Spectrum Pricing Games with Arbitrary Bandwidth Availability Probabilities

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***Abstract*—We consider price competition among multiple primary users in a Cognitive Radio Network with multiple secondary users. In every slot, each primary has unused bandwidth with some probability, possibly different for different primaries, which he would like to lease to a secondary. For the case in which all the primaries and secondaries are in a single location, we explicitly compute a Nash equilibrium and show its uniqueness. Then we consider the game with spatial reuse of spectrum, and for a special class of conflict graphs, explicitly compute a Nash equilibrium and show its uniqueness in a natural sub-class of Nash equilibria.**

I. Introduction

Cognitive Radio Technology [1] is a newly emerging technique for using the available radio spectrum more efficiently. In Cognitive Radio Networks (CRNs), there are two types of spectrum users: (i) *primary* users who lease portions (channels or bands) of the spectrum directly from the regulator, and (ii) *secondary* users who lease channels from primaries and can use a channel when it is not in use by the primary. Time is slotted, and in every slot, each primary has unused bandwidth with some probability, which he would like to sell to secondaries. Now, secondaries would like to buy bandwidth from the primaries that offer it at a low price, which results in *price competition* among the primaries. If a primary quotes a low price, it will attract buyers, but at the cost of reduced revenues. This is a common feature of an *oligopoly* [2], in which multiple firms sell a common good to a pool of buyers. Price competition in an oligopoly is naturally modeled using *game theory* [17], and has been extensively studied in economics using, for example, the classic *Bertrand game* [2] and its variants.

However, a CRN has several distinguishing features, which makes the price competition very different from oligopolies encountered in economics. First, in every slot, each primary may or may not have unused bandwidth available. So a primary who *has* unused bandwidth is uncertain about the number of primaries from whom he will face competition. A low price will result in unnecessarily low revenues in the event that very few other primaries have unused bandwidth, because even with a higher price the primary's bandwidth would have been bought, and vice versa. Second, spectrum is a commodity that allows *spatial reuse*: the same band can be simultaneously used at far-off locations without interference; on the other hand, simultaneous transmissions at neighboring locations on the same band interfere with each other.

Pricing related issues have been extensively studied in the context of wired networks and the Internet; see [10] for an overview. Price competition among spectrum providers in wireless networks has been studied in [11], [12], [13], [16], [14], [15]. However, neither uncertain bandwidth availability, nor spatial reuse is modeled in any of the above papers. Also, most of these papers do not explicitly find a NE (exceptions are [12], [14]).

In the Economics literature, the Bertrand game [2] and several of its variants [5], [6], [7], [8], [9] have been used to study price competition. The closest to our work are [8], [9], which analyze price competition where each seller may be inactive with some probability and find a Nash equilibrium [2] (NE), which they show to be unique. However, the results in [8], [9] are restricted to the case of one buyer; but, a CRN is likely to have multiple secondaries, which we seek to consider. In our prior work [21], [22], we analyzed price competition in a CRN with multiple primaries and secondaries. However, with the exception of [21], all the above papers [8], [9], [22] analyze only the *symmetric* model where the bandwidth availability probability of each seller is the same. Even in [21], the asymmetric case is considered only for a toy model with two primaries and one secondary. Also, in [8], [21], [22] it is only shown that the NE is unique in the class of *symmetric* NE. In [9], uniqueness in the class of *all* NE is shown only for the case of a single buyer (and symmetric bandwidth availability probabilities). In addition, none of the above papers (except [21], [22], which focus on the symmetric model) consider spatial reuse.

We consider price competition in a CRN with multiple primaries and multiple secondaries, where each primary has available bandwidth in a slot with a certain probability, which may be different for different primaries. First, we analyze the case of primaries and secondaries in a single location (Section II). Since prices can take real values, the *strategy sets of players are continuous*. Thus, classical results do not establish the existence and uniqueness of NE in the resulting game, and there is no standard algorithm for finding a NE, unlike when each player's strategy set is finite [17]. Nevertheless, we are able to explicitly compute a NE and show that it is unique in the class of *all* NE, allowing for player strategies that are arbitrary mixtures of continuous and discrete probability distributions (Section III). Our explicit NE computations reveal that asymmetry in bandwidth availability probabilities of different primaries may lead to fundamental structural differences in equilibrium pricing strategies as compared to the symmetric scenario (Section III-C). We subsequently model the scenario where each primary owns bandwidth across multiple locations using a conflict graph in which there is an edge between each pair of mutually interfering locations. Each primary must simultaneously select a set of mutually non-interfering

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locations (independent set) at which to offer bandwidth and the prices at those locations (Section IV). We first show that there exist multiple NE even for a simple two-node graph; then, for a special, but general class of conflict graphs, we explicitly compute a NE and shows its uniqueness among the class of NE with symmetric independent set selection strategies of the primaries. All proofs are deferred until the Appendix.

Finally, our results apply to any setting where the sellers' supply is uncertain. In particular, *microgrids* [19] are a newly emerging technology for distributed electricity generation, which consist of a connected network of generators (*e.g.*, solar panels, wind turbines) and loads (*e.g.*, households, factories). There is uncertainty in the power generated by a generator at a given time, *e.g.*, the power produced by a solar panel on a given day depends on the availability of sunlight. Our results characterize NE in pricing games in such electricity markets.

