MAC Layer Multicast: Theory, Approaches and Protocols

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Abstract

The paper is directed towards designing bandwidth efficient strategies for MAC layer multicast. Bandwidth efficiency of wireless multicast can be improved substantially by exploiting the fact that several receivers can be reached at the MAC layer by a single transmission. However, this feature can be effectively exploited only by resolving several decision problems and protocol challenges. Interestingly, fundamental relations between qos parameters such as throughput and loss change on account of the multicast nature of transmissions, e.g., a strategy which maximizes system throughput does not necessarily minimize packet loss. We explore this trade-off from a theoretical standpoint, and provide optimal transmission strategies which maximize system throughput subject to loss constraints. Simulation based performance evaluations demonstrate significant improvement in system performance as compared to existing MAC layer multicast strategies.

Method Keywords: Mathematical programming/optimization, Stochastic processes/Queueing theory

I. Introduction

A. Motivation

Wireless packet networks are gaining popularity since they are suitable for many applications. The applications include conference meetings, sensor networks, emergency operation in case of a natural disaster and military operations. Many of these applications need one to many (*multicast*) communication, e.g. group leader giving instructions to the group in case of military operation. In this paper we study various MAC layer issues pertaining to multicast communication in wireless networks.

Wireless communication is inherently broadcast in nature, i.e. all nodes in the transmission range of a sender can receive a transmission from the sender. This broadcast nature of the transmissions can be used to enhance bandwidth efficiency. Figure 1 illustrates the point. A single transmission from the sender S_1 can reach all its receivers $R_1, \dots, R_4$. Hence at the MAC layer, it suffices to transmit each packet only once in order to reach all the intended receivers. It is interesting to note that multicast in the wireless case can be more efficient than the wireline counterpart as in the latter case a packet has to be transmitted on each link (from sender to each of the receivers) separately and exploiting this may provide significant bandwidth and power advantage. *In this paper, we focus on designing an optimal MAC strategy for bandwidth efficient wireless multicast.*

Though the broadcast nature of wireless transmissions provides a possible approach to improve the efficiency of the multicast communication, it also imposes various difficulties. A multicast specific challenge is that some but *not all* the receivers may be ready to receive. For example, when sender S_2 is transmitting to receiver R_5, receiver R_2 can not receive the transmission from sender S_1 as both the transmissions will collide at R_2. However, receivers R_1, R_3 and R_4 can still receive the transmission. The readiness state of a receiver depends on the network load. In this scenario, it is not clear which transmission policy will be efficient. The policy decision in this case is whether the sender S_1 should transmit or it should wait till all the receivers are ready. A transmission policy, which does not transmit until all the receivers are ready, may provide low system throughput. On the other hand, if the sender transmits when only a few receivers are ready, then the transmitted packet will be lost at the receivers that were not ready. This packet loss may not be acceptable. *The challenge is to design a transmission policy that maximizes sender's throughput, while limiting the packet loss at every receiver below a required level.*

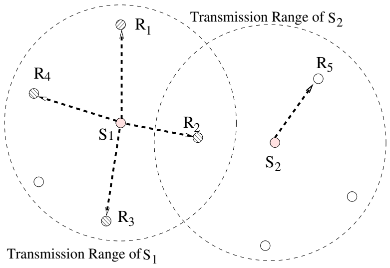

Fig. 1. An example to demonstrate the advantages and the challenges associated with wireless multicast. The figure shows two senders S_1, S_2 and 5 receivers R_1 to R_5. Dashed circle indicates the communication range of a sender.

Further, power constraint in the wireless case can make this problem even more challenging. A sender may achieve required loss characteristics by transmitting a packet several times till sufficient number of receivers receive the packet, e.g. in Figure 1 sender may transmit a packet to the receivers R_1, R_3 and R_4 even when the receiver R_2 is not ready and then retransmit the packet when receiver R_2 becomes ready. But each additional transmission causes additional power consumption at the sender, which imposes a limit on the number of such transmissions.

The above example shows that the design of MAC layer transmission strategy involves the following issues.

1) When should a sender transmit to maximize its throughput, while maintaining the packet loss at every receiver and the power consumption at the sender below an acceptable level? This is a *decision problem* at the sender. In general, an optimal transmission policy will depend on the network load, statistics of the packet arrival process and the power availability at the sender.
2) What information about the receiver's state should be available at the sender to obtain efficient solution for the decision problem? How can this information be made available? These are the *protocol design problems*.

We observe that a good solution to the issues mentioned above has a potential to improve the efficiency of the multicast communication in wireless networks. Surprisingly, these issues have not received much attention from the research community. In IEEE 802.11, multicast can be implemented in two possible ways. First, a multicast message is transmitted without exchanging *handshake* messages like Request to Send (RTS), Clear to Send (CTS) and Acknowledgment (ACK). Second, multicast can be implemented as several unicasts, i.e. a packet is transmitted to each receiver sequentially. We refer to the former as *broadcast based multicast* and the latter as *unicast based multicast*. The broadcast based multicast scheme may suffer from data packet collision as the handshakes are disabled. Hence, the scheme may incur high packet loss. Subsequent data recovery at a higher layer is not efficient if the packet loss at the MAC layer is high. In the unicast based multicast scheme, a sender transmits the same packet many times. This may result in a low system throughput and a high power consumption. In this paper, we compare the performance of these schemes with the proposed scheme and show that the proposed scheme provides a considerable performance improvement.

B. Literature Review

Multicasting has generated considerable interest in the wireless community. Most of the work in this area is focused on the network layer. Many effective multicast routing schemes have been proposed in the literature [2], [3], [4], [7], [10], [12], [15]. Further, energy efficient multicast routing schemes are proposed in [14], [18], [19]. Content based multicast in adhoc networks is proposed in [**?**]. In cellular networks, multicast is studied in [**?**]. A scheme for reliable multicast in wireless LANs is proposed in [6]. Issues in wireless LANs are much different than in general wireless networks as in former case every node is in the transmission range of every other node in the network. At the MAC layer, reliability of multicast communication is studied in [16], in which authors have proposed a MAC protocol that transmits a packet to each neighbor in a round robin fashion. We note that this scheme is conceptually similar to the unicast based multicast scheme.

C. Our Contributions

In this paper, first we consider a case, where power constraint does not allow packet retransmissions. For this case we propose a *throughput optimal transmission policy* under certain symmetry assumption on the network load. The proposed

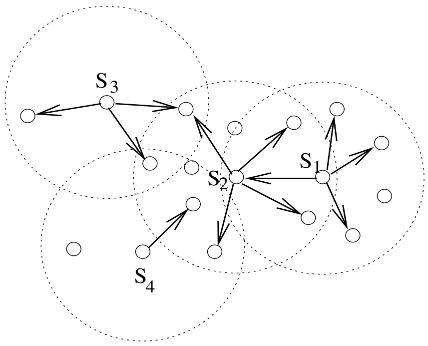

Fig. 2. Figure shows four MAC layer multicast sessions. Nodes S_1 to S_4 are the multicast senders that multicast packets to the receivers in their neighborhood. Arrows from a sender point to its designated receivers. Note that the node S_2 is a sender for multicast group 2, while it is a receiver for multicast group 1.

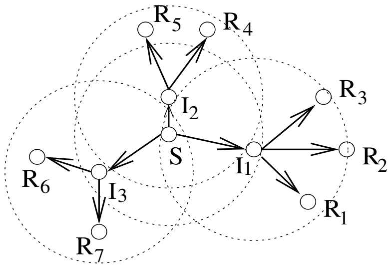

Fig. 3. Figure shows a multicast tree in a multi-hop wireless network. We observe that at the MAC layer, sender S multicasts to intermediate nodes I_1 to I_3, I_1 multicasts to the receivers R_1 to R_3, etc. So, there are four separate MAC sessions, namely, S to $[I_1, I_2, I_3]$, I_1 to $[R_1, R_2, R_3]$, I_2 to $[R_4, R_5]$ and I_3 to $[R_6, R_7]$.

throughput optimal transmission policy is threshold based, in which a sender transmits only when a certain number of receivers are ready to receive. This result is desirable because of two reasons. First, it provides a simple rule for a sender to decide when to transmit in order to achieve the maximum throughput. Second, it shows that a sender only needs the number of ready receivers and need not know which particular receivers are ready. As we show, this simplifies the protocol design problem significantly. We provide a simple way to compute the threshold values that achieve the maximum throughput.

Next we consider the case, where power constraint allows at most a fixed number of retransmissions. In this case, we propose a Dynamic Programming (DP) based approach to decide the number of retransmissions and the transmission strategy at each retransmission. This proposed approach can adapt the transmission strategy dynamically for different transmissions based on the outcomes of the past transmissions. This scheme achieves substantial performance gain even with a limited number of retransmissions.

Further, we note that the proposed transmission strategies are threshold based. So, for these strategies a sender needs a feedback about the number of ready receivers. We propose a few possible MAC protocols that allow a sender to estimate the number of ready receivers and subsequently decide the transmission strategy based on the threshold schemes. Performance evaluation via simulations demonstrate that the proposed MAC protocols attain a significant performance improvement over unicast and broadcast based multicast schemes.

The paper is arranged as follows. In section II we define the system model and notation. In section III we propose an optimal transmission policy when power constraint does not allow retransmissions. In section IV we propose a systematic approach using dynamic programming that improves the system throughput when the retransmissions are allowed. In section V we propose MAC protocols for obtaining feedback about the number of ready receivers. In section VI we present simulation results for the proposed MAC strategies.

II. System Model

We consider a wireless network with several MAC layer multicast sessions. A typical scenario is shown in Figure 2. Each multicast session comprises of a sender and a set of receivers (multicast group). Since we are considering MAC

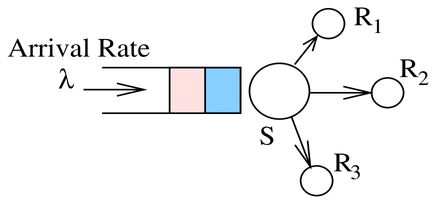

Fig. 4. Figure shows an isolated MAC layer multicast session that we study analytically. Node S is a sender, where packets arrive at the rate λ. Sender S transmits the packets to receivers R_1 to R_3. The receivers are ready to receive w.p. p in any slot.

layer multicast, all the receivers are within the transmission range of the sender. A sender of one multicast session can be a receiver for another multicast session. The scenario described above is motivated by the network level view of the multicast communication in a multi-hop wireless network. Figure 3 shows an example multicast routing tree from a source S to destinations R_1 to R_7 and the corresponding MAC layer multicasts.

We assume that all nodes in the system have omni-directional antennas. Receivers are not always ready to receive a transmission from a sender because of other transmissions in their neighborhood. Further, a receiver may not receive a transmission on account of power saving operations, which disable all communication abilities.

We primarily consider transmission of data traffic and assume FIFO selection of packets for transmission. We consider two Quality of Service (QoS) measures: (a) *system throughput* (b) *MAC layer packet loss at a receiver*. System throughput is defined as the expected reward that a transmission policy obtains per unit time, where *reward* is the number of receivers that receive a transmission successfully. MAC layer packet loss at a receiver is the fraction of packets, which are either not received or received in error at the receiver. A packet loss constraint specifies an upper bound (L) on the acceptable packet loss at any receiver in the multicast group. In multicast, small loss tolerance increases throughput significantly. The lost information can be recovered by using coding redundancy in packets. If the loss is constrained to a small value, then a higher layer protocol can also be used to retrieve the lost packets without significant performance penalty.

We make certain simplifying assumptions to obtain an optimal transmission strategy, which satisfies the desired QoS objectives. Subsequently, using simulations we demonstrate a significant performance gain of these strategies with respect to the traditional MAC layer multicast schemes in the generic network scenario described so far. We describe our analytical model next.

We assume a slotted time axis, with unit length slots. The number of packets arriving in a slot (expectation denoted as λ) and the packet lengths are independent and identically distributed (iid) random variables (r.v.), with arbitrary probability distributions.

For analytical purposes, we consider a single multicast session in isolation (refer to Figure 4). The impact of the network on the multicast session is that the receivers are not always ready to receive. This can be captured by assuming that in any given slot, every receiver is ready independent of its state in other slots with probability (w.p.) p. The probability p depends on the network load and the power saving scheme. We assume that the effect of the network load and the power saving scheme is same on each receiver. These assumptions simplify the design of the optimal strategy while capturing the network details, which will be evident from the simulation results.

In the analysis we ignore bursty channel errors and dynamic changes in multicast group membership. We consider all these generalities in simulation.

In the MAC layer protocol that we propose, a sender queries the readiness state of receivers by control packet exchange. After each query, depending on the transmission strategy and the result of the query, the sender either transmits or backs off for a random duration before querying again. We assume that every receiver maintains its readiness state throughout the transmission. We show that the assumption holds for the MAC layer strategy that we propose (in absence of channel errors which we do not consider in analysis). After every transmission the sender goes backs off for a random duration, so as to allow other senders to use the shared wireless medium. Random back-offs are also useful when the querying control messages collide, e.g. collision of RTS messages in IEEE 802.11. We consider the effects of back-off in both analysis and simulations.

We use simulations to evaluate the performance of the optimal transmission strategy in *network settings*, This allows us to relax the uniformity assumption and to study the effects of the bursty channel errors and the dynamic group mem-

TABLE I

NOTATION USED IN THE SECTION

Notation	Definition
G	Multicast group size
λ	Packet arrival rate at the sender
L	Required loss constraint
K	Upper bound on the number of transmissions per packet
$E[S]$	The expected service time of a transmission policy
$E[R]$	The expected reward per transmission obtained by a transmission policy

berships.

A. Motivation for an upper bound on the number of retransmissions

We would like to clarify that "retransmission" in our notation is retransmission on account of the readiness state of the receivers, pertaining to other transmissions in the neighborhood, power saving mode, etc. Transmission decisions can be taken after assessing these readiness states by control message exchange. Thus retransmission is really a matter of choice, i.e. whether a sender wishes to wait long enough till all receivers are ready, or sends a packet each time a small group is ready. This decision will have important bearing on bandwidth and power efficiency of the scheme. We note that each time a sender retransmits, it consumes additional power. Hence, large number of retransmissions corresponds to large power consumption. This motivates an upper bound K on the number of retransmissions. The upper bound does not apply to the retransmissions triggered by the conditions which can not be assessed a priori, e.g. channel errors that may happen during a data transmission. Corresponding error recovery may incur additional retransmissions. The upper bound that we consider does not apply to these cases.

