ or more ready receivers and transmits the remaining packets for samples with exactly T ready receivers. ■

Lemma 6: Let $\mathcal{F}$ be the set of transmission policies whose loss is less than or equal to L in a saturated system. If $\mathcal{F} \neq \phi$, there exists a two-threshold policy which is in $\mathcal{F}$ and attains the maximum throughput in $\mathcal{F}$.

Proof: Let Δ_1 be an arbitrary transmission policy in $\mathcal{F}$. Let ω be a non-trivial sample path for this policy. The quantities $\Omega^{\Delta_1}, n^{\Delta_1}(t)/t, n_u^{\Delta_1}(t)/t, z^{\Delta_1}(t)/t$ are those for the sample path w. All of these quantities or their limits need not be equal or even exist for every non-trivial sample path. We assume that the reward per packet in sample path ω is lower bounded by $G - L$, i.e.,

$$\frac{\Omega^{\Delta_1}}{\lim_{t\to\infty}(z^{\Delta_1}/t)} \geq G - L. \tag{47}$$

Given the loss constraint, (47) holds for any ergodic transmission policy, and may hold even otherwise.

Now we construct a two-threshold policy(T, q) Δ_2, and show that $\Delta_2 \in \mathcal{F}$ and $\Omega^{\Delta_2} \geq \Omega^{\Delta_1}$. We choose

$$T = \arg\max_{0\leq i\leq G} \left\{ \sum_{u=i}^{G} b_u > \frac{\mathbf{E}[X]}{\lim_{t\to\infty} \frac{t}{z^{\Delta_1}(t)} - \mathbf{E}[V]} \right\}$$

$$q = \frac{1}{b_T}\left[\frac{\mathbf{E}[X]}{\lim_{t\to\infty} \frac{t}{z^{\Delta_1}(t)} - \mathbf{E}[V]} - \sum_{u=T+1}^{G} b_u \right].$$

From Property 7, for the above choice of parameters,

$$\lim_{t\to\infty} \frac{z^{\Delta_2}(t)}{t} = \lim_{t\to\infty} \frac{z^{\Delta_1}(t)}{t} \quad \text{w.p. 1.} \tag{48}$$

From (48) and Property 4,

$$\lim_{t\to\infty} \frac{n^{\Delta_1}(t)}{t} = \lim_{t\to\infty} \frac{n^{\Delta_2}(t)}{t} \quad \text{w.p. 1.} \tag{49}$$

From Property 6, for every $u \in \{0, \ldots, G\}$,

$$\lim_{t\to\infty} \frac{n_u^{\Delta_1}(t)}{n^{\Delta_1}(t)} = \lim_{t\to\infty} \frac{n_u^{\Delta_2}(t)}{n^{\Delta_2}(t)} \quad \text{w.p. 1.} \tag{50}$$

From Lemma 5, it follows that

$$\Omega^{\Delta_2} = \lim_{t\to\infty} \frac{\sum_{u=T+1}^{G} u n_u^{\Delta_2}(t)}{t} + \lim_{t\to\infty} \frac{T\max\{0, z^{\Delta_2}(t) - \sum_{u=T+1}^{G} n_u^{\Delta_2}(t)\}}{t} \quad \text{w.p. 1.}$$

Now, from Lemma 4,

$$\Omega^{\Delta_1} \leq \lim_{t\to\infty} \frac{\sum_{u=T+1}^{G} u n_u^{\Delta_1}(t)}{t} + \lim_{t\to\infty} \frac{T\max\{0, z^{\Delta_1}(t) - \sum_{u=T+1}^{G} n_u^{\Delta_1}(t)\}}{t}.$$

From (48), (49) and (50),

$$\Omega^{\Delta_2} \geq \Omega^{\Delta_1} \quad \text{w.p. 1} \tag{51}$$

Next, we note that

$$\begin{aligned} R^{\Delta_2} &= \frac{\Omega^{\Delta_2}}{\lim_{t\to\infty}(z^{\Delta_2}/t)} \quad \text{w.p. 1} \\ &\geq \frac{\Omega^{\Delta_1}}{\lim_{t\to\infty}(z^{\Delta_1}/t)} \quad \text{w.p. 1 (from (48) and (51))} \\ &\geq G - L \quad \text{w.p. 1} \quad \text{(from (47))}. \end{aligned}$$

Thus $\Delta_2 \in \mathcal{F}$.