II. Single Location Model

Suppose there are $n \geq 2$ primaries and $k \geq 1$ secondaries in a region. Each secondary may constitute a customer who requires 1 unit of bandwidth, or may simply be a demand for 1 unit of bandwidth. Time is divided into slots of equal duration. In every slot, primary $i \in \{1, \ldots, n\}$ has unused bandwidth with probability $q_i \in (0, 1)$, where we assume without loss of generality that:

$$q_1 \geq q_2 \geq \ldots \geq q_n. \tag{1}$$

A primary i who has unused bandwidth in a slot can lease it out to a secondary for the duration of the slot, in return for an access fee of p_i. Leasing in a slot incurs a cost of $c \geq 0$. This cost may arise, for example, if the secondary uses the primary's infrastructure to access the Internet. We assume that $p_i \leq v$ for each primary, for some constant $v > c$. This upper bound v may either be a regulator imposed limit to ensure that primaries do not excessively overprice bandwidth or the valuation of each secondary for 1 unit of bandwidth.

Secondaries buy bandwidth from the primaries that offer the lowest price. More precisely, in a given slot, let Z be the number of primaries who offer unused bandwidth. Then the bandwidth of the $\min(Z, k)$ primaries that offer the lowest prices is bought (ties are resolved at random).

We formulate the above price competition among primaries as a *game* [17], in which the primaries are the players, and the action of primary i is the price p_i that he chooses [1]. If primary i has unused bandwidth and primary $j \in \{1, \ldots, n\}$ sets a price of p_j, then the *utility* or *payoff* of primary i, $u_i(p_1, \ldots, p_n)$, is defined to be his net revenue [2]. Thus:

$$u_i(p_1, \ldots, p_n) = \begin{cases} p_i - c & \text{if primary } i \text{ sells his bandwidth} \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

We allow each primary i to choose his price p_i randomly from a set of prices using an arbitrary distribution function [3] (d.f.) $\psi_i(.)$, which is referred to as the *strategy* of primary i. The vector $(\psi_1(.), \ldots, \psi_n(.))$ of strategies of the primaries is called a *strategy profile* [2]. Let $\psi_{-i} = (\psi_1(.), \ldots, \psi_{i-1}(.), \psi_{i+1}(.), \ldots, \psi_n(.))$ denote the vector of strategies of the primaries other than i. Let $E\{u_i(\psi_i(.), \psi_{-i})\}$ denote the expected utility of primary i when he adopts strategy $\psi_i(.)$ and the other primaries adopt ψ_{-i}.

A *Nash equilibrium* (NE) is a strategy profile such that no player can improve his expected utility by unilaterally deviating from his strategy [2]. Thus, $(\psi_1^*(.), \ldots, \psi_n^*(.))$ is a NE if for each primary i: $E\{u_i(\psi_i^*(.), \psi_{-i}^*)\} \geq E\{u_i(\tilde{\psi}_i(.), \psi_{-i}^*)\}, \ \forall \ \tilde{\psi}_i(.)$. When players other than i play ψ_{-i}^*, $\psi_i^*(.)$ maximizes i's expected utility and is thus his *best-response* [2] to ψ_{-i}.

If $k \geq n$, then the number of buyers is always greater than or equal to the number of sellers. So a primary i will sell his unused bandwidth even when he chooses the maximum possible price v. So the strategy profile under which all primaries deterministically choose the price v is the unique NE. So henceforth, we assume that $k \leq n - 1$.

III. Single Location Nash Equilibrium Analysis

For convenience, we introduce the notion of a "pseudo-price". The pseudo-price of primary $i \in \{1, \ldots, n\}$, denoted as p_i', is the price he selects if he has unused bandwidth and $p_i' = v + 1$ otherwise [4]. Let $\phi_i(.)$ be the d.f. of p_i'. For $c \leq x \leq v$, $p_i' \leq x$ for a primary i iff he has unused bandwidth and sets a price $p_i \leq x$. So $\phi_i(x) = q_i P(p_i \leq x) = q_i \psi_i(x)$. Thus, $\psi_i(.)$ and $\phi_i(.)$ differ only by a constant factor on $[c, v]$ and we use them interchangeably wherever applicable.

For a function $f(.)$, we denote the left and right hand side limits at a point a, $\lim_{x \uparrow a} f(x)$ and $\lim_{x \downarrow a} f(x)$ by $f(a-)$ and $f(a+)$ respectively [4].

A. Necessary Conditions for a NE

Consider a NE under which the d.f. of the price (respectively, pseudo-price) of primary $i \in \{1, \ldots, n\}$ is $\psi_i(.)$ (respectively, $\phi_i(.)$). In Theorem 1, we show that the NE strategies must have a particular structure. First, we describe some basic properties of the NE strategies.

Property 1: $\phi_2(.), \ldots, \phi_n(.)$ are continuous on $[c, v]$. $\phi_1(.)$ is continuous at every $x \in [c, v)$, has a jump [5] of size $q_1 - q_2$ at v if $q_1 > q_2$ and is continuous at v if $q_1 = q_2$.

In particular, there does not exist a pure strategy NE (one in which every primary selects a single price with probability (w.p.) 1).

Now, let $u_{i,max}$ be the expected payoff that primary i gets in the NE and L_i be the lower endpoint of the support set [6] of $\psi_i(.)$, *i.e.*:

$$L_i = \inf\{x : \psi_i(x) > 0\}. \tag{3}$$

[1] If primary i has no unused bandwidth, it does not matter what price p_i he sets. Yet, for convenience, we speak of p_i as being his action.

[2] If instead, $u_i(p_1, \ldots, p_n)$ were defined to be primary i's