In the following section we design efficient transmission strategies for a single transmission per packet $(K = 1)$, i.e., when retransmissions are not allowed and the bounded retransmission $(K > 1)$ case.

III. Transmission Policy without Retransmissions

In this section we obtain a throughput optimal transmission policy for multicasting in a wireless network when power constraints do not allow packet retransmissions. The optimal transmission policy should maximize the system throughput, while satisfying the loss constraint. Recall that for analysis we are considering a multicast session in isolation as described in the previous section (refer to Figure 4), while capturing the effect of the network by randomizing the receiver readiness states. In the following subsection we present important definitions and notations.

A. Definitions and Notation

Notations are given in Table I. Now, we present some important definitions.

Definition 1: A *transmission policy* is an algorithm at a sender node that decides when to transmit a packet.

Definition 2: The *service time* (S) of a transmission policy is the expected time duration starting from an instance when a packet reaches head of line till it departs from the system.

Definition 3: A *reward* (R) for a transmission is the number of receivers that receive the packet successfully.

In general, the expected service time $(E[S])$ and the expected reward $(E[R])$ depend on the transmission policy, the number of receivers and their respective readiness probabilities.

Definition 4: A *single-threshold transmission policy(T)* is a transmission policy that transmits a packet only when T or more receivers are ready. We shall refer to the parameter T as *threshold*.

Definition 5: A *two-threshold transmission policy(T,q)* is a transmission policy that sets threshold T for a given transmission w.p. q or a threshold $T+1$ w.p. $1-q$ and transmits in accordance with the threshold.

Definition 6: A *multi-threshold transmission policy($\vec{a}$)* is a transmission policy that sets threshold i w.p. a_i, i can take any value from 1 to G, $\vec{a}$ is a probability mass vector.

We note that the multi-threshold transmission policy is a generalization of the two-threshold transmission policy, which is a generalization of the single-threshold transmission policy.

Now, we present some observations regarding the relation between the two QoS parameters, namely throughput and loss, for a stable system. *A system is said to be stable if the mean queue length is bounded.* In general, the queue length will be bounded if the expected service time is smaller than the expected inter-arrival time. This happens when the packet departure rate at a sender is greater than or equal to the packet arrival rate λ. Hence in the stable system, the throughput of a transmission policy can be given as $\lambda E[R]$. This shows that we need to maximize the expected reward $E[R]$ in order to maximize the throughput subject to the system stability. Further, the expected loss experienced by all the receivers in the system is $G - E[R]$, i.e. if L_i is the expected loss of a receiver i then $\sum_{i=1}^{G} L_i = G - E[R]$. Because of the symmetric load assumption, we can conclude that the expected loss at any given receiver is $1 - \frac{E[R]}{G}$*. Thus a throughput optimal policy minimizes the expected loss at a receiver as it maximizes the expected reward. Hence if the loss for the throughput optimal transmission policy is more than the required loss constraint, it implies that the required loss constraint can not be guaranteed by any transmission policy, while maintaining the system stability. Though system stability is desirable, limiting packet loss is necessary for a meaningful communication. Hence we design a transmission policy that satisfies the loss constraint even at the cost of system stability, *if* the loss constraint can not be achieved while maintaining system stability.

An unstable system is saturated, in the sense that a sender always has a packet to transmit. We show that in a saturated system, maximizing the throughput is not equivalent to minimizing the loss at a receiver. For this case also, we obtain a throughput optimal transmission policy that satisfies the loss constraint.

B. Throughput Optimal Transmission Policy for Stable Systems

In this section, we obtain a throughput optimal transmission policy subject to the system stability constraint. As discussed earlier, maximizing the system throughput is equivalent to minimizing the expected packet loss at a receiver when the system is stable. Hence we do not consider the packet loss constraint explicitly in this section.

We observe that there are numerous choices possible for the transmission policy at a sender. These choices include (a) A policy that makes the transmission decisions based on the queue length at the sender (b) A policy that fixes the service time and then decides the transmission policy in every slot based on the time elapsed since the last transmission (c) a threshold based policy and its generalizations (refer to Definitions 4, 5 and 6). In the following proposition we show that the two-threshold based transmission policy is throughput optimal subject to the stability constraint.

Proposition 1: A two-threshold based policy(T, q) is a throughput optimal transmission policy, where

$$\begin{aligned} T &= \arg\max_{0 \le j \le G} \{E[S_j] \le \frac{1}{\lambda}\} \text{ and} &(1)\\ q &= \frac{E[S_{T+1}] - \frac{1}{\lambda}}{E[S_{T+1}] - E[S_T]} \text{ if } T < G &(2)\\ &= 1 \text{ o.w..} \end{aligned}$$

Here $E[S_T]$ is the expected service time for a packet given that the threshold value is T.

Technical Note: In general, the stability constraint requires the expected service time to be strictly smaller than the expected inter-arrival time. Hence, the condition in equation (1) should be $E[S_j] \le \frac{1}{\lambda} - \epsilon$, for some small $\epsilon > 0$. Subsequently, a numerator in the right hand side of equation (2) should be $E[S_{T+1}] - \frac{1}{\lambda} + \epsilon$. We note that it is possible to approximate the values obtained in equations (1) and (2) with any desired accuracy by choosing ϵ small enough, stability holds as long as $\epsilon > 0$. Hence, we omit ϵ all through for the sake of clarity. Also, to be more precise, the notion of throughput optimality considered is that of ϵ-optimality. More specifically, if some policy achieves throughput TH then

*Because of the system symmetry, any policy that minimizes the loss should treat all the receivers equally.

the ϵ-optimal policy should achieve throughput greater than or equal to $TH - \epsilon$ for arbitrary $\epsilon > 0$. We omit these technical details for clarity.

Why is the result intuitive? As discussed above, a threshold based policy sets a threshold for each transmission and then it transmits only if the number of ready receivers (reward) is greater than or equal to the threshold value. For a stable system, we note that as the threshold value increases, the policy achieves more reward per transmission and hence the system throughput increases. Unfortunately, higher threshold values result in higher service times because when the threshold is high, the sender has to wait till a large number of receivers are simultaneously ready. Hence the threshold can not be increased arbitrarily without violating the stability constraint. So, intuitively a threshold based policy that achieves the maximum throughput should set threshold T such that T is the maximum threshold value that satisfies $E[S_T] \leq \frac{1}{\lambda}$. y. In this case the system throughput is $E[R_T]\lambda$, where $E[R_T]$ is the expected reward obtained for a transmission when the threshold is set to T. Since the thresholds are discrete, the expected service time can not be made arbitrarily close to the expected inter-arrival time. This motivates a two-threshold policy(T, q) instead of a single-threshold policy(T). We observe that with the two-threshold policy(T, q) the throughput can be increased further, while maintaining the system stability. Further, to maintain stability, q should be chosen such that $qE[S_T] + (1-q)E[S_{T+1}] \leq \frac{1}{\lambda}$.

Why is the result not immediate? The two-threshold policy (T, q) achieves a better throughput than the single-threshold policy(T). As a natural extension, a multi-threshold based policy$(\vec{a})$ may provide a better throughput than the two-threshold based policy(T, q). So it seems logical to obtain a throughput optimal threshold based policy by solving the following linear program (LP).

Objective Function : $\sum_{T=0}^{G} E[R_T]a_T$ $/*$Reward of a multi-threshold policy$*/$

Constraints :

1) Stability Condition: $\sum_{T=0}^{G} E[S_T]a_T \leq \frac{1}{\lambda}$
2) Normalization Constraint: $\sum_{T=0}^{G} a_T = 1$
3) Variable Constraints: $a_T \geq 0$, for every $T \in \{1, 2, \dots, G\}$.

We note that the objective function is the expected reward of a multi-threshold policy$(\vec{a})$. Now, Proposition 1 states that there is an optimal solution for the LP, in which $a_j = 0$ for every $j < i$ and $j > i + 1$ for some i. We note that the expected service time $E[S_i]$ is not a convex function of the threshold value i in general. So, Proposition 1 is not immediate from the LP.

Now, if the throughput optimal policy obtained in the above proposition does not satisfy the loss constraint, then no transmission policy can satisfy it without violating the system stability.

The next logical question is the following. "is it always possible to find a transmission policy that satisfies the required loss constraint?" The answer is yes if we relax the stability requirement, e.g. a transmission policy that sets threshold G w.p. 1 has zero packet loss, but it can make the system unstable. Since meeting the loss constraint is imperative for a meaningful communication, we shall design a transmission policy that achieves required loss characteristics even at the cost of stability, *if* necessary. An unstable system is saturated in the sense that a sender always has a packet to transmit. A throughput optimal transmission policy in the saturated case that satisfies the loss constraint is obtained in the following subsection.

C. Throughput Optimal Transmission Policy for Saturated Systems

We will assume a saturated system throughout this subsection. In the main result of this subsection, we show that for appropriate choice of parameters T, q a two-threshold based policy(T, q) is throughput optimal for any given loss constraint. First, we obtain a throughput expression for a single-threshold policy(T) and its generalizations.

Lemma 1: In the saturated case, the throughput for a threshold based policy is $\frac{E[R]}{E[S]}$ w.p. 1.

In the saturated case, the sender always has a packet to transmit. So, the packet transmission rate in the system is equal to the service rate ($\frac{1}{E[S]}$). Further, $E[R]$ is the expected reward per transmission. Intuitively, the reward per unit time is $\frac{E[R]}{E[S]}$, which is the system throughput. Next, we obtain a throughput optimal transmission policy for the saturated case.

Procedure Optimal_Transmission_Policy()
begin

Find $T = \arg\max_{0 \le j \le G}\{E[R_j] \le G(1-L)\}$
Find $T' = \arg\max_{T \le j \le G}\{\frac{E[R_j]}{E[S_j]}\}$
if $(T < T')$ **then**
  Transmission policy is to set threshold T' w.p. 1;
else
  Find $q = \frac{E[R_{T+1}]-G(1-L)}{E[R_{T+1}]-E[R_T]}$
  Transmission policy is to set threshold T and $T+1$ w.p. q and $1-q$, respectively.

end

Fig. 5. Pseudo code of an algorithm to obtain throughput optimal transmission policy that satisfies the loss constraint in the saturated case

Proposition 2: A single-threshold policy(T) is throughput optimal in the saturated case, if $T = \arg\max_{0 \le u \le G}\left\{\frac{E[R_u]}{E[S_u]}\right\}$.

The main message of the proposition is that a throughput optimal policy lies in the class of single-threshold based policies(T). The choice of the parameter T is immediate from the Lemma 1.

We note that similar to the stable systems, the expected loss at a receiver is $1-\frac{E[R]}{G}$ in the saturated case. The difference is that the throughput in the saturated case is not a linear function of the expected reward, which was the case for the stable systems. Hence a throughput optimal policy may not be the one that maximizes the expected reward per packet. This implies that unlike stable systems, throughput optimal transmission policy need not minimize the loss in the saturated case. So the policy proposed in Proposition 2 may not satisfy the required loss constraint. In the following proposition we present a transmission policy, which is throughput optimal subject to satisfying the loss constraint.

Proposition 3: In the saturated case, the algorithm in Figure 5 maximizes the throughput subject to satisfying the required loss constraint L.

The algorithm in Figure 5 is motivated by the following observations. Intuitively, if a throughput optimal policy obtained in Proposition 2 satisfies the loss constraint, then it is the required policy. If not then we need to choose a two threshold-based transmission policy(T, q) with appropriate parameters. Next we explain the algorithm.

As a first step, the proposed algorithm obtains the smallest threshold value (denoted $T+1$ in the pseudo code) that would satisfy the loss constraint. If this value is smaller than the one obtained by Proposition 2 (denoted by T' in the pseudo code), then the throughput optimal transmission policy satisfies the desired loss constraint. Hence in this case, we just choose a single-threshold based policy(T') given by Proposition 2. But if T' is smaller than $T+1$, then we choose a two-threshold policy(T, q). This choice is motivated by the following observations. The throughput in the saturated case is a $\cap$-shaped (cap-shaped) function of the threshold values with the maximum at T'. As noted earlier $T' < T+1$. Hence the throughput with threshold value T will be more than the throughput with threshold value $T+1$. Threshold values smaller than $T+1$ do not provide the required loss characteristics. Since the threshold values are discrete, we can increase the system throughput by randomizing between T and $T+1$ while keeping the loss at the required level.

In this section, we have shown a way to obtain a throughput optimal transmission policies when power constraint does not allow packet retransmissions. We note that the computation of optimal threshold values requires a knowledge of the expected service time $E[S_T]$ for every threshold value T. In general, the expected service time depends on the network load (or receiver readiness probability). The network load is hard to obtain in reality. In the next section, we propose a heuristic to set threshold values based on the local information at the sender.

D. Threshold Values Based on Sender Queue Length

In this section we propose an asymptotically throughput optimal transmission policy that does not need any information about system statistics. Our policy chooses threshold values based on the current queue length at the sender. Broadly, the policy sets small(large) threshold values when queue length is large(small). More precisely, the threshold values are chosen by the following rule. Let Q denote the queue length at the sender and Δ be some fixed positive integer. For $T \ge 2$, threshold is set to T if $(G-T)\Delta \le Q < (G-T+1)\Delta$ and threshold is set to 1 if $Q \ge (G-1)\Delta$. We note that the throughput under the proposed policy is a function of chosen Δ given the arrival process and the receiver readiness process. Lets denote the throughput under proposed policy by $\Omega(\Delta)$ and the optimal throughput by Ω_{opt}. Then we have the following result.

Proposition 4: If $E[S_1] < \frac{1}{\lambda}$, then the throughput of the proposed queue length based policy approaches optimal throughput w.p. 1 as Δ goes to infinity, i.e.

$$\lim_{\Delta \to \infty} \Omega(\Delta) = \Omega_{opt} \text{ w.p. } 1. \tag{3}$$

Intuition behind the result is as follows. For a given system, let the optimal policy be a two-threshold policy(T, q). Then we know that $E[S_t]$ is less than or equal to $\frac{1}{\lambda}$ for every $t \leq T$, while $E[S_t]$ is greater than $\frac{1}{\lambda}$ for every $t \geq T+1$. This implies that for threshold values greater than or equal to $T+1$ the queue length process has positive drift and as a result queue length increases. On the other hand for threshold values less or equal to T the queue length process has a negative drift and hence queue length decreases. Hence we observe that when Δ is large enough the thresholds T and $T+1$ are chosen most of the time. We show that as Δ goes to infinity, thresholds T and $T+1$ are set w.p. q and $1-q$, respectively, which is exactly the same as optimal two threshold based policy(T, q).