Now consider the two-threshold strategy Δ which has the maximum throughput among all two-threshold policies in $\mathcal{F}$. There exists a two threshold policy which attains this maximum given the expressions for the throughput and expected reward per packet obtained in Proposition 1. It follows from (51) that the throughput under Δ is greater than or equal to that attained in any non-trivial sample path of an arbitrary transmission policy in $\mathcal{F}$. The result follows. ■

Henceforth, $\Delta_{(T,q)}$ will refer to an arbitrary two-threshold policy(T, q).

Lemma 7: If $T_1 > T_2$, or $T_1 = T_2, q_1 < q_2$, $R^{\Delta_{(T_1,q_1)}} \geq R^{\Delta_{(T_2,q_2)}}$. The inequality is strict in the last case.

Proof: If $T_1 = G$, then $q_1 = 1$, $R^{\Delta_{(T_1,q_1)}} = G \geq R^{\Delta_{(T_2,q_2)}}$, irrespective of the values of T_2 and q_2. In this case $q_1 \geq q_2$. Thus the lemma holds. Let $T_1 \leq G$. Therefore, $T_2 \leq G$.

Now, we state a property that we use in the following discussion. Let x, y and u be the real numbers. Then for every $u \in [0, 1]$,

$$\min\{x, y\} \leq ux + (1-u)y \leq \max\{x, y\}. \tag{52}$$

Let $T < G$ and $N_T = \sum_{r=T}^{G} r b_r$ and $D_T = \sum_{r=T}^{G} b_r$. Note that D_T is defined differently here. Note that

$$\frac{N_T}{D_T} < \frac{N_{T+1}}{D_{T+1}}. \tag{53}$$

Thus,

$$\begin{aligned} R^{\Delta_{(T,q)}} &= \frac{qN_T + (1-q)N_{T+1}}{qD_T + (1-q)D_{T+1}} \quad \text{w.p. 1} \quad (54) \\ &\qquad \text{(from Proposition 1),} \\ &\leq \max\left(\frac{N_T}{D_T}, \frac{N_{T+1}}{D_{T+1}}\right) \quad \text{(from (52))} \\ &= \frac{N_{T+1}}{D_{T+1}} \quad \text{(from (53)).} \quad (55) \end{aligned}$$

In addition,

$$\begin{aligned} R^{\Delta_{(T,q)}} &\geq \min\left(\frac{N_T}{D_T}, \frac{N_{T+1}}{D_{T+1}}\right) \quad \text{w.p. 1} \\ &\qquad \text{(from (54) and (52))} \\ &= \frac{N_T}{D_T} \quad \text{(from (53)).} \quad (56) \end{aligned}$$

From (56), it follows that

$$\begin{aligned} R^{\Delta_{(T_1,q)}} &\geq \frac{N_{T_1}}{D_{T_1}} \quad \text{(since } T_1 < G) \\ &\geq \frac{N_{T_2+1}}{D_{T_2+1}} \quad \text{(from (53) since } G > T_1 > T_2) \\ &\geq R^{\Delta_{(T_2,q)}} \quad \text{(from (55) since } T_2 < G) \quad (57) \end{aligned}$$

Now, we note that

$$\begin{aligned}
&R^{\Delta_{(T,q_1)}} - R^{\Delta_{(T,q_2)}} \\
&= \frac{D_T D_{T+1}(q_2 - q_1)\left(\frac{N_{T+1}}{D_{T+1}} - \frac{N_T}{D_T}\right)}{(q_1 D_T + (1-q_1)D_{T+1})(q_2 D_T + (1-q_2)D_{T+1})} \\
&> 0 \text{ if } T < G, \text{ (from (53) and since } q_1 < q_2). \qquad (58)
\end{aligned}$$

The lemma follows from (57) and (58) since $T_1 < G$. ■

Lemma 8: Let $0 \leq T < G$. Then, $\Omega^{\Delta_{(T,q)}} \leq \max(\Omega^{\Delta_{(T,1)}}, \Omega^{\Delta_{(T+1,1)}})$, for any $q \in [0,1]$. In addition, $\Omega^{\Delta_{(T,q_1)}} > \Omega^{\Delta_{(T,q_2)}}$ if $q_1 > q_2$ and $\Omega^{\Delta_{(T,1)}} > \Omega^{\Delta_{(T+1,1)}}$.

Proof: Let, $N_T = \sum_{r=T}^{G} r b_r$ and $D_T = \mathbf{E}[X] + \mathbf{E}[V]\sum_{r=T}^{G} b_r$. Note that the definition of D_T is different than that in Lemma 7.

$$\begin{aligned}
\Omega^{\Delta_{(T,q)}} &= \frac{qN_T + (1-q)N_{T+1}}{qD_T + (1-q)D_{T+1}} \quad \text{w.p. } 1 \qquad (59) \\
&\quad \text{(from Proposition 1)} \\
&\leq \max\left(\frac{N_T}{D_T}, \frac{N_{T+1}}{D_{T+1}}\right) \\
&= \max\left(\Omega^{\Delta_{(T,1)}}, \Omega^{\Delta_{(T+1,1)}}\right). \qquad (60)
\end{aligned}$$

Now, we note that

$$\begin{aligned}
&\Omega^{\Delta_{T,q_1}} - \Omega^{\Delta_{T,q_2}} \\
&= \frac{D_T D_{T+1}(q_2 - q_1)\left(\frac{N_{T+1}}{D_{T+1}} - \frac{N_T}{D_T}\right)}{(q_1 D_T + (1-q_1)D_{T+1})(q_2 D_T + (1-q_2)D_{T+1})} \\
&= \frac{D_T D_{T+1}(q_2 - q_1)\left(\Omega^{\Delta_{(T+1,1)}} - \Omega^{\Delta_{(T,1)}}\right)}{(q_1 D_T + (1-q_1)D_{T+1})(q_2 D_T + (1-q_2)D_{T+1})} \\
&\quad \text{(from (59))} \\
&> 0 \quad \text{(since } q_1 > q_2, \Omega^{\Delta_{(T,1)}} > \Omega^{\Delta_{(T+1,1)}}). \qquad (61)
\end{aligned}$$

The lemma follows from (60) and (61). ■

Lemma 9: Let $T_u = \arg\max_{T \in \{P,\ldots,G\}} \Omega^{\Delta_{(T,1)}}$. Then, $\Omega^{\Delta_{(T_u,1)}} = \max_{T \in \{P,\ldots,G\}} \Omega^{\Delta_{(T,q)}}$, for any $q \in [0,1]$.

Proof: The lemma follows if we show that $\Omega^{\Delta_{(T,q)}} \leq \Omega^{\Delta_{(T_u,1)}}$ for arbitrary $T \geq P$ and q. If $q = 1$, then the inequality holds from the choice of T_u. Let $q < 1$. Thus, $T < G$. Now, we note that

$$\begin{aligned}
\Omega^{\Delta_{(T,q)}} &\leq \max\left(\Omega^{\Delta_{(T,1)}}, \Omega^{\Delta_{(T+1,1)}}\right) \\
&\quad \text{(from Lemma 8 since } T < G) \\
&\leq \Omega^{\Delta_{(T_u,1)}} \\
&\quad \text{(from the choice of } T_u \text{ and since } T \geq P).
\end{aligned}$$

The lemma follows. ■

E. Proof of Theorem 3

Assume that there is no limitation on loss, i.e., $L = G$. Lemma 6 states that the throughput optimal policy lies in the class of two-threshold based policies. The result follows using $P = 0$ in Lemma 9. ■

F. Proof of Theorem 4

Proof: Refer to the algorithm in Figure 6. If $R^{\Delta_{(0,1)}} \geq G - L$, then $R^{\Delta_{(T,q)}} \geq G - L$, for any T, q, from Lemma 7. Thus, any single-threshold policy satisfies the loss constraint. Thus Δ_S^* is the desired policy from Theorem 3.

Now we assume that $R^{\Delta_{(0,1)}} < G - L$. Thus, T_M can be computed. Note that $R^{\Delta_{(G,1)}} = G \geq G - L$. Thus $T_M < G$. Hence, Δ_1 is a valid two-threshold policy. We will later show that $q_2 \in [0,1)$. Thus, Δ_2 is a valid two-threshold policy.

Using Proposition 1, it can be shown that $R^{\Delta_{(T_M,q_2)}} = G - L$. From Lemma 7, $R^{\Delta_{(T,q)}} \geq G - L$, only if $T > T_M$, or $T = T_M, q \leq q_2$. From Lemma 6, a two-threshold policy which maximizes throughput among the two-threshold policies $\Delta_{(T,q)}$ such that $T > T_M$, or $T = T_M, q \leq q_2$ is the desired policy.

$$\Omega^{\Delta_1} \geq \Omega^{\Delta_{(T,q)}} \text{ for any } q, T > T_M \quad \text{(from Lemma 9).} \qquad (62)$$

In view of (62), the result follows if we show that $\max\left(\Omega^{\Delta_1}, \Omega^{\Delta_2}\right) \geq \Omega^{\Delta_{(T_M,q)}}$ for any q, such that $q < q_2$. First let $\max(\Omega^{\Delta_{(T_M,1)}}, \Omega^{\Delta_{(T_M+1,1)}}) = \Omega^{\Delta_{(T_M+1,1)}}$. From Lemma 8, for any q,

$$\begin{aligned}
\Omega^{\Delta_{(T_M,q)}} &\leq \Omega^{\Delta_{(T_M+1,1)}} \\
&\leq \Omega^{\Delta_1} \quad \text{(from the choice of } \Delta_1). \qquad (63)
\end{aligned}$$

Now let $\Omega^{\Delta_{(T_M,1)}} > \Omega^{\Delta_{(T_M+1,1)}}$. From Lemma 8, since $\Delta_2 = \Delta_{(T_M,q_2)}$,

$$\Omega^{\Delta_{(T_M,q)}} \leq \Omega^{\Delta_2}, \text{ for any } q < q_2. \qquad (64)$$

The result follows from (63) and (64).

Finally we show that $q_2 \in [0,1)$. Note that $q_2 \geq 0$, if $\sum_{r=T_M+1}^{G}(r + L - G)b_r \geq 0$ and $b_{T_M}(G - L - T_M) > 0$. Let $\sum_{r=T_M+1}^{G}(r + L - G)b_r < 0$. Thus,

$$\begin{aligned}
\Rightarrow \quad \frac{\sum_{r=T_M+1}^{G} r b_r}{\sum_{r=T_M+1}^{G} b_r} &< G - L. \\
\Rightarrow \quad R^{\Delta_{(T_M+1,1)}} &< G - L.
\end{aligned}$$

This contradicts the choice of T_M. Thus, $\sum_{r=T_M+1}^{G}(r + L - G)b_r \geq 0$. Note that $b_{T_M} > 0$ from the choice of T_M. Now, let $b_{T_M}(G - L - T_M) \leq 0$. Then, $T_M \geq G - L$. Since $R^{\Delta_{(T_M,1)}} \geq T_M$, $R^{\Delta_{(T_M,1)}} \geq G - L$. This contradicts the choice of T_M. Thus, $b_{T_M}(G - L - T_M) > 0$. Now we show that $q_2 < 1$. If $q_2 \geq 1$, then

$$\begin{aligned}
\frac{\sum_{r=T_M+1}^{G}(r + L - G)b_r}{b_{T_M}(G - L - T_M)} &\geq 1 \\
\Rightarrow \quad R^{\Delta_{(T_M,1)}} &\geq G - L.
\end{aligned}$$

This contradicts the choice of T_M. Thus, $q_2 < 1$.

Finally, the expression for throughput follows from Proposition 1. ■

THROUGHPUT ANALYSIS FOR THRESHOLD-0 MULTICAST POLICY Δ_B (THEOREM 5)

The throughput and reward for Δ_B can be quantified by analyzing the DTMC representing the evolution of the system state under Δ_B.

Proof: We assume that the stability condition (1) holds, i.e., $\mathbf{E}[A] + \mathbf{E}[\bar{A}] < 1$. We model the process observed by the sender at the sampling instances under Δ_B as $Y_n^{\Delta_B} = (k, l, \vec{j})$, where k is the queue length, $l \in \mathcal{A}$ is the state of the arrival process, and $\vec{j} \in \mathcal{S}$ is the receiver readiness vector at the n^{th} sample, $\vec{j} = [j_1\ j_2\ \ldots\ j_G]$. Here, $\{Y_n^{\Delta_B} : n \geq 0\}$ is a Discrete Time Markov Chain (DTMC).

We assume that the number of packets arriving and the number of ready receivers are mutually independent in any slot. This assumption was not required in the earlier proofs. Let $P = [p_{\vec{i}\vec{j}}]$ be the TPM for the sampled readiness process. Here, P does not depend on the transmission strategy since the readiness process does not change during packet transmission. Note that P can be obtained from $\hat{P}$. Let q_{ijk} ($\hat{q}_{ijk}$) be the probability that the state of the arrival process is j at the end of a random backoff (packet transmission and subsequent backoff) interval when the state was i at the beginning of the backoff (packet transmission and subsequent backoff) interval and k packets arrive during the backoff (packet transmission and subsequent backoff) interval. The quantities $q_{ijk}, \hat{q}_{ijk}$ are well-defined as the packet lengths and the backoff intervals are i.i.d. and independent of the transmission policy.

Note that Δ_B is a two-threshold policy with $T = 0$ and $q = 1$. From Lemma 1, Δ_B is stable if

$$\mathbf{E}[A] + \mathbf{E}[\bar{A}] < 1.$$

Thus the DTMC $\{Y_n^{\Delta_B} : n \geq 1\}$ is positive recurrent.

Let $\vec{\tilde{\pi}} = [\tilde{\pi}(k,l,\vec{j})]$ be the unique stationary distribution of the DTMC $\{Y_n^{\Delta_B} : n \geq 1\}$. Then $\vec{\tilde{\pi}}$ is the unique solution of the following balance equations. For every $\vec{j} \in \mathcal{S}$, $l \in \mathcal{A}$,

$$\begin{aligned} \tilde{\pi}(0,l,\vec{j}) &= \sum_{m\in\mathcal{A}} \sum_{\vec{i}\in\mathcal{S}} \tilde{\pi}(0,m,\vec{i}) q_{ml0} p_{\vec{i}\vec{j}} \\ &\quad + \sum_{m\in\mathcal{A}} \sum_{\vec{i}\in\mathcal{S}} \tilde{\pi}(1,m,\vec{i}) \hat{q}_{ml0} p_{\vec{i}\vec{j}}. \qquad (65) \\ \tilde{\pi}(k,l,\vec{j}) &= \sum_{m\in\mathcal{A}} \sum_{\vec{i}\in\mathcal{S}} \tilde{\pi}(0,m,\vec{i}) q_{mlk} p_{\vec{i}\vec{j}} \\ &\quad + \sum_{d=0}^{k-1} \sum_{m\in\mathcal{A}} \sum_{\vec{i}\in\mathcal{S}} \tilde{\pi}(k-d,m,\vec{i}) \hat{q}_{ml(d+1)} p_{\vec{i}\vec{j}} \\ &\quad + \sum_{m\in\mathcal{A}} \sum_{\vec{i}\in\mathcal{S}} \tilde{\pi}(k+1,m,\vec{i}) \hat{q}_{ml0} p_{\vec{i}\vec{j}},\ \forall k > 0. \qquad (66) \end{aligned}$$

Normalization condition is the following.

$$\sum_{k=0}^{\infty} \sum_{\vec{j}\in\mathcal{S}} \sum_{l\in\mathcal{A}} \tilde{\pi}(k,l,\vec{j}) = 1. \qquad (67)$$

We next show that

$$\tilde{\pi}(k,l,\vec{j}) = \pi_{k,l} B_{\vec{j}}, \qquad (68)$$

where $\pi_{k,l}$ is a stationary measure for the following DTMC. Packets arrive at a server as per a Markov process with TPM Q ($\hat{Q}$) if the sender has an empty (nonempty) queue. The server serves packets every slot if it has a packet. The following balance equations describe the DTMC.

$$\begin{aligned} \pi_{0,l} &= \sum_{m\in\mathcal{A}} \pi_{0,m} q_{ml0} + \sum_{m\in\mathcal{A}} \pi_{1,m} \hat{q}_{ml0}. \qquad (69) \\ \pi_{k,l} &= \sum_{m\in\mathcal{A}} \pi_{0,m} q_{mlk} + \sum_{d=0}^{k-1} \sum_{m\in\mathcal{A}} \pi_{k-d,m} \hat{q}_{ml(d+1)} \\ &\quad + \sum_{m\in\mathcal{A}} \pi_{k+1,m} \hat{q}_{ml0},\ \forall k > 0. \qquad (70) \end{aligned}$$

Normalization condition is the following.

$$\sum_{k=0}^{\infty} \sum_{l\in\mathcal{A}} \tilde{\pi}(k,l) = 1. \qquad (71)$$

Using Lyapunov functions and Foster's Theorem [26], it can be shown that this DTMC is positive recurrent whenever $\mathbf{E}[A] + \mathbf{E}[\bar{A}] < 1$.

Now, we show that $\tilde{\pi}(k,j)$ given in (68) is a unique solution to the balance equations (65) to (67) of the DTMC $\{Y_n^{\Delta_B} : n \geq 1\}$. The claim follows from the following observations. (a) Since $\vec{B}$ is the stationary distribution of the receiver readiness process, $\sum_{\vec{i}\in\mathcal{S}} B_{\vec{i}} p_{\vec{i}\vec{j}} = B_{\vec{j}}$ and $\sum_{\vec{i}\in\mathcal{S}} B_{\vec{i}} = 1$. (b) If we substitute (68) in (65) (66) and (67), then we obtain (69), (70) and (71) respectively, by applying observation (a). (c) Since, $\vec{\pi}$ is the unique solution for (69) to (71), $\vec{\tilde{\pi}}$ is the unique solution for the balance equations of the DTMC $\{Y_n^{\Delta_B} : n \geq 1\}$.

Now, we obtain the throughput of the policy Δ_B. Let $\mathcal{C}$ be an event that a packet is transmitted under policy Δ_B. The expected reward for a transmission is given as

$$\begin{aligned} R^{\Delta_B} &= \sum_{u=0}^{G} u \mathbf{P}\{\text{reward} = u | \mathcal{C}\} \\ &= \sum_{u=0}^{G} u \mathbf{P}\{\text{number of ready receivers} = u | \mathcal{C}\}. \end{aligned}$$

From equation (68), the events $\mathcal{C}$ and that u receivers are ready are mutually independent under the steady state distribution of $\{Y_n^{\Delta_B} : n \geq 1\}$. Hence, $\mathbf{P}\{\text{number of ready receivers} = u|\mathcal{C}\} = b_u$ under the steady state distribution of $\{Y_n^{\Delta_B} : n \geq 1\}$. Thus,

$$R^{\Delta_B} = \sum_{u=0}^{G} u b_u. \qquad (72)$$

Since Δ_B is stable,

$$\begin{aligned} \Omega^{\Delta_B} &= \lambda R^{\Delta_B} \qquad \text{w.p. } 1 \\ &= \lambda \left(\sum_{u=0}^{G} u b_u \right). \qquad (73) \end{aligned}$$

The lemma follows from (72) and (73). ∎

Throughput Analysis for Threshold-1 Multicast Policy Δ_C (Theorem 6)

The proof follows from the stability condition obtained in Lemma 1 and the lower bound for throughput for arbitrary two-threshold policies obtained in (35) in the proof of Lemma 2.

Proof: The policy Δ_C is a two-threshold$(1,1)$ policy. Using $T=1, q=1$, in Lemma 1, Δ_C is stable if $\frac{\mathbf{E}[A]}{(1-\mathbf{E}[\bar{A}])} < \sum_{u=1}^{G} b_u$.

Let $\frac{\mathbf{E}[A]}{(1-\mathbf{E}[\bar{A}])} < \sum_{u=1}^{G} b_u$. The sender decides to transmit whenever at least one receiver is ready. However, the sender may not transmit even if it decides to, if its queue is empty. Thus, from (35), since $\tilde{n}_u^{\Delta_C}(t) = n_u^{\Delta_C}(t)$ and $\tilde{n}_{u,I}^{\Delta_C}(t) = n_{u,I}^{\Delta_C}(t)$,

$$\Omega^{\Delta_C} \geq \lim_{t\to\infty} \frac{1}{t}\sum_{u=1}^{G} u n_u^{\Delta_C}(t) - G \lim_{t\to\infty}\frac{1}{t}\sum_{u=1}^{G} \tilde{n}_{u,I}^{\Delta_C}(t). \quad (74)$$

$$\begin{aligned} \lim_{t\to\infty}\frac{1}{t}\sum_{u=1}^{G} u n_u^{\Delta_C}(t) &= \lim_{t\to\infty}\frac{n^{\Delta_C}(t)}{t}\lim_{t\to\infty}\sum_{u=1}^{G} u\frac{n_u^{\Delta_C}(t)}{n^{\Delta_C}(t)} \\ &= \sum_{u=1}^{G} u b_u \frac{1-\mathbf{E}[\bar{A}]}{\mathbf{E}[X]} \quad (75) \\ &\quad \text{(from (10) and Property 6).} \end{aligned}$$

$$\Omega^{\Delta_C} \geq \frac{\sum_{u=1}^{G} u b_u (1-\mathbf{E}[\bar{A}])}{\mathbf{E}[X]} - G \lim_{t\to\infty}\frac{1}{t}\sum_{u=1}^{G} \tilde{n}_{u,I}^{\Delta_C}(t).$$

The last inequality follows from (74) and (75). ■

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