Let $\mu_1 + \mu_2 + \lambda_1 + \lambda_2 = 1$

Proposition 1 *Properties for the cost function* $J(x, y)$:

1. *Monotonicity and Boundedness in* x :

$$0 \leq J(x+1, y) - J(x, y) \leq 1, 0 \leq x \leq B-1$$

2. *Monotonicity and Boundedness in* y :

$$0 \leq J(x, y+1) - J(x, y) \leq 1, 0 \leq y \leq B-1$$

3. *Concavity along* x :

$$J(x+1, y) - J(x, y) \leq J(x, y) - J(x-1, y), 1 \leq x \leq B-1$$

4. *Concavity along* y :

$$J(x, y+1) - J(x, y) \leq J(x, y) - J(x-1, y), 1 \leq x \leq B-1$$

5. *Concavity along the line* $x + y = b$, $2 \leq b \leq B$: $J(x+1, y-1) - J(x, y) \leq J(x, y) - J(x-1, y+1)$, $1 \leq x \leq B-1$, $1 \leq y \leq B-1$

Proof:

- By Induction
- assume the property holds for $J_k(a, b)$ in the value iteration sequence
- prove the property for $J_{k+1}(a, b)$
- verify the property holds for the function $f(a, b) = 0$

proof of (1): Let

$$0 \leq J_k(x+1, y) - J_k(x, y) \leq 1, 0 \leq x \leq B-1$$

The result holds for the function $f(a, b) = 0$

case 1: $x + y \leq B - 2$

$$\begin{aligned} J_{k+1}(x,y) &= \mu_1 \left(I(x \geq 0) + \beta J_k \left((x-1)^+, y \right) \right) + \mu_2 \left(I(y \geq 0) + \beta J_k \left(x, (y-1)^+ \right) \right) \\ &+ \lambda_1 \beta J_k(x+1,y) + \lambda_2 \beta J_k(x,y+1) \end{aligned}$$

$$\begin{aligned} J_{k+1}(x+1,y) &= \mu_1 \left(1 + \beta J_k \left(x,y \right) \right) + \mu_2 \left(I(y \geq 0) + \beta J_k \left(x, (y-1)^+ \right) \right) \\ &+ \lambda_1 \beta J_k(x+2,y) + \lambda_2 \beta J_k(x+1,y+1) \end{aligned}$$

$$\begin{aligned} J_{k+1}(x+1,y) - J_{k+1}(x,y) &= \mu_1 \left(I(x=0) + \beta J_k \left(x,y \right) - J_k \beta \left((x-1)^+, y \right) \right) \\ &+ \mu_2 \beta \left(J_k \left(x+1, (y-1)^+ \right) - J_k \left(x, (y-1)^+ \right) \right) \\ &+ \lambda_1 \beta \left(J_k(x+2,y) - J_k(x+1,y) \right) \\ &+ \lambda_2 \beta \left(J_k(x+1,y+1) - J_k(x,y+1) \right) \end{aligned}$$

$$0 \leq J_{k+1}(x+1,y) - J_{k+1}(x,y) \leq \mu_1 + \mu_2 + \lambda_1 + \lambda_2 \leq 1$$

case 2: $x + y = B - 1$

$$\begin{aligned} J_{k+1}(x+1,y) &= \mu_1 \left(1 + \beta J_k \left(x,y \right) \right) + \mu_2 \left(I(y \geq 0) + \beta J_k \left(x, (y-1)^+ \right) \right) \\ &+ \lambda_1 \beta \max(J_k(x+1,y), J_k(x+2,y-1)) + \lambda_2 \beta \max(J_k(x,y+1), J_k(x+1,y)) \end{aligned}$$

$$\begin{aligned} J_{k+1}(x+1,y) - J_{k+1}(x,y) &= \mu_1 \left(I(x=0) + \beta J_k \left(x,y \right) - J_k \beta \left((x-1)^+, y \right) \right) \\ &+ \mu_2 \beta \left(J_k \left(x+1, (y-1)^+ \right) - J_k \left(x, (y-1)^+ \right) \right) \\ &+ \lambda_1 \beta \left(\max(J_k(x+2,y-1), J_k(x+1,y)) - J_k(x+1,y) \right) \\ &+ \lambda_2 \beta \left(\max(J_k(x,y+1), J_k(x+1,y)) - J_k(x,y+1) \right) \end{aligned}$$

$$\text{Note that } J_k(x+2,y-1) \leq J_k(x+2,y) \leq 1 + J_k(x+1,y)$$

$$\text{Note that } J_k(x+1,y) \leq J_k(x+1,y+1) \leq 1 + J_k(x,y)$$

Thus

$$0 \leq J_{k+1}(x+1,y) - J_{k+1}(x,y) \leq \mu_1 + \mu_2 + \lambda_1 + \lambda_2 \leq 1$$

Proof of (2) is similar

Proof of (3):