We note that unlike threshold based policies, queue length based policies are optimal only in asymptotic sense, i.e. when Δ approaches infinity. Further, when Δ is very large, the time required for the convergence may become large, which is undesirable. Summarizing, a large convergence time is the cost we pay for obtaining throughput optimal transmission policy that does not require system statistics.

In the following section we consider a case, where at most K retransmissions are allowed for each packet.

IV. Transmission Policy with Retransmissions

In this section, we allow limited retransmissions, up to K transmissions per packet, where K is a design parameter. We show that the system throughput can indeed be improved significantly even for small values of K. *Our aim is to design a strategy that maximizes the throughput subject to the stability constraint.* For stable systems, maximizing loss minimizes system throughput, and thus we do not consider loss constraints explicitly.

The following example demonstrates that retransmissions can improve the system performance. Assume that sender S_1 wants to achieve reward 4 in Figure 1. All receivers except R_2 are ready to receive. If no retransmissions are allowed then the sender has to wait. Further, it may so happen that when R_2 becomes ready, receiver R_1 is no longer ready. So, the sender has to wait more. But if retransmissions were allowed, then S_1 could have transmitted when R_1, R_3 and R_4 were ready and then it could have retransmitted the packet when receiver R_2 became ready, thereby reducing the total time to deliver the packet to the desired number of receivers.

Thus retransmissions can be used to reduce the expected time required for achieving a certain reward. Reducing the expected time will allow us to obtain higher reward per transmission without violating system stability, thereby increasing the system throughput. Motivated by these observations, we propose the following strategy for utilizing the retransmissions efficiently to improve throughput subject to stability.

(1) Obtain the minimum expected time $E[S_Z]$ to reach at least Z receivers for each packet using at most K retransmissions for any given Z, $Z = 1, 2, \ldots, G$.

(2) Obtain a two-threshold based policy(Z,q) as follows. Choose the maximum value Z that provides the expected service time $(qE[S_Z]+(1-q)E[S_{Z+1}])$ less than the expected inter-arrival time. The parameter $q \in [0, 1]$ is chosen appropriately depending on $E[S_Z]$, $E[S_{Z+1}]$ and the expected inter-arrival time. Intuitively the above policy provides near-maximum expected reward per packet subject to stability because the expected reward and the expected service time $E[S_Z]$ are monotonically increasing functions of the reward-parameter Z.

The basic building block of the above heuristic based approach is the algorithm that minimizes the expected time to reach a certain number of receivers (Step (1)). A sender has to judiciously use the available transmissions so as to reach required number of receivers in the minimum possible time. We propose a Dynamic Programming (DP) based policy to achieve the required reward Z using at most K transmissions in the minimum expected time. Pseudo code for the DP is given in Figure 6. The input for the DP are (a) the upper bound K on the number of transmissions (b) the minimum number of receivers that a packet should reach (Z). The proposed DP maintains a *system state*, which is a two dimensional vector (k, z) comprising of the number of transmissions already over (k) and the total reward collected in those transmissions (z), i.e. a packet has already been transmitted k times and z receivers have received the packet.

Procedure DP_for_Retransmission_Thresholds(K, Z)
begin

/*System State (k, z): k is the number of transmissions already over and z is the total reward collected in these k transmissions.*/
/* Termination State: Program terminates when $z \geq Z$.*/
/* $\mathcal{S}_{k,z}$ is the minimum expected time to terminate from state (k, z). */
/* $T_{k,z}$ is the optimal threshold that a transmission policy should set in state (k, z).*/
/* $P_{t,u}(k, z)$ is the probability of getting reward u in state (k, z) when the threshold is t */
/* $s_{Z-z}(k, z)$ is the expected time to transmit in state (k, z) when the threshold is t*/
Initialize for every $z \in \{1, 2, \ldots, G\}$

$$\begin{aligned}\mathcal{S}_{K-1,z} &= s_{Z-z}(K-1, z) \text{ if } z < Z \\ &= 0 \text{ o.w.;} \\ T_{K-1,z} &= Z - z \text{ if } z < Z;\end{aligned}$$

for $(k = K - 2$ to $1)$ **do**
  for $(z = 1$ to $Z)$ **do**

$$\mathcal{S}_{k,z} = \min_{0 \leq t \leq Z-z} \left\{ s_t(k, z) + \sum_{u=t}^{G-z} P_{t,u}(k, z) \mathcal{S}_{k-1,z+u} \right\} \quad (4)$$

$$T_{k,z} = \arg \min_{0 \leq t \leq Z-z} \left\{ s_t(k, z) + \sum_{u=t}^{G-z} P_{t,u}(k, z) \mathcal{S}_{k-1,z+u} \right\} \quad (5)$$

$\mathcal{S}_{0,0} = \min_{0 \leq t \leq Z} \left\{ s_t(0, 0) + \sum_{u=t}^{G} P_{t,u}(0, 0) \mathcal{S}_{1,u} \right\}$;
$T_{0,0} = \arg \min_{0 \leq t \leq Z} \left\{ s_t(0, 0) + \sum_{u=t}^{G} P_{t,u}(0, 0) \mathcal{S}_{1,u} \right\}$;
end

Fig. 6. Pseudo code for a DP that obtains optimal threshold values to achieve the reward greater than or equal to Z in the minimum expected time with at most K transmissions. A value $T_{k,z}$ provides the optimal threshold value that a sender should set if it has achieved total reward z in k transmissions. A value $\mathcal{S}_{0,0}$ is the minimum expected time to achieve the reward Z in at most K transmissions.

Based on the system state, the DP chooses a threshold $T_{k,z}$ at every transmission so that it attempts transmission only when more than $T_{k,z}$ receivers are ready to receive. After a transmission, it updates the state and again decides the threshold accordingly. We shall refer this policy as *Dynamic Threshold* (DT) policy.

The thresholds are computed as follows. The proposed DP maintains the following two tables (1) the expected termination time from the given system state (denoted by $\mathcal{S}_{k,z}$) (the system terminates when Z or more receivers receive the packet) (2) the optimal threshold value that a sender should set in a given system state (denoted by $T_{k,z}$). The DP builds these tables in the following manner. First, it considers the states after $K - 1$ transmissions, e.g. a state $(K - 1, z)$ for some $z \leq Z$. In such a state, the sender can transmit the packet only one more time and needs at least a reward $Z - z$. Hence the required threshold in this state is $T_{k,z} = Z - z$. The expected time, in which $Z - z$ (or more) receivers, who have not received the packet yet, are ready to receive is $s_{Z-z}(K - 1, z)$. This is the minimum expected time to terminate from a state $(K - 1, z)$ and hence $\mathcal{S}_{K-1,z} = s_{Z-z}(K - 1, z)$.

After computing $\mathcal{S}_{K-1,z}$ for every $z \leq Z$, DP considers states, where $K - 2$ transmissions are over, e.g. a state $(K - 2, z)$. Now, consider a threshold t. So, the sender will transmit a packet when t or more receivers, who have not received the packet yet are ready. Hence the expected time to transmit in this case is $s_t(K - 2, z)$ and after transmission the system state can be $(K - 1, z + t)$,$(K - 1, z + t + 1)$,. . .,$(K - 1, G)$ with probabilities $P_{t,t}(K - 2, z)$,$P_{t,t+1}(K - 2, z)$,. . .,$P_{t,G-z}(K - 2, z)$, respectively. Recall that we have already computed the minimum expected termination time from each one of these states. So, for threshold t, the minimum expected service time to terminate from the state $(K-2, z)$ is given by equation $\mathcal{S}_{K-2,z} = s_t(K - 2, z) + \sum_{u=t}^{G-z} P_{t,u}(K - 2, z)\mathcal{S}_{K-1,z+u}$. Now, there are various possible threshold values that a sender can choose in the state $(K-2, z)$. The optimal threshold $T_{K-2,z}$ is one, which provides the minimum expected time for termination (equation (5)). Similarly, the DP computes $\mathcal{S}_{k,z}$ and $T_{k,z}$ for all $k \leq K, z \leq Z$ sequentially, that is first $\mathcal{S}_{K-1,z}$ and $T_{K-1,z}$ for every z, then $\mathcal{S}_{K-2,z}$ and $T_{K-2,z}$ for every z, and so on. Finally, $\mathcal{S}_{0,0}$ provides the

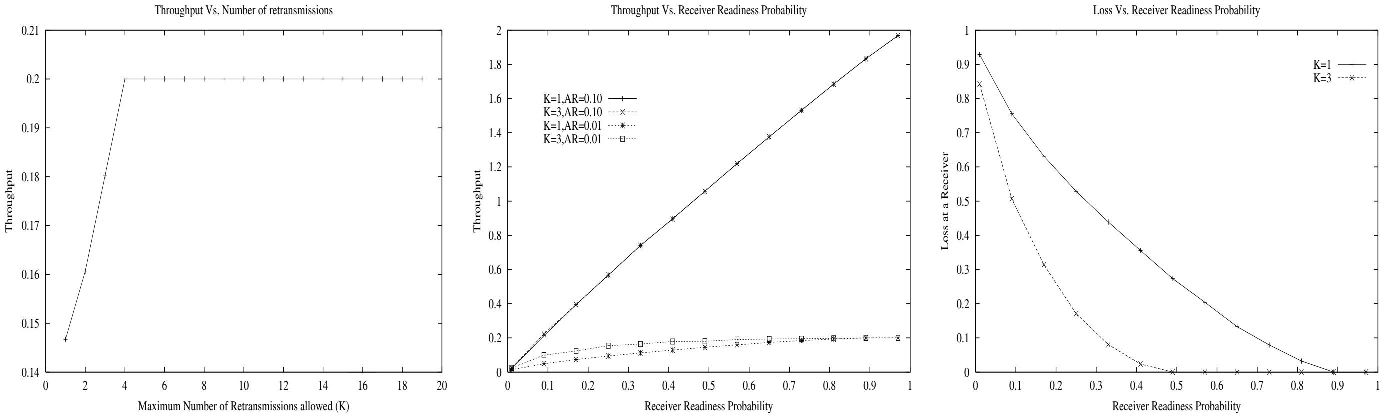

Fig. 7. Figure compares the performance of strategies with and without retransmissions for multicast group size $G = 20$. We consider only stable systems in this figure. The first plots throughput as a function of the allowed number of transmissions K, for receiver readiness probability $p = 0.5$, AR $= 0.01$ (AR is the arrival rate in this figure). The second plots throughput as a function of p, for different arrival rates, and $K = 1, 3$. The third investigates loss performance as a function of p for AR $= 0.01$, $K = 1, 3$.

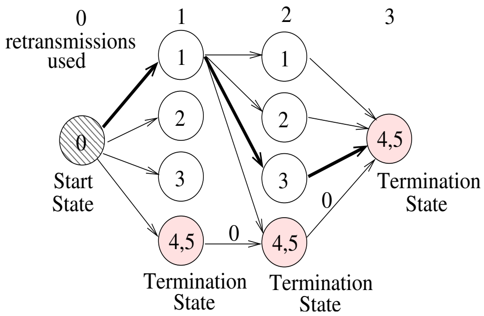

Fig. 8. The figure presents an example illustrating the dynamic program for a multicast group of size 5, $K = 3, Z = 4$. The DP first computes the expected minimum termination times and the optimal thresholds for states in column 2, these correspond to system states $(2, z)$ for different z. Using these values, the expected minimum termination times and the optimal thresholds are computed for the states in column 1 (these correspond to system states $(1, z)$ for different z), and finally for the initial state labeled 0 (which represents the state $(0, 0)$).

minimum expected service time to achieve the desired reward Z in at most K transmissions. Figure 8 explains the DP using an example.

Proposition 5: DT policy minimizes the expected service time to obtain a required reward Z in up to K transmissions for each packet.

A. Numerical Computations

We numerically compare the performances of the proposed optimal strategies with and without retransmissions, using Propositions 1, 3, DP Figure 8. First, we present certain expressions required to perform such computations.

$$E[S_T] = \frac{E[X]+1}{\sum_{i=T}^{G}\binom{G}{i}p^i(1-p)^{G-i}} + E[V] \quad (6)$$

$$E[R_T] = \frac{\sum_{i=T}^{G} i\binom{G}{i}p^i(1-p)^{G-i}}{\sum_{i=T}^{G}\binom{G}{i}p^i(1-p)^{G-i}}, \quad (7)$$

$$s_t(k,z) = \frac{E[X]+1}{\sum_{i=t}^{G-z}\binom{G-z}{i}p^i(1-p)^{G-z-i}} + E[V].$$

$$P_{t,u}(k,z) = \frac{\binom{G-z}{u}p^u(1-p)^{G-z-i}}{\sum_{i=t}^{G-z}\binom{G-z}{i}p^i(1-p)^{G-z-i}}$$

Here $E[X]$ is the expected length of back-off interval and $E[V]$ is the expected packet length plus the expected back-off interval after transmission.

Figure 7 shows the throughput and loss curves for stable systems. We observe that limited number of retransmissions exhibit substantial improvement in throughput, while the performance advantage somewhat saturates with further increase in K (Figure 7(a)). Performance advantage brought about by additional retransmissions decrease with increase in arrival rates (Figure 7(b)). This is because higher arrival rates demand higher service rates for stability, and higher service rates allow fewer retransmissions in spite of a higher allowed upper bound K. Note that the throughput advantage appears somewhat small in Figure 7(b) for arrival rate 0.01 since throughput is $0.01 * ER$ in this case (ER is the expected reward per transmission). However, higher number of retransmissions produce significant gain in reward per packet in this case, as brought out by the performance difference in loss rates (Figure 7(c)).

V. MAC Layer Protocol

We note that the transmission policies proposed so far need information about the number of receivers ready to receive the data. In this section, our aim is to explore various possible approaches to allow the sender to attain this knowledge.

First we present the system challenges experienced by the existing IEEE 802.11 hand-shakes in the case of multiple receivers. Recall that IEEE 802.11 uses RTS-CTS-DATA-ACK handshake for communication. The handshake works well if there is only one intended receiver. In a multicast scenario though, there are many receivers. So, if all the nodes in a multicast group send CTS simultaneously in response to a RTS from a sender, then these CTS messages will collide at the sender. Hence, the sender will not know whether the receivers are ready to receive the data. If the sender knew (by some means) that the colliding messages were CTS responses from the receivers, then the sender can infer the number of ready receivers by measuring the power. Power is additive and power information is not destroyed even in the case of collision. But the power measurement does not solve the problem entirely because of the following reasons. CTS message is not only for the sender but for the nodes in the neighborhood of the intended receivers. CTS message not only conveys that it is going to receive data but also the duration of this data transfer to the neighboring nodes. Because of this information, these neighboring nodes defer their transmissions and hence a receiver can receive the data without collision. This is no longer possible if CTS collides and hence data packets can be corrupted on account of the transmission in the neighborhood. Similar problems exist for ACK message.

IEEE 802.11 can be modified to provide a possible solution, e.g. when the receivers receive RTS message aimed for their multicast group, they back-off for random intervals before transmitting the CTS. This scheme potentially reduces the chances of CTS collisions but does not eliminate it. Further, the sender has to wait for a certain amount of time for the CTS response. The back-off duration will depend upon the group size because to reduce the probability of CTS collisions the range of back-off interval should be large for a the large multicast group. Thus the system could be inefficient for large multicast groups.

An alternative approach is to use a busy-tone based scheme[17]. In this scheme the total available bandwidth is divided into two channels (a) a message channel and (b) a busy-tone channel. A sender initiates a data transfer to a multicast group by sending an RTS message on a message channel as specified in IEEE 802.11 standard. After receiving RTS, all the receivers in the multicast group, which are ready to receive, start sending a busy-tone. This busy-tone acts as a CTS message. A sender estimates the number of ready receivers by measuring the power† of the busy-tone signal. If the estimated number of ready receivers is greater than a threshold value, it starts transmitting the data packet. Receivers keep sending the busy tone until the packet transmission is complete. Busy-tone does not interfere with the data transmission as both are on different frequency bands. Neighboring nodes defer their transmission till the busy-tone stops.

The sender can infer the number of ready receivers from the received busy-tone power only if the received power levels are the same for all the receivers. This is not possible without any power regulation as the receivers are at different distances from the sender. If all the receivers know their distances from the sender, then they can regulate the power so that the busy tones from all the receivers reach the sender at the same power level. The receivers can estimate their distances from the power level of the received RTS (assuming that all the senders transmit at the same power). With this

†We note that the total received power can be measured using standard circuits [11].

information receivers can decide the power level at which they should transmit in order to uniformize the reception power at the sender.

We note that accurate power measurement and regulation are difficult in reality. Hence the sender may make errors in estimating the number of ready receivers. The proposed MAC layer protocol is robust in presence of estimation errors.

The proposed busy tone based scheme provides the feedback about the number of ready receivers at the sender. The approach also solves the problems originating from CTS and ACK collisions in IEEE 802.11 based schemes by providing unambiguous and instantaneous feedback. But these desirable features comes at the cost of the following limitations. (1) The scheme is not totally compatible with IEEE 802.11, though it works on the same philosophy. Hence implementation will need protocol changes. But some other recent papers have advocated busy tone based schemes as well [5], [8]. (2) Since the receiver nodes need to transmit busy tone throughout the data transmission, this can result in an inefficient use of power at the receivers. This problem can be addressed with an additional busy tone channel (a dual busy tone based scheme has been proposed in [5]). On one busy tone channel, receivers can send a short burst of signal indicating CTS message and on the other busy tone channel receivers can send short burst indicating ACK. All the nodes in the neighborhood will not initiate any communication till they receive ACK busy tone for every CTS busy tone they received.

We note that the various proposed schemes have some advantages and some limitations. Designing a scheme that is efficient in all respects is a topic of on going research. A need for such research is motivated by the performance gain achieved by the threshold based transmission policy over the traditional unicast and broadcast based multicast transmission schemes for the MAC layer multicast. The performance comparison in the network case is given in the following section.

VI. Simulation Results and Discussion

In Sections III and IV, we have studied an isolated multicast session, where the effect of network was captured by assuming random readiness states. In this case, we obtained the throughput optimal transmission policies for MAC layer multicast in wireless network. For analytical ease, we made some simplifying assumptions about network load, static group membership and the bursty channel errors. In this section, our aim is to study the performance of the proposed schemes in realistic network settings where we relax these assumptions, by using simulations.

In the following subsection we explain our simulation setup.

A. *Simulation Setup*

In our simulation settings, we place 100 nodes uniformly in a square of side 100. At the beginning of the simulations, we decide the number of senders, and choose them randomly among these nodes. The transmission power of the senders is chosen so that it can communicate in a circle of radius 15 units. A node in the neighborhood of a sender is a member of multicast group w.p. 0.5. Thus different senders have different group sizes, and hence some transmissions are unicast, broadcast etc.

We assume that the arrival process at each of the senders is Bernoulli with arrival probability A_p. The packet lengths are exponential (rounded to the closest integer) with mean 5. Back-off intervals are uniformly distributed between 1 and 5. We model error-prone wireless channel as an ON-OFF Markov process. Transition probabilities from ON state to OFF state and from OFF state to ON are denoted by α and β, respectively. We choose α equal to 0.001 and β equal to 0.1. These channel parameters correspond to Raleigh fading channel of mean fade duration 10 slots and channel being good for 99% of time [13].

We consider the following mobility model[9]. Each node moves linearly with an uniform speed in a random direction for a fixed duration. At the end of this fixed interval, each node again chooses a random direction and moves in the chosen direction. Multicast tree at the network layer changes on account of node mobility, and this is reflected in group membership changes at the MAC layer. The group membership changes will be available to the sender at the end of fixed intervals. This corresponds to periodic topology discovery phase in mobile networks.

Throughout the analysis we assumed that the network load is known at the sender, but in reality the network load is hard to obtain. We have proposed a heuristic to set threshold values based on the local information at the sender. The proposed heuristic samples sender's queue length periodically and updates the threshold value depending on the increase

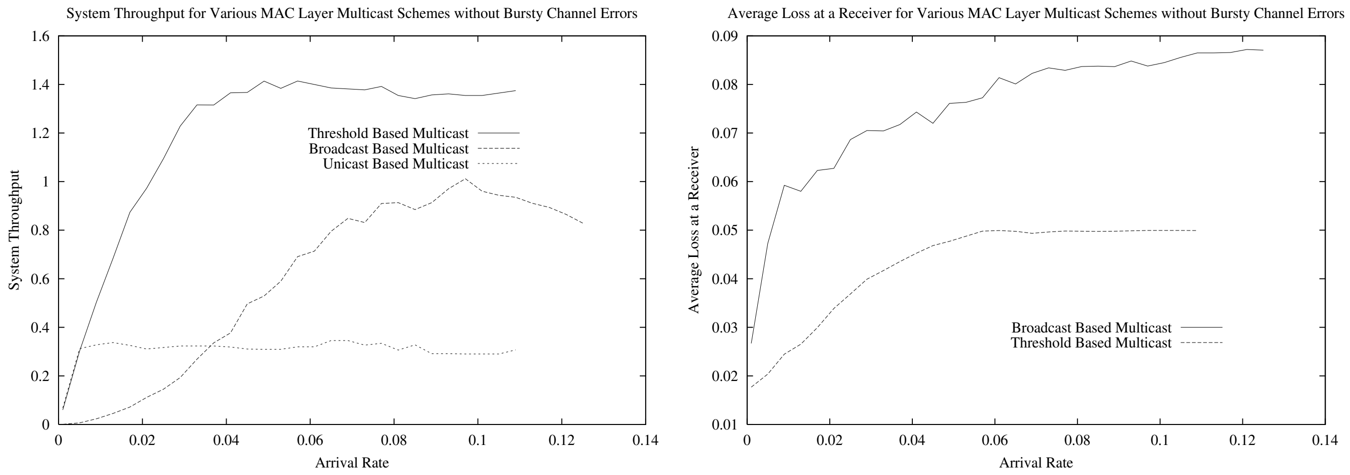

Fig. 9. The system performance for the various multicast schemes as a function of the expected arrival rate at a sender without channel errors.

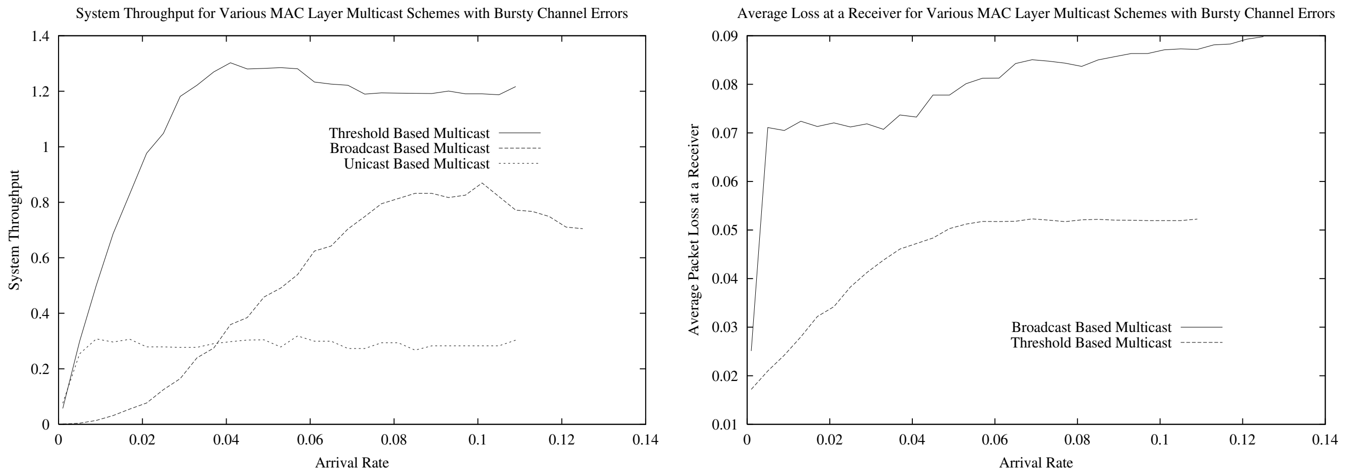

Fig. 10. The system performance for the various multicast schemes as a function of the expected arrival rate at a sender with channel errors.

or decrease in the queue length. In general, it increments(decrements) threshold by 1 in response to decrease(increase) in queue length. We use this heuristic based transmission policy in our simulations. We perform the simulations for 10^6 time slots.

We simulate the busy tone based MAC multicast scheme with $K = 1$ (i.e., no retransmissions other than for error recovery). We refer to this as threshold based multicast. We compare its performance with two IEEE 802.11 based schemes, namely unicast based multicast and broadcast based multicast. No other MAC based multicast scheme has been proposed in the literature to the best of our knowledge.

B. Discussion on the Simulation Results

We have considered two performance measures, first the *system throughput* and second the *average loss* at a receiver. We define the system throughput as the reward achieved by all senders per unit time. The average packet loss at a receiver is the total loss per transmission divided by the current multicast group size.

At first we ignore the effect of bursty channel errors. Figure 9 compares the performances of the three schemes in this case. We observe that the threshold based multicast strategy obtains significantly higher throughput than the other two schemes (Figure 9(a)). The performance difference with the broadcast based scheme can be explained as follows. The broadcast based scheme does not exchange any hand shake messages, and thus results in frequent data packet collisions. Furthermore, it transmits a data packet even when only a few receivers are ready to receive (as it does not know the readiness states of the receivers before the transmission decision). Thus the reward per data packet may be low resulting in low system throughput. This explains the higher packet loss experienced by the broadcast based mechanism as well (Figure 9(b)). The threshold based multicast scheme uses control packet exchange to learn the readiness states of the receivers, and takes intelligent transmission decisions accordingly. The handshake mechanism prevents any data packet collision, as also data transmission when only a few receivers are ready to receive. For broadcast based multicast, data collision dominates under heavy load (high arrival rates) and thus the system throughput substantially degrades with increase in system load. Collision avoidance prevents such throughput degradation for threshold based multicast scheme even during heavy load.

We now explain the performance difference with unicast based multicast (Fig 9(a)). Unicast based multicast uses separate transmissions to reach different members of the multicast group even when they can be reached using one transmission. This increases channel contention for neighboring sources, and reduces the overall system throughput. However, unicast based multicast does not exhibit severe performance degradation during heavy load on account of the control message exchange and subsequent collision avoidance. The threshold based transmission scheme may transmit a packet even when a few receivers are not ready to receive, and thus experiences limited packet loss (Figure 9(b)). Note that unicast based multicast delivers every packet to every receiver by virtue of separate transmission for each receiver. Thus it does not suffer from any packet loss, but this zero-loss performance is attained at the cost of significant throughput reduction (Figure 9(a)). The threshold based multicast scheme exploits limited loss tolerance of the system so as to increase the system throughput substantially (coding redundancy and/or higher layer retransmission offers limited loss tolerance at the MAC layer). Furthermore, it has provisions to control the loss below any desired level by choosing the threshold suitably, and can thereby attain any desired throughput-loss trade-off.

Now we consider the effect of bursty channel errors. The error model has been explained previously. The threshold based multicast scheme and the unicast schemes use retransmission based error recovery (the number of error recovery related retransmissions are not affected by the transmission upper bound of K per packet as explained previously, pp. 5). Every erroneous transmission (as detected by lack of acknowledgment) induces a retransmission in the unicast based scheme. We explain the error recovery process in the threshold based multicast scheme briefly. The sender estimates from the acknowledgments the number of receivers who received the packet without error (either from the randomized ACKs or from power measurement of the busy tone based ACKS). If this number is less than the threshold for the packet (the desired minimum number of receivers who must receive the packet), then the packet is retransmitted: otherwise there is no retransmission. The broadcast based scheme can not incorporate any error recovery in the MAC layer since there is no feedback after any transmission (control message exchange is completely disabled).

Figure 10 compares the performance of the three schemes under bursty channel errors. The channel errors do not affect the performance trends. The performance differences can be explained as before. As anticipated the system throughput decreases for all the three schemes, as compared to the error-free counterparts. As before, the unicast based scheme does not have any packet loss on account of error recovery. Packet loss increases substantially for the broadcast based scheme. The error recovery maintains the packet loss experienced by the threshold based scheme around the same level as when the channel is error free.

VII. Conclusion

This paper is directed towards exploring the basic theory, design trade offs and protocol challenges in MAC layer wireless multicast. We have proposed transmission strategies which improve the system throughput substantially, while satisfying desired packet loss constraints. Future research will be directed towards integrating the MAC layer multicast strategies with wireless multicast routing approaches.

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TABLE II

NOTATION USED IN THE SECTION

Notation	Definition
λ	The expected arrival rate
V_k	Length of the k^{th} packet
X_u	Length of the u^{th} back-off interval
$E[V]$	The expected packet length
$E[X]$	The expected length of the back-off interval
$E[S_T]$	The expected service time of a single-threshold transmission policy(T)
$E[R_T]$	The expected reward per transmission for a single-threshold transmission policy(T)
$E[S^\pi]$	The expected service time of a transmission policy π
$E[R^\pi]$	The expected reward per transmission obtained by a transmission policy π
Ω^π	Throughput achieved by transmission policy π
$z_r^\pi(t)$	The total number of packets serviced and which received reward r under policy π till time t
$z^\pi(t)$	The total number of packets departed by time t under policy π
$n_r^\pi(t)$	The total number of times the sender queries receiver status and finds exactly r ready receivers under policy π till time t
$n^\pi(t)$	The total number of times the sender queries receiver status under policy π till time t
p_r	Probability that exactly r receivers are ready in any given slot
P_T	Probability that T or more receivers are ready in any given slot, $P_T = \sum_{r=T}^{G} p_r$

APPENDIX

Table II gives all the notation used in this section.

A. Preliminary Results

Lemma 2: Following statements hold for the transmission policies in the the class of threshold based policies.

1) For a single-threshold based policy(T),

$$\begin{aligned} E[S_T] &= E[V] + \frac{E[X]}{P_T} \text{ and} \\ E[R_T] &= \frac{\sum_{r=T}^{G} r p_r}{P_T}. \end{aligned} \tag{8}$$

2) The expected service time $E[S_T]$ and the expected reward $E[R_T]$ are monotonically increasing functions of the threshold value T, i.e. $E[S_T] \leq E[S_{T+1}]$ and $E[R_T] \leq E[R_{T+1}]$.
3) For a multi-threshold policy($\vec{a}$) (denoted by π)

$$\begin{aligned} E[S^\pi] &= \sum_{T=0}^{G} a_T E[S_T] \text{ and} \\ E[R^\pi] &= \sum_{T=0}^{G} a_T E[R_T]. \end{aligned} \tag{9}$$

4) For a two-threshold policy(T,q) (denoted by π^*),

$$\begin{aligned} E[S^{\pi^*}] &= qE[S_T] + (1-q)E[S_{T+1}] \text{ and} \\ E[S^{\pi^*}] &= qE[R_T] + (1-q)E[R_{T+1}]. \end{aligned} \tag{10}$$

5) The expected service time $E[S^{\pi^*}]$ and the expected reward $E[R^{\pi^*}]$ are the monotonic functions of parameter T and they are also the monotonically decreasing functions of parameter q for a given threshold value T.

Lemma 3: If two transmission policies π and π^* have the same service time, then the average number of times both the policies query the receivers' status per packet are the same. More formally,

$$\lim_{t\to\infty} \frac{n^{\pi}(t)}{z^{\pi}(t)} = \lim_{t\to\infty} \frac{n^{\pi^*}(t)}{z^{\pi^*}(t)} \quad \text{w.p. } 1. \tag{11}$$

Proof: Let service time for policy π be denoted as $E[S^{\pi}]$.

By definition, the expected service time for policy π is

$$E[S^{\pi}] = \lim_{t\to\infty} \frac{\int_0^t 1_B^{\pi}(u)du}{z^{\pi}(t)} \quad \text{w.p. } 1,$$

where $1_B^{\pi}(u)$ is a indicator variable that takes value 1 if system is busy at time u under policy π and 0 otherwise. We consider non-trivial service policy, where $z^{\pi}(t) \to \infty$ as $t \to \infty$ w.p. 1. Furthermore, whenever packet departs from the system sender goes on random back-off. This implies that for every packet system takes back-off at-least once. Hence $n^{\pi}(t) \geq z^{\pi}(t)$ for every t. Hence $n^{\pi}(t) \to \infty$ as $t \to \infty$ w.p. 1.

We note that

$$\sum_{k=1}^{z^{\pi}(t)} V_k + \sum_{k=1}^{n^{\pi}(t)} X_k \leq \int_0^t 1_B^{\pi}(u)du \leq \sum_{k=1}^{z^{\pi}(t)+1} V_k + \sum_{k=1}^{n^{\pi}(t)+1} X_k.$$

Now we observe that V_k's and X_k's are iid. Hence by Kolmogorov's Strong Law of Large Numbers (KSLLN) we conclude that

$$\lim_{t\to\infty} \frac{1}{z^{\pi}(t)} \sum_{k=1}^{z^{\pi}(t)} V_k + \lim_{t\to\infty} \frac{1}{z^{\pi}(t)} \sum_{k=1}^{n^{\pi}(t)} X_k = E[V] + E[X] \lim_{t\to\infty} \frac{n^{\pi}(t)}{z^{\pi}(t)} \quad \text{w.p. } 1.$$

Similarly,

$$\begin{aligned} &\lim_{t\to\infty} \frac{1}{z^{\pi}(t)} \sum_{k=1}^{z^{\pi}(t)+1} V_k + \lim_{t\to\infty} \frac{1}{z^{\pi}(t)} \sum_{k=1}^{n^{\pi}(t)+1} X_k \\ &= \lim_{t\to\infty} \frac{z^{\pi}(t)+1}{z^{\pi}(t)} \lim_{t\to\infty} \frac{1}{z^{\pi}(t)+1} \sum_{k=1}^{z^{\pi}(t)+1} V_k + \lim_{t\to\infty} \frac{n^{\pi}(t)+1}{z^{\pi}(t)} \lim_{t\to\infty} \frac{1}{n^{\pi}(t)+1} \sum_{k=0}^{n^{\pi}(t)+1} X_k \\ &= E[V] + E[X] \lim_{t\to\infty} \frac{n^{\pi}(t)}{z^{\pi}(t)} \quad \text{w.p. 1 by KSLLN.} \end{aligned}$$

Hence,

$$E[S^{\pi}] = \lim_{t\to\infty} \frac{\int_0^t 1_B^{\pi}(u)du}{z^{\pi}(t)} = E[V] + E[X] \lim_{t\to\infty} \frac{n^{\pi}(t)}{z^{\pi}(t)} \quad \text{w.p. 1 by KSLLN.} \tag{12}$$

Similarly,

$$E[S^{\pi^*}] = E[V] + E[X] \lim_{t\to\infty} \frac{n^{\pi^*}(t)}{z^{\pi^*}(t)} \quad \text{w.p. 1 by KSLLN.} \tag{13}$$

Since expected service times are the same for policies π and π^*, from equations (12) and (13) we can conclude that

$$\lim_{t\to\infty} \frac{n^{\pi}(t)}{z^{\pi}(t)} = \lim_{t\to\infty} \frac{n^{\pi^*}(t)}{z^{\pi^*}(t)} \quad \text{w.p. 1 by KSLLN.}$$

■

B. Proof of Proposition 1

Proof: We prove the Proposition in the following two steps. We show that

1) For any given scheduling policy π such that $E[S^\pi] < \frac{1}{\lambda}$, there exists a two-threshold policy(T, q) (denoted by π^*) such that $E[S^\pi] = E[S^{\pi^*}]$ and policy π^* provides throughput greater than or equal to the throughput by policy π, i.e. $\Omega^\pi \leq \Omega^{\pi^*}$ w.p. 1.
2) If π_1 and π_2 are two-threshold based policies such that $E[S^{\pi_2}] < E[S^{\pi_1}] < \frac{1}{\lambda}$, then throughput achieved by transmission policy π_1 is greater than or equal to the throughput achieved by the transmission policy π_2, i.e. $\Omega^{\pi_1} \geq \Omega^{\pi_2}$ w.p. 1.

STEP 1): We consider arbitrary transmission policy π with average service time $E[S^\pi]$. We note that the throughput for any policy that stabilizes the system is λR^π, where R^π is the expected reward per packet obtained by the policy π.

Next we find a two-threshold based policy(T, q) as follows. Choose

$$\begin{aligned} T &= \arg \max_{0 \leq j \leq G} \{E[S_j] \leq E[S^\pi]\} \text{ and} \\ q &= \frac{E[S_{T+1}] - E[S^\pi]}{E[S_{T+1}] - E[S_T]} \quad \text{if } T < G \\ &= 1 \quad \text{o.w.} \end{aligned}$$

If chosen $T = G$, then the chosen policy is trivially optimal as no policy can achieve throughput more than $G\lambda$. So, without loss of generality we consider a case with $T < G$ and $q > 0$.

We observe that since both the policies stabilizes the system

$$\lim_{t \to \infty} \frac{z^\pi(t)}{t} = \lim_{t \to \infty} \frac{z^{\pi^*}(t)}{t} = \lambda \quad \text{w.p. 1} \tag{14}$$

Now we will show that the number of times sender queries the receivers' status and finds that more than or equal to $T+1$ receivers are ready is less than the number of packets departed in the system. We observe that

$$\begin{aligned} &\lim_{t \to \infty} \frac{z^\pi(t)}{t} - \lim_{t \to \infty} \frac{\sum_{r=T+1}^{G} n_r^\pi(t)}{t} \\ &= \lim_{t \to \infty} \frac{z^\pi(t)}{t} - \lim_{t \to \infty} \frac{\sum_{r=T+1}^{G} n_r^\pi(t)}{n^\pi(t)} \lim_{t \to \infty} \frac{n^\pi(t)}{z^\pi(t)} \lim_{t \to \infty} \frac{z^\pi(t)}{t}. \end{aligned}$$

We note that since receiver states are iid and since $n^\pi(t) \to \infty$ w.p. 1 as $t \to \infty$, by KSSLN we can conclude that $\lim_{t\to\infty} \frac{\sum_{r=T+1}^{G} n_r^\pi(t)}{n^\pi(t)}$ is equal to the probability that $T+1$ or more receivers are ready w.p. 1. Let the probability that $T+1$ or more receivers are ready be P_{T+1}, then we get the following expression.

$$\begin{aligned} &\lim_{t \to \infty} \frac{z^\pi(t)}{t} - \lim_{t \to \infty} \frac{\sum_{r=T+1}^{G} n_r^\pi(t)}{t} \\ &= \lambda - \lambda P_{T+1} \lim_{n^\pi(t)} z^\pi(t) \quad \text{w.p. 1 by KSLLN} \\ &= \lambda \left(1 - P_{T+1} \lim_{t \to \infty} \frac{n^\pi(t)}{z^\pi(t)}\right). \end{aligned}$$

Now we show that

$$1 - P_{T+1} \lim_{t \to \infty} \frac{n^\pi(t)}{z^\pi(t)} > 0. \tag{15}$$

We prove the statement (15) by contradiction. Let

$$1 - P_{T+1} \lim_{t \to \infty} \frac{n^\pi(t)}{z^\pi(t)} \leq 0.$$

From the Lemma 3 we know that

$$\lim_{t \to \infty} \frac{n^\pi(t)}{z^\pi(t)} = \frac{E[S^\pi] - E[V]}{E[X]}.$$

Furthermore, $E[S^\pi] = E[S^{\pi^*}] < E[S_{T+1}]$. Now, $E[S_{T+1}] = E[V] + \frac{E[X]}{P_{T+1}}$. Hence,

$$\begin{aligned} 0 &\geq 1 - P_{T+1} \lim_{t\to\infty} \frac{n^\pi(t)}{z^\pi(t)} \\ &= 1 - P_{T+1} \frac{E[S^\pi] - E[V]}{E[X]} \\ &= 1 - P_{T+1} \frac{E[S^{\pi^*}] - E[V]}{E[X]} \\ &> 1 - P_{T+1} \frac{E[S_{T+1}] - E[V]}{E[X]} \end{aligned}$$

This implies that

$$\begin{aligned} 0 &> E[X] - P_{T+1}(E[S_{T+1}] - E[V]) \\ &= E[X] - P_{T+1}\left(E[V] + \frac{E[X]}{P_{T+1}} - E[V]\right) \\ &= 0 \quad \text{(Contradiction)}. \end{aligned}$$

This implies that $\left(1 - P_{T+1} \lim_{t\to\infty} \frac{n^\pi(t)}{z^\pi(t)}\right) > 0$ w.p. 1. Hence,

$$\lim_{t\to\infty} \frac{z^\pi(t)}{t} > \lim_{t\to\infty} \frac{\sum_{r=T+1}^G n_r^\pi(t)}{t} \quad \text{w.p. 1.} \tag{16}$$

Now we show that

$$\lim_{t\to\infty} \frac{z^\pi(t)}{t} - \lim_{t\to\infty} \frac{\sum_{r=T}^G n_r^\pi(t)}{t} < 0 \quad \text{w.p. 1.}$$

Using the similar analysis as above we conclude that

$$\lim_{t\to\infty} \frac{z^\pi(t)}{t} - \lim_{t\to\infty} \frac{\sum_{r=T}^G n_r^\pi(t)}{t} = \lambda \left(1 - P_T \frac{E[S^{\pi^*}] - E[V]}{E[X]}\right) \quad \text{w.p. 1.}$$

Hence to prove the desired result, it suffices to show that $1 - P_T \frac{E[S^{\pi^*}] - E[V]}{E[X]} < 0$. We show this using contradiction. So, lets assume

$$\begin{aligned} 0 &\leq 1 - P_T \frac{E[S^{\pi^*}] - E[V]}{E[X]} \\ &< 1 - P_T \frac{E[S_T] - E[V]}{E[X]} \quad \text{Since } E[S_T] < E[S^{\pi^*}] \\ &< E[X] - P_T(E[S_T] - E[V]) \\ &< E[X] - P_T(E[V] + \frac{E[X]}{P_T} - E[V]) \\ &< 0. \ (\textit{Contradiction}). \end{aligned}$$

Thus we have shown that

$$\lim_{t\to\infty} \frac{\sum_{r=T+1}^G n_r^\pi(t)}{t} < \lim_{t\to\infty} \frac{z^\pi(t)}{t} < \lim_{t\to\infty} \frac{\sum_{r=T+1}^G n_r^\pi(t)}{t} \quad \text{w.p. 1.} \tag{17}$$

Equation (17) implies that there exist a time instant $\hat{t}_1$ such that $\sum_{r=T+1}^G n_r^\pi(t) < z^\pi(t) < \sum_{r=T}^G n_r^\pi(t)$ for every $t \geq \hat{t}_1$ w.p. 1. Heuristically this implies that for sufficiently large t, number of times sender queries the receivers' status and finds $T+1$ or more receivers are ready is smaller that the total packets departed under policy π, which in turn smaller than the number of times sender queries receivers' status and finds T or more receivers are ready.

Similarly, it follows that $\lim_{t\to\infty} \frac{\sum_{r=T+1}^{G} n_r^{\pi^*}(t)}{t} < \lim_{t\to\infty} \frac{z^{\pi^*}(t)}{t} < \lim_{t\to\infty} \frac{\sum_{r=T}^{G} n_r^{\pi^*}(t)}{t}$ w.p. 1. So, there exists $\hat{t}_2$ such that for every $t \geq \hat{t}_2$, $\sum_{r=T+1}^{G} n_r^{\pi^*}(t) < z^{\pi^*}(t) < \sum_{r=T}^{G} n_r^{\pi^*}(t)$ w.p. 1.

Now consider throughput under policy π. By definition throughput is given as

$$\Omega^{\pi} = \lim_{t\to\infty} \frac{\sum_{r=0}^{G} r z_r^{\pi}(t)}{t} \quad \text{w.p. 1.}$$

Since, for every $t > \hat{t}_1$ number of samples for which number of ready receivers is greater than or equal to $T+1$ is less than the number of packets departed. Hence,

$$\Omega^{\pi} \leq \lim_{t\to\infty} \frac{\sum_{r=T+1}^{G} r n_r^{\pi}(t)}{t} + \lim_{t\to\infty} \frac{T\left[z^{\pi}(t) - \sum_{r=T+1}^{G} n_r^{\pi}(t)\right]}{t} \quad \text{w.p. 1}$$

Right hand side in the above expression corresponds to the throughput any policy would achieve if it transmits whenever $T+1$ or more receivers are ready and the remaining packets when T receivers are ready. This is the maximum throughput a policy can achieve, when the number of slots in which sender queries and finds $T+1$ or more ready receivers is less than or equal to number of packets departed. We note that the number of packets departed till time t is decided by the expected service time of the policy.

Further, we observe that

$$\Omega^{\pi^*} = \lim_{t\to\infty} \frac{\sum_{r=T+1}^{G} r n_r^{\pi^*}(t)}{t} + \lim_{t\to\infty} \frac{T\left[z^{\pi^*}(t) - \sum_{r=T+1}^{G} n_r^{\pi^*}(t)\right]}{t} \quad \text{w.p. 1}$$

Now we consider,

$$\begin{aligned} &\Omega^{\pi} - \Omega^{\pi^*} \\ \leq \; &\lim_{t\to\infty} \left[\frac{\sum_{r=T+1}^{G} r n_r^{\pi}(t)}{t} + \frac{T\left[z^{\pi}(t) - \sum_{r=T+1}^{G} n_r^{\pi}(t)\right]}{t}\right] - \lim_{t\to\infty} \left[\frac{\sum_{r=T+1}^{G} r n_r^{\pi^*}(t)}{t} + \frac{T\left[z^{\pi^*}(t) - \sum_{r=T+1}^{G} n_r^{\pi^*}(t)\right]}{t}\right]. \end{aligned}$$

Now we note that $\lim_{t\to\infty} \frac{z^{\pi}(t)}{t} = \lim_{t\to\infty} \frac{z^{\pi^*}(t)}{t}$ w.p. 1 and $\lim_{t\to\infty} \frac{n_r^{\pi}(t)}{t} = \lim_{t\to\infty} \frac{n_r^{\pi^*}(t)}{t}$ w.p. 1 for every $r = 0, 1, \ldots, G$. Hence,

$$\Omega^{\pi} - \Omega^{\pi^*} \leq 0 \quad \text{w.p. 1.}$$

STEP 2): We consider two two-threshold based policies, namely π_1 and π_2, such that $E[S^{\pi_2}] < E[S^{\pi_1}] < \frac{1}{\lambda}$. Furthermore, we assume that $\pi_1(\pi_2)$ chooses threshold $T_1(T_2)$ w.p. $q_1(q_2)$ and threshold $T_1+1(T_2+1)$ w.p. $1-q_1(1-q_2)$. We consider a case where π_1 and π_2 stabilize the system.

Now we will show that policy π_1 achieves better throughput than policy π_2, i.e. $\Omega^{\pi_1} \geq \Omega^{\pi_2}$ w.p. 1. To show this, we will show that the policy π_1 achieves higher expected reward per transmission than the expected reward per transmission under policy π_2.

From Lemma 2, we note that $E[R_T]$ and $E[S_T]$ are the monotonically increasing function of the threshold value T. Hence $E[S^{\pi_1}] > E[S^{\pi_2}]$ implies that either $T_1 > T_2$ or $T_1 = T_2$ and $q_1 < q_2$. We consider a first case, i.e. $T_1 > T_2$.

$$\begin{aligned} E[R^{\pi_1}] &= q_1 E[R_{T_1}] + (1-q_1) E[R_{T_1+1}] \\ &\geq E[R_{T_1}] \\ &\geq q_2 E[R_{T_2}] + (1-q_2) E[R_{T_2+1}] \text{ since } T_1 > T_2 \text{ and the expected reward is monotone function of } T \\ &= E[R^{\pi_2}]. \end{aligned}$$

In the second case, where $T_1 = T_2$ and $q_1 < q_2$, we obtain the following result.

$$\begin{aligned} E[R^{\pi_1}] &= q_1 E[R_{T_1}] + (1-q_1)E[R_{T_1+1}] \\ &\geq q_2 E[R_{T_1}] + (1-q_2)E[R_{T_1+1}] \text{ since } q_1 < q_2 \\ &\qquad\qquad\qquad\qquad \text{and } E[R_{T_1}] < E[R_{T_1+1}] \\ &= E[R^{\pi_2}]. \end{aligned}$$

Since the system is stable under both the transmission policies pi_1 and π_2,

$$\begin{aligned} \Omega^{\pi_1} &= \lambda E[R^{\pi_1}] \text{ and} \\ \Omega^{\pi_2} &= \lambda E[R^{\pi_2}] \text{ w.p. 1 by KSLLN} \\ \Rightarrow \Omega^{\pi_1} &\geq \Omega^{\pi_2} \text{ w.p. 1, whenever } E[R^{\pi_1}] \geq E[R^{\pi_2}]. \end{aligned}$$

From the above two steps, we can conclude that a two-threshold based policy(T, q) that achieves the highest possible expected service time without violating stability is throughput optimal. For stability we need service time less than or equal to $\frac{1}{\lambda} - \epsilon$ for some $\epsilon > 0$. A particular choice of ϵ will determine the exact two-threshold based policy(T, q) that is throughput optimal. ■

We note that in the proof of Proposition 1, we have proved a stronger result. In particular, we have shown that for any given service time (or for any desired system utilization) there exists a two-threshold based policy(T, q) that provides optimum throughput.

C. Proof of Lemma 1

Proof: We prove Lemma 1 in the following two steps. For a general multi-threshold transmission policy($\vec{a}$) (denoted by π), we show that when the system is saturated, following holds.

1) the packet transmission process is a renewal process, where packet departure epochs (time instant when last bit of the packet is transmitted) are the renewal points
2) by applying *Renewal Reward Theorem* (RRT) the desired result will be obtained

STEP 1): Consider two consecutive packet departure epochs t_1 and t_2 and define service time for a packet to be $t_2 - t_1$. Let S_T denote a random variable indicating service time for a packet given that the threshold is T.

Now, we note that a multi-threshold transmission policy($\vec{a}$) sets threshold for a packet independent of the threshold set for any other packet according to the probability mass vector $\vec{a}$.

Hence for arbitrary packet u the service time (denoted by r.v. S_u^π) distribution is given as

$$\begin{aligned} Pr\{S_u^\pi = k\} &= \sum_{T=0}^{G} a_T Pr\{S_u^\pi = k/\text{Threshold is chosen to be } T\} \\ &= \sum_{T=0}^{G} a_T Pr\{S_T = k\}. \end{aligned}$$

We note that the distribution of S_u^π does not depend on u, which implies that S_u^π's are identically distributed.

Since the thresholds are set independently for every packet when it reaches head of line, the service time random variables S_u^π's are independent for every u. This shows that the service times are iid. Lets define $E[S^\pi]$ to be the expected service time under policy π .

Hence the packet transmission process is a renewal process with packet departure epochs as renewal epochs.

Now, lets define *reward* to be the number of receivers, who received the transmitted packet successfully. Further, let $E[R^\pi]$ be the expected reward achieved per transmission by policy π.

Further, let $R(t)$ is the total reward collected in time t. Then by definition throughput of the transmission policy is $\lim_{t\to\infty} \frac{R(t)}{t}$.Now, by RRT

$$\lim_{t\to\infty} \frac{R(t)}{t} = \frac{E[R^\pi]}{E[S^\pi]} \qquad \text{w.p. } 1.$$

We note that the single-threshold p[olicy(T) and the two-threshold policy(T, q) are just the special cases of multi-threshold policy($\vec{a}$). ■

Proposition 6: For any given transmission policy π with the expected service time $E[S^{\pi}]$, there exists a two-threshold based policy(T, q) (denoted by π^*) such that $\Omega^{\pi} \leq \Omega^{\pi^*}$ and $E[S^{\pi}] \geq E[S^{\pi^*}]$.

We note that there is no assumption about the system stability. In other words the above result holds for stable system $(E[S^{\pi}] < \frac{1}{\lambda})$ and also for saturated systems $((E[S^{\pi}] \geq \frac{1}{\lambda})$.

Proof: We have proved the result for the stable systems in STEP 1 of Proposition 1. We further note that the proof for the saturated system follows exactly as STEP 1 of Proposition 1, the only difference in this case is that for the saturated systems

$$\lim_{t\to\infty} \frac{z^{\pi}(t)}{t} = \frac{1}{E[S^{\pi}]} \text{ w.p. 1 by RRT.}$$

Hence it follows that

$$\lim_{t\to\infty} \frac{z^{\pi}(t)}{t} = \lim_{t\to\infty} \frac{z^{\pi^*}(t)}{t} \text{ w.p. } 1,$$

whenever $E[S^{\pi}] = E[S^{\pi^*}]$. We have shown an algorithm to obtain two threshold based policy with the desired expected service time in STEP 1 of Proposition 1.

With these observations the remaining proof follows exactly as STEP 1 of Proposition 1. ■

D. Proof of Proposition 2

Proof: From Proposition 6, it is evident that for any given service time two-threshold based transmission policy(T, q) maximizes the expected reward per transmission. Since throughput is the expected reward per unit time, we can conclude that the two-threshold based policy(T, q) is throughput optimal for saturated case as well.

Now, we will show that a throughput optimal policy is a single-threshold based policy(T) or it is a two-threshold based policy(T, q) with $q = 1$.

Lets consider two single-threshold based policies such that one policy sets threshold i for every packet and other sets threshold j for every packet. j can either be $i - 1$ or $i + 1$, since we need to consider only two-threshold based policies, which randomize between two consecutive thresholds. Without loss of generality we assume that $j = i + 1$.

From Lemma 1, the throughputs achieved by a policy that sets threshold i and j for every packet are $\frac{E[R_i]}{E[S_i]}$ and $\frac{E[R_j]}{E[S_j]}$, respectively. Without loss of generality we assume that $\frac{E[R_i]}{E[S_i]} \geq \frac{E[R_j]}{E[S_j]}$.

We claim that $\frac{E[R_i]}{E[S_i]} \geq \frac{qE[R_i]+(1-q)E[R_j]}{qE[S_i]+(1-q)E[S_j]}$ for every $q \in [0, 1]$. In other words, we claim that a single-threshold based policy(i) achieves equal or more throughout than the two-threshold based policy(i, q). We prove the claim by contradiction.

Let

$$\begin{aligned}
\frac{E[R_i]}{E[S_i]} &< \frac{qE[R_i] + (1-q)E[R_j]}{qE[S_i] + (1-q)E[S_j]} \\
\Rightarrow E[R_i]E[S_j] &< E[R_j]E[S_i] \\
\Rightarrow \quad \frac{E[R_i]}{E[S_i]} &< \frac{E[R_j]}{E[S_j]} \qquad \textit{Contradiction.}
\end{aligned}$$

Furthermore, $\frac{E[R_i]}{E[S_i]} \leq \frac{E[R_T]}{E[S_T]}$ for every i, where T is chosen as shown in the statement of Proposition. Hence, a single-threshold based policy(T) is throughput optimal. ■

Lemma 4: Throughput for a single-threshold policy(T), when viewed as a function of T, first increases monotonically and then decreases monotonically as T increases.

The above Lemma states that the throughput curve as a function of threshold T is a bell-shaped curve. In other words, their exists a unique maximum value for the throughput in saturated systems.

Proof: To prove the above Lemma, it suffices to show that *if throughput attained at threshold value T is greater than or equal to the throughput attained at threshold value $T+1$, then throughput attained at threshold value $T+1$ is greater than or equal to the throughput attained at threshold value $T+2$.*

From Lemma 1, we know that the throughput for threshold value T is $\frac{E[R_T]}{E[S_T]}$ w.p. 1. From Lemma 2, we conclude that

$$\frac{E[R_T]}{E[S_T]} = \frac{\sum_{i=T}^{G} ip_i}{E[X] + \sum_{i=T}^{G} E[V]p_i}.$$

Now we know that

$$\begin{aligned} \frac{E[R_T]}{E[S_T]} &> \frac{E[R_{T+1}]}{E[S_{T+1}]} \\ \Rightarrow T(E[X]) &> \left[E[V] \sum_{i=T+1}^{G} (i-T)p_i \right]. \end{aligned} \tag{18}$$

Now lets assume that $\frac{E[R_{T+2}]}{E[S_{T+2}]} > \frac{E[R_{T+1}]}{E[S_{T+1}]}$. Then using the similar algebra as above it follows that

$$\begin{aligned} E[V] \sum_{i=T+2}^{G} (i-T-1)p_i &> (T+1)E[X] \\ E[V] \sum_{i=T+2}^{G} (i-T-1)p_i &> TE[X]. \end{aligned}$$

Substitute $E[V] \sum_{i=T+1}^{G} (i-T)p_i$ (refer equation (18)) in the above equation for $TE[X]$. After simplification we get

$$p_{T+1} < -P_{T+2},$$

which is not possible as both p_{T+1} and P_{T+2} are positive. This provides a *contradiction* for the assumption that $\frac{E[R_{T+2}]}{E[S_{T+2}]} > \frac{E[R_{T+1}]}{E[S_{T+1}]}$. ■

E. *Proof of Proposition 4*

Proof: We prove the results with some simplifying assumptions. We assume that when queue is not empty, sender queries the receivers' status in every slot and based on the result of the query and transmission policy the sender decides to transmit or query again. In short, we assume that the sender does not go on the back-off. Further, we assume that the packet length is one and the arrivals are Bernoulli in a slot, i.e. a packet arrives with probability p in any given slot. The result holds without these assumptions. The system analysis without these assumption becomes complicated without providing any additional insight. We make these assumptions for the better readability.

We model the system as a Discrete Time Markov Chain (DTMC) with sender's queue length as a system state. Lets X_k be the queue-length in k^{th} time slot. Let the transition probability matrix be denoted by $\mathcal{A} = [a_{i,j}]$. The transition probabilities are follows.

$$
\begin{aligned}
a(i,j) &= pP_G + (1-p) \quad \text{if } i=j=0,\\
&= pP_T + (1-p)(1-P_T) \quad \text{if } i=j\neq 0 \text{ and } (G-T)\Delta \leq i < (G-T+1)\Delta, \text{ where } T\geq 2,\\
&= p(1-P_T) \quad \text{if } j=i+1 \text{ and } (G-T)\Delta \leq i < (G-T+1)\Delta, \text{ where } T\geq 2,\\
&= P_T(1-p) \quad \text{if } j=i-1, i\neq 0 \text{ and } (G-T)\Delta \leq i < (G-T+1)\Delta, \text{ where } T\geq 2,\\
&= pP_1 + (1-p)(1-P_1) \quad \text{if } i=j \text{ and } (G-1)\Delta \leq i,\\
&= p(1-P_1) \quad \text{if } j=i+1 \text{ and } (G-1)\Delta \leq i,\\
&= P_1(1-p) \quad \text{if } j=i-1 \text{ and } (G-1)\Delta \leq i.
\end{aligned}
$$

Let $\pi = [\pi_1\ \pi_2 \ldots]$ be the steady state distribution. If the system is stable then π is the solution of matrix equation $\pi = \pi\mathcal{A}$ and the normalization equation $\sum_{i=0}^{\infty} \pi_i = 1$.

Lets define

$$
\begin{aligned}
\zeta_j &= \frac{p(1-P_j)}{(1-p)P_j} \text{ and}\\
\beta_j &= \frac{P_{j+1}}{P_j} \text{ for all } j = 1,2,\ldots,G.
\end{aligned}
$$

Here we assume that $P_{G+1} = P_G$, i.e. $\beta_G = 1$.

Now using these notation and some elementary algebra it can be seen that solution of matrix equation $\pi = \pi\mathcal{A}$ is as follows.

$$
\begin{aligned}
\pi_{k\Delta+i} &= \pi_0 \zeta_{G-k}^{i} \prod_{j=G-k}^{G} \beta_j \prod_{j=G-k+1}^{G} \zeta_j^{\Delta}, \quad \text{where } k = 0,1,\ldots,G-2 \text{ and } i = 0,1,\ldots,\Delta-1\\
\pi_{(G-1)\Delta+i} &= \pi_0 \zeta_1^{i} \prod_{j=1}^{G} \beta_j \prod_{j=2}^{G} \zeta_j^{\Delta}, \quad \text{where } i = 0,1,\ldots.
\end{aligned}
$$

Now lets consider a threshold value T such that $P_T > p$ and $P_{T+1} < p$. Such T always exists and it is unique because P_T is a monotonically decreasing function of threshold value T and we know that $P_1 > p$. For the given choice of threshold value, we note that

$$
\begin{aligned}
\zeta_j &< 1 \quad \text{whenever } j = 0,1,\ldots,T\\
\zeta_j &> 1 \quad \text{whenever } j = T+1,\ldots,G.
\end{aligned}
$$

Now we note that

$$
\sum_{i=0}^{\Delta-1} \pi_{k\Delta+i}
$$

$$
\begin{aligned}
&= \pi_0 \left[\prod_{j=G-k}^{G} \beta_j \prod_{j=G-k+1}^{G} \zeta_j^{\Delta} \right] \sum_{i=0}^{\Delta-1} \zeta_{G-k}^{i} \\
&= \pi_0 \left[\prod_{j=G-k}^{G} \beta_j \prod_{j=G-k+1}^{G} \zeta_j^{\Delta} \right] \frac{\zeta_{G-k}^{\Delta}-1}{\zeta_{G-k}-1} \quad \text{whenever } k = 0,1,\dots,G-(T+1) \\
&= \pi_0 \left[\prod_{j=G-k}^{G} \beta_j \prod_{j=G-k+1}^{G} \zeta_j^{\Delta} \right] \frac{1-\zeta_{G-k}^{\Delta}}{1-\zeta_{G-k}} \quad \text{whenever } k = G-T,\dots,G-2.
\end{aligned}
$$

Furthermore, we note that

$$
\begin{aligned}
\sum_{i=0}^{\infty} \pi_{(G-1)\Delta+i} &= \pi_0 \left[\prod_{j=1}^{G} \beta_j \prod_{j=2}^{G} \zeta_j^{\Delta} \right] \sum_{i=0}^{\infty} \zeta_1^{i} \\
&= \pi_0 \left[\prod_{j=1}^{G} \beta_j \prod_{j=2}^{G} \zeta_j^{\Delta} \right] \frac{1}{1-\zeta_1}.
\end{aligned}
$$

Now we consider normalization equation.

$$
\begin{aligned}
1 &= \sum_{i=0}^{\infty} \pi_i \\
&= \sum_{k=0}^{G-2} \sum_{i=0}^{\Delta-1} \pi_{k\Delta+i} + \sum_{i=0}^{\infty} \pi_{(G-1)\Delta+i} \\
&= \sum_{k=0}^{G-(T+1)} \pi_0 \left[\prod_{j=G-k}^{G} \beta_j \prod_{j=G-k+1}^{G} \zeta_j^{\Delta} \right] \frac{\zeta_{G-k}^{\Delta}-1}{\zeta_{G-k}-1} + \sum_{k=G-T}^{G-2} \pi_0 \left[\prod_{j=G-k}^{G} \beta_j \prod_{j=G-k+1}^{G} \zeta_j^{\Delta} \right] \frac{1-\zeta_{G-k}^{\Delta}}{1-\zeta_{G-k}} \\
&\quad + \pi_0 \left[\prod_{j=1}^{G} \beta_j \prod_{j=2}^{G} \zeta_j^{\Delta} \right] \frac{1}{1-\zeta_1}.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&\pi_0^{-1} \\
&= \sum_{k=0}^{G-(T+1)} \left[\prod_{j=G-k}^{G} \beta_j \prod_{j=G-k+1}^{G} \zeta_j^{\Delta} \right] \frac{\zeta_{G-k}^{\Delta}-1}{\zeta_{G-k}-1} + \sum_{k=G-T}^{G-2} \left[\prod_{j=G-k}^{G} \beta_j \prod_{j=G-k+1}^{G} \zeta_j^{\Delta} \right] \frac{1-\zeta_{G-k}^{\Delta}}{1-\zeta_{G-k}} + \left[\prod_{j=1}^{G} \beta_j \prod_{j=2}^{G} \zeta_j^{\Delta} \right] \frac{1}{1-\zeta_1}.
\end{aligned}
$$

We note that the steady state probability that the threshold is chosen to be $\hat{T} \geq 2$ is $\sum_{i=0}^{\Delta} \pi_{(G-\hat{T})\Delta+i}$ and the steady state probability that the threshold is chosen to be 1 is $\sum_{i=0}^{\infty} \pi_{(G-1)\Delta+i}$.

Now with some algebra it can be shown that the steady state probability that the threshold value is greater than or equal to $T+2$ or less than or equal to $T-1$ approaches zero as Δ approaches ∞, i.e.,

$$
\begin{aligned}
\lim_{\Delta\to\infty} \sum_{k=0}^{G-(T+2)} \sum_{i=0}^{\Delta-1} \pi_{k\Delta+i} &= 0 \quad \text{and} \\
\lim_{\Delta\to\infty} \sum_{k=G-(T-1)}^{G-2} \sum_{i=0}^{\Delta-1} \pi_{k\Delta+i} + \sum_{i=0}^{\infty} \pi_{(G-1)\Delta+i} &= 0.
\end{aligned}
$$

Furthermore, it can be shown that the probabilities that the threshold is set to T or $T+1$ approaches the following values as Δ approaches ∞.

$$
\begin{aligned}
\lim_{\Delta\to\infty}\sum_{i=0}^{\Delta-1}\pi_{G-(T+1)+i} &= \frac{\prod_{j=T+1}^{G}\beta_j\left[\frac{1}{\zeta_{T+1}-1}\right]}{\prod_{j=T+1}^{G}\beta_j\left[\frac{1}{\zeta_{T+1}-1}\right]+\prod_{j=T}^{G}\beta_j\left[\frac{1}{1-\zeta_T}\right]}\\
&= \frac{P_{T+1}(P_T-p)}{p(P_T-P_{T+1})} \quad \text{and}\\
\lim_{\Delta\to\infty}\sum_{i=0}^{\Delta-1}\pi_{G-T+i} &= \frac{\prod_{j=T}^{G}\beta_j\left[\frac{1}{1-\zeta_T}\right]}{\prod_{j=T+1}^{G}\beta_j\left[\frac{1}{\zeta_{T+1}-1}\right]+\prod_{j=T}^{G}\beta_j\left[\frac{1}{1-\zeta_T}\right]}\\
&= \frac{P_T(p-P_{T+1})}{p(P_T-P_{T+1})}.
\end{aligned}
$$

For better readability, we denote the steady state probability that the threshold value is chosen to be t for a fixed Δ by $q_t(\Delta)$, i.e.

$$
q_t(\Delta)=\sum_{i=0}^{\Delta-1}\pi_{(G-t)\Delta+i}.
$$

Using this notation, we note that

$$
\begin{aligned}
\lim_{\Delta\to\infty} q_t(\Delta) &= 0 \quad \text{if } t\in\{1,2,\ldots,T-1,T+2,\ldots,G\}\\
&= \frac{P_{T+1}(P_T-p)}{p(P_T-P_{T+1})} \quad \text{if } t=T \quad \text{and}\\
&= \frac{P_T(p-P_{T+1})}{p(P_T-P_{T+1})} \quad \text{if } t=T+1.
\end{aligned}
$$

We observe that an optimal two-threshold policy(T,q) will also choose thresholds T and $T+1$ w.p. q and $1-q$, respectively, where

$$
\begin{aligned}
q &= \frac{P_{T+1}(P_T-p)}{p(P_T-P_{T+1})} \quad \text{and hence}\\
1-q &= \frac{P_T(p-P_{T+1})}{p(P_T-P_{T+1})}.
\end{aligned}
$$

The above values are computed using equations (1) and (2). We note that in the special case, where no back-offs are considered and the packet lengths are identically 1, $E[S_t]=\frac{1}{P_t}$ for any threshold value $t=1,2,\ldots,G$.

From the above observations, we conclude that in steady state as $\Delta\to\infty$, queue length based policy sets threshold T w.p. q and $T+1$ w.p. $1-q$, which is exactly the same as that of optimal two-threshold based policy. In other words,

$$
\begin{aligned}
\lim_{\Delta\to\infty} q_T(\Delta) &= q \quad \text{and}\\
\lim_{\Delta\to\infty} q_{T+1}(\Delta) &= 1-q.
\end{aligned}
$$

This implies that for every $\epsilon>0$, there exist a $\hat{\Delta}$ such that for every $\Delta\geq\hat{\Delta}$,

$$
\begin{aligned}
q_T(\Delta) &\geq q-\frac{\epsilon}{2G} \quad \text{and}\\
q_{T+1}(\Delta) &\geq (1-q)-\frac{\epsilon}{2G}.
\end{aligned}
$$

Lets denote the queue length based policy and optimal two-threshold policy(T,q) by π and π^*, respectively. We fix Δ greater than or equal to $\hat{\Delta}$. Now, we observe that the receiver readiness process and queue length process are independent.

Now, using the ergodicity of the queue length process and the KSLLN for the receiver readiness process, we can compute the expected reward per transmission achieved by policy π in steady state as follows.

$$\begin{aligned}
E[R^{\pi}] &= \sum_{k=0}^{G} kPr\{\text{Reward for a transmission} = k\} \\
&= \sum_{t=0}^{G}\sum_{k=0}^{G} kPr\{\text{Reward for a transmission} = k/ \;\text{ threshold} = t\}Pr\{\text{threshold} = t\} \\
&= \sum_{t=0}^{G}\sum_{k=0}^{G} kPr\{\text{Reward} = k/ \;\text{ Number of ready receivers} \geq t\}Pr\{\text{threshold} = t\} \\
&= \sum_{t=0}^{G}\sum_{k=t}^{G} k\frac{p_k}{P_t}q_t(\Delta) \quad \text{w.p. 1} \\
&\geq \sum_{k=T}^{G} k\frac{p_k}{P_T}q_T(\Delta) + \sum_{k=T+1}^{G} k\frac{p_k}{P_{T+1}}q_{T+1}(\Delta) \quad \text{Since } q_t = 0 \text{ whenever } t \notin \{T, T+1\} \\
&= qE[R_T] - \frac{\epsilon}{2G}E[R_T] + (1-q)E[R_{T+1}] - \frac{\epsilon}{2G}E[R_T] \\
&\geq qE[R_T] + (1-q)E[R_{T+1}] - \epsilon \\
&= E[R^{\pi^*}] - \epsilon.
\end{aligned}$$

Now, we observe that the system is stable and hence the throughput is the product of the arrival rate p and the expected reward w.p. 1.

$$\begin{aligned}
\Omega^{\pi} &\geq pE[R^{\pi}] \quad \text{w.p. 1} \\
&\geq p\left(E[R^{\pi^*}] - \epsilon\right) \\
&\geq \Omega^{\pi^*} - \epsilon
\end{aligned}$$

Since this is true for every $\epsilon > 0$, we conclude that

$$\Omega^{\pi} \geq \Omega^{\pi^*} \quad \text{w.p. 1.}$$

This implies that the throughput obtained by policy π is the same as the throughput obtained by the optimal policy π^* w.p. 1, since π^* is an optimal policy. Hence π is an optimal policy. ■

F. *Proof of Proposition 3*

We note that to satisfy the required loss constraint L, we need expected reward $E[R^{\pi}] = G(1-L)$ as the expected loss at the receiver is $1 - \frac{E[R^{\pi}]}{G}$.

Now a single-threshold policy(T') obtained in Proposition 2 provides the expected reward higher than $G(1-L)$ then it is the throughput optimal policy that satisfies the required loss constraint. Otherwise, we need to find a transmission policy that maximizes the ratio of the expected reward and the expected service time, where the expected reward should be greater than or equal to $G(1-L)$. Now, from Proposition 6, we know that for any given service time two-threshold based transmission policy(T, q) maximizes the expected reward per transmission. This shows that a two-threshold based policy(T, q) will be optimal. We need to find parameter values T and q that maximizes throughput while achieving the desired loss characteristics.

Now we will show that the parameter values obtained in the algorithm given in Figure 5 are optimal. Recall that the values obtained are

$$\begin{aligned} T &= \arg \max_{0 \le t \le G} \{E[R_t] \le G(1-L)\} \\ q &= \frac{E[R_{T+1}] - G(1-L)}{E[R_{T+1}] - E[R_T]}. \end{aligned}$$

We observe that the two-threshold based policy(T, q) specified by above parameters obtains the expected reward $G(1-L)$ which is the minimum expected reward required to satisfy the loss constraint. Hence it suffices to show that any two-threshold policy that achieves the higher expected reward provides lesser throughput.

We note that a single-threshold policy(T') obtained in Proposition 2 does not provide the required expected reward. Hence $T \ge T'$ and $\frac{E[R_T]}{E[S_T]} \le \frac{E[R_{T'}]}{E[S_{T'}]}$. Now we consider any two-threshold policy$(T+i, \hat{q})$. Now, from Lemma 4 we know that $\frac{E[R_{T+1}]}{E[S_{T+1}]} \ge \frac{E[R_{T+i}]}{E[S_{T+i}]} \ge \frac{E[R_{T+i+1}]}{E[S_{T+i+1}]}$. Further in Proposition 2 we have shown that a single threshold based policy$(T+1)$ will obtain higher throughput that a two-threshold policy$(T+i, \hat{q})$. This shows that choosing higher threshold values can only degrade the throughput. And the smaller threshold values can not satisfy the loss constraints. Hence the algorithm provides optimal choice. ■

G. *Proof of Proposition 5*

Proof: The problem of obtaining required reward in the minimum time with at most K retransmissions can be formulated as Stochastic Shortest Path problem [1]. Now we present the problem formulation.

In our formulation, the system state is two dimensional and it is denoted by $(k, \vec{A})$, where k is the number of retransmissions already over and $\vec{A} = \{(I_1, L_1), (I_2, L_2), \ldots, (I_G, L_G)\}$. Here I_i and R_i are indicators. I_i takes value 1 if receiver i has already received the packet and 0 otherwise. Similarly, L_i is equal to 1 if i^{th} receiver is ready to receive and 0 otherwise. Given a system state, transmission policy decides either to transmit or to back-off. Hence in each state control decision is whether to transmit or back-off. Further, corresponding to each decision there is a cost. In particular, if transmission policy decides to transmit then the cost is packet transmission time plus 1, otherwise if policy decides to back-off then the cost is back-off duration plus 1. Furthermore, as a result of control decision system makes transition from one state to another with certain probability. Let $Pr\{(k, \vec{A}) \to (k_1, \vec{A}_1)\}$ be the probability of transition from state $(k, \vec{A})$ to the state $(k_1, \vec{A}_1)$. If the total reward to be achieved in at most K retransmissions is Z, then the states like $(k, \vec{A})$ such that $\sum_{i=1}^{G} I_i \geq Z$ are termination states. If the total reward achieved in first $K-1$ transmissions is $z < Z$, then we assume that any transmission policy will transmit only if the number of ready receivers is greater than or equal to $Z - z$. This ensures that every transmission policy will always achieve reward greater than or equal to Z in K retransmissions as desired. So, the problem is to obtain a sequence of control decisions in order to reach any of the termination states in minimum time.

Lets define by $J^*(k, \vec{A})$ to be the minimum expected cost (time) to terminate from the state $(k, \vec{A})$. Then, these costs can be obtained as follows [1].

$$\begin{aligned} J^*(k, \vec{A}) &= \lim_{l \to \infty} J_l(k, \vec{A}), \quad \text{where} \\ J_l(k, \vec{A}) &= \min\{J_{l,B}(k, \vec{A}), J_{l,T}(k, \vec{A})\}. \end{aligned} \tag{19}$$

We note that $J_l(k, \vec{A})$ is the optimal cost to terminate in the l^{th} iteration. This is iterative approach, which is guaranteed to converge to the actual optimal costs [1]. The terms $J_{l,B}(k, \vec{A})$ and $J_{l,T}(k, \vec{A})$ are the optimal costs to terminate from state $(k, \vec{A})$ it the decision were to back-off and transmit in l^{th} iteration, respectively. The costs at each iteration are obtained as follows.

$$J_{l,B}(k, \vec{A}) = E[X] + 1 + \sum_{(k, \vec{A}_1)} Pr\{(k, \vec{A}) \to (k, \vec{A}_1)\} J_{l-1}(k, \vec{A}_1) \tag{20}$$

$$J_{l,T}(k, \vec{A}) = E[V] + 1 + \sum_{(k+1, \vec{A}_1)} Pr\{(k, \vec{A}) \to (k+1, \vec{A}_1)\} J_{l-1}(k+1, \vec{A}_1). \tag{21}$$

Without loss of generality, we initialize costs $J_0(k, \vec{A}) = 0$ for every state $(k, \vec{A})$.

Now, for each state $(k, \vec{A})$ lets define three important quantities S_A, z_A and t_A as follows. S_A is the set of receivers that have already received the packet, z_A is the number of receivers that have already received the packet and t_A is the number of ready receivers that have not received the packet yet. Mathematically,

$$\begin{aligned} S_A &= \{i : I_i = 1\} \\ z_A &= \sum_{i=0}^{G} I_i \\ t_A &= \sum_{i \notin S_A} L_i. \end{aligned}$$

Now, consider any permutations $\vec{A}'$ of $\vec{A}$. We note that any such permeation will satisfy $z_{A'} = z_A$ and $t_{A'} = t_A$. Furthermore, using a fact that receiver readiness states are iid we can conclude that $J_l(k, \vec{A}) = J_l(k, \vec{A}')$ for every iteration l. Thus, to design an optimal transmission policy it suffices to consider any representative permutation of $\vec{A}$. Without loss of generality we consider a following representative. For the given values of z and t we consider a representative $\vec{A}$ such that $I_i = 1$ for every $1 \leq i \leq z$ and 0 otherwise. Similarly, $R_i = 1$ for every $z+1 \leq i \leq z+t$ and

0 otherwise. Now in view of this new problem structure, a system state can be simply defined as (k, z, t), which implies that k retransmissions are already over and first z receivers have received the packet and $z+1$ to $z+t$ receivers are ready to receive the packet. We illustrate this notation using an example. We consider a group size of 4. Now, state $(1, 2, 1)$ will mean that one transmission is already over, in this transmission two receivers have received the packet successfully and 1 receiver that has not received a packet is ready to receive. In particular, $\vec{A} = \{(1, \times), (1, \times), (0, 1), (0, 0)\}$. We note that readiness states for the receivers that have already received the packet is immaterial (denoted by $\times$).

With these notation we now prove that the proposed DT policy is optimal. The proof is by induction on number of iterations. For each iteration we will prove that

1) if $J_{l,T}(k, z, t) \leq J_{l,B}(k, z, t)$, then $J_{l,T}(k, z, t+1) \leq J_{l,B}(k, z, t+1)$. This shows that the threshold based decision is optimal.
2) $J_l(k, z+1, t) \leq J_l(k, z, t)$.
3) $J_l(k, z+1, t-1) \leq J_l(k, z, t)$.

Recall that the cost $J_0(k, z, t)$ for every state (k, z, t) is initialized to 0. Hence 1,2 and 3 holds for $l = 0$. By induction hypothesis we assume that 1,2, and 3 hold till l^{th} iteration. As a final step, we need to prove these three statements for the iteration $l+1$.

Proof of statement 1): We observe from equation (20) that

$$\begin{aligned} J_{l+1,B}(k, z, t) &= E[X] + 1 + \sum_{i=0}^{G-z} p_i^{G-z} J_l(k, z, i) \\ &= J_{l+1,B}(k, z, t+1), \end{aligned} \tag{22}$$

where p_i^{G-z} is the probability that i receivers are ready out of $G-z$ receivers that have not received the packet yet.

Now, if the control decision is to transmit from state $(k, z, t+1)$, then system makes transition from $(k+1, z+t+1, i)$ with probability $p_i^{G-z-t-1}$. For simplicity of notation, lets denote $p_i^{G-z-t-1}$ by p_i. We note that $(k+1, z+t+1, i)$ is just a representative state, which represents states where total reward received in $k+1$ transmissions is $z+t+1$ and i receivers that have not received the packet yet are ready to receive. Hence,

$$J_{l+1,T}(k, z, t+1) = E[V] + 1 + \sum_{i=0}^{G-z-t-1} p_i J_l(k+1, z+t+1, i). \tag{23}$$

Now, again consider a state $(k+1, z+t+1, i)$. We note that in this state i receivers among the receivers $z+t+2$ to G are ready to receive, but nothing has been said about the receiver $z+t+1$. We consider two cases, first the receiver $z+t+1$ is ready and second the receiver $z+t+1$ is not ready. Now, consider transmission from state (k, z, t). If $z+t+1^{th}$ receiver were ready, then the transition because of the control decision would be to state $(k+1, z+t, i+1)$, otherwise is the $z+t+1^{th}$ receiver were not ready then the transition would be to the state $(k+1, z+t, i)$. Thus,

$$\begin{aligned} J_{l+1,T}(k, z, t) &= E[V] + 1 + \sum_{i=0}^{G-z-t-1} pp_i J_l(k+1, z+t, i+1) + (1-p)p_i J_l(k+1, z+t, i) \\ &= E[V] + 1 + \sum_{i=0}^{G-z-t-1} p_i \left[pJ_l(k+1, z+t, i+1) + (1-p)J_l(k+1, z+t, i) \right]. \end{aligned} \tag{24}$$

Now we compare each term in the summation of equations (23) and (24) separately. By induction hypothesis, we note that $J_l(k+1, z+t+1, i) \leq J_l(k+1, z+t, i+1)$ and $J_l(k+1, z+t+1, i) \leq J_l(k+1, z+t, i)$. Hence, $J_l(k+1, z+t+1, i) \leq pJ_l(k+1, z+t, i+1) + (1-p)J_l(k+1, z+t, i)$. This proves statement 1) for iteration $l+1$.

Proof of statement 2): We need to prove that $J_{l+1}(k, z+1, t) \leq J_{l+1}(k, z, t)$. To prove the desired result we will show that $J_{l+1,B}(k, z+1, t) \leq J_{l+1,B}(k, z, t)$ and $J_{l+1,T}(k, z+1, t) \leq J_{l+1,T}(k, z, t)$. Now,

$$J_{l+1,B}(k, z+1, t) = E[X] + 1 + \sum_{i=0}^{G-z-1} p_i^{G-z-1} J_l(k, z+1, i). \tag{25}$$

Furthermore, as discussed in the proof of statement 1),

$$\begin{aligned} J_{l+1,B}(k,z,t) &= E[X]+1+\sum_{i=0}^{G-z-1} pp_i^{G-z-1}J_l(k,z,i+1)+(1-p)p_i^{G-z-1}J_l(k,z,i) \\ &= E[X]+1+\sum_{i=0}^{G-z-1} p_i^{G-z-1}\left[pJ_l(k+1,z,i+1)+(1-p)J_l(k+1,z,i)\right]. \end{aligned} \quad (26)$$

Again comparing individual terms in the summation of equations (25) and (26) and using the induction hypothesis as in the proof of statement 1) to obtain the desired result that $J_{l+1,B}(k,z+1,t) \le J_{l+1,B}(k,z,t)$.

Now, we consider

$$\begin{aligned} J_{l+1,T}(k,z+1,t) &= E[V]+1+\sum_{i=0}^{G-z-t-1} p_i^{G-z-t-1}J_l(k+1,z+t+1,i) \\ &= J_{l+1,T}(k,z,t+1). \end{aligned}$$

Hence, using statement 1), we conclude that $J_{l+1,T}(k,z+1,t) \le J_{l+1,T}(k,z,t)$.

Thus we obtained the desired result $J_{l+1}(k,z+1,t) \le J_{l+1}(k,z,t)$.

Proof of statement 3): We need to show that $J_{l+1}(k,z+1,t-1) \le J_{l+1}(k,z,t)$. To show the desired result, we will show that $J_{l+1,B}(k,z+1,t-1) \le J_{l+1,B}(k,z,t)$ and $J_{l+1,T}(k,z+1,t-1) \le J_{l+1,T}(k,z,t)$.

We observe that

$$\begin{aligned} J_{l+1,B}(k,z+1,t-1) &= E[X]+1+\sum_{i=0}^{G-z-1} p_i^{G-z-1}J_l(k,z+1,i) \\ &= J_{l+1,B}(k,z+1,t). \end{aligned}$$

In the proof of statement 2), we have shown that $J_{l+1}(k,z+1,t) \le J_{l+1,B}(k,z,t)$. Hence, $J_{l+1,B}(k,z+1,t-1) \le J_{l+1}(k,z,t)$.

Furthermore, $J_{l+1,T}(k,z+1,t-1) = J_{l+1,T}(k,z,t)$. This is because after transmission, system makes transition from both the states to the same state with equal probability.

Thus, statement 3) is proved for $l+1^{th}$ iteration.

By induction principle we can conclude that statements 1), 2) and 3) hold for all the iterations. Hence they will hold for the final optimal solution. Thus, DT policy indeed minimizes the time to achieve certain reward in the given number of transmissions. ∎