- By Induction
- assume the property holds for $J_k(a,b)$ in the value iteration sequence
- prove the property for $J_{k+1}(a,b)$
- verify the property holds for the function $f(a,b) = 0$

case 1: $x + y \leq B - 2$

$$\begin{aligned} J_{k+1}(x+1,y) - J_{k+1}(x,y) &= \mu_1 \left(\beta J_k\left(x,y\right) - J_k \beta\left(x-1,y\right)\right) \\ &+ \mu_2 \beta \left(J_k\left(x+1,(y-1)^+\right) - J_k\left(x,(y-1)^+\right)\right) \\ &+ \lambda_1 \beta \left(J_k(x+2,y) - J_k(x+1,y)\right) \\ &+ \lambda_2 \beta \left(J_k(x+1,y+1) - J_k(x,y+1)\right) \end{aligned}$$

$$\begin{aligned} J_{k+1}(x,y) - J_{k+1}(x-1,y) &= \mu_1 \left(I(x=1) + \beta J_k\left(x-1,y\right) - J_k \beta\left((x-2)^+,y\right)\right) \\ &+ \mu_2 \beta \left(J_k\left(x,(y-1)^+\right) - J_k\left(x-1,(y-1)^+\right)\right) \\ &+ \lambda_1 \beta \left(J_k(x+1,y) - J_k(x,y)\right) + \lambda_2 \beta \left(J_k(x,y+1) - J_k(x-1,y+1)\right) \end{aligned}$$

The concavity results hold from comparing the corresponding terms and the induction hypothesis.

case 2: $x + y = B - 1$

$$\begin{aligned}
J_{k+1}(x+1,y) - J_{k+1}(x,y) &= \mu_1 \left(\beta J_k(x,y) - J_k\beta(x-1,y)\right) \\
&\quad + \mu_2\beta\left(J_k\left(x+1,(y-1)^+\right) - J_k\left(x,(y-1)^+\right)\right) \\
&\quad + \lambda_1\beta\left(\max(J_k(x+2,y-1), J_k(x+1,y)) - J_k(x+1,y)\right) \\
&\quad + \lambda_2\beta\left(\max(J_k(x,y+1), J_k(x+1,y)) - J_k(x,y+1)\right)
\end{aligned}$$

$$\begin{aligned}
J_{k+1}(x,y) - J_{k+1}(x-1,y) &= \mu_1\left(I(x=1) + \beta J_k(x-1,y) - J_k\beta\left((x-2)^+, y\right)\right) \\
&\quad + \mu_2\beta\left(J_k\left(x,(y-1)^+\right) - J_k\left(x-1,(y-1)^+\right)\right) \\
&\quad + \lambda_1\beta\left(J_k(x+1,y) - J_k(x,y)\right) + \lambda_2\beta\left(J_k(x,y+1) - J_k(x-1,y+1)\right)
\end{aligned}$$

Show that $\max(J_k(x+2,y-1), J_k(x+1,y)) - J_k(x+1,y) \le J_k(x+1,y) - J_k(x,y)$.

- Let $\max(J_k(x+2,y-1), J_k(x+1,y)) = J_k(x+1,y)$. Then L.H.S $= 0$ and R.H.S is nonnegative.

- Let $\max(J_k(x+2,y-1), J_k(x+1,y)) = J_k(x+2,y-1)$. Then L.H.S. $\le J_k(x+2,y)$. The result now follows

Similarly $\max(J_k(x,y+1), J_k(x+1,y)) - J_k(x,y+1) \le J_k(x,y+1) - J_k(x-1,y+1)$. Thus concavity result holds.

Proof of (4) is similar

Proof of (5): Lets work it out!

Proposition 2 *If* $\mu_1 = \mu_2$ *and* $\lambda_1 \ge \lambda_2$ $J(x,y) \ge J(y,x)$, $x \ge y$

Proof: By Induction

case 1: $x + y \le B - 1$

$$\begin{aligned} J_{k+1}(x,y) &= \mu\left(I(x \ge 0) + \beta J_k\left((x-1)^+, y\right)\right) + \mu\left(I(y \ge 0) + \beta J_k\left(x, (y-1)^+\right)\right) \\ &\quad + \lambda_1 \beta J_k(x+1,y) + \lambda_2 \beta J_k(x, y+1) \end{aligned}$$

$$\begin{aligned} J_{k+1}(y,x) &= \mu\left(I(x \ge 0) + \beta J_k\left((y-1)^+, x\right)\right) + \mu\left(I(y \ge 0) + \beta J_k\left(y, (x-1)^+\right)\right) \\ &\quad + \lambda_1 \beta J_k(y+1,x) + \lambda_2 \beta J_k(y, x+1) \end{aligned}$$

The result follows comparing each term with the others, and using induction.

case 2: $x + y = B$

$$\begin{aligned} J_{k+1}(x,y) &= \mu\left(I(x \ge 0) + \beta J_k\left((x-1)^+, y\right)\right) + \mu\left(I(y \ge 0) + \beta J_k\left(x, (y-1)^+\right)\right) \\ &\quad + \lambda_1 \beta \max\left(J_k(x+1,y-1), J_k(x,y)\right) + \lambda_2 \beta \max\left(J_k(x-1,y+1), J_k(x,y)\right) \end{aligned}$$

$$\begin{aligned} J_{k+1}(y,x) &= \mu\left(I(x \ge 0) + \beta J_k\left((y-1)^+, x\right)\right) + \mu\left(I(y \ge 0) + \beta J_k\left(y, (x-1)^+\right)\right) \\ &\quad + \lambda_1 \beta J_k(y,x) + \lambda_2 \beta J_k(y-1, x+1) \end{aligned}$$

$$\begin{aligned} \max\left(J_k(x,y), J_k(x+1,y-1)\right) - J_k(y,x) &\ge J_k(x+1,y-1) - J_k(x,y) \\ &\ge J_k(y-1,x+1) - \max\left(J_k(x,y), J_k(x-1,y+1)\right) \end{aligned}$$

The result follows since $\lambda_1 \ge \lambda_2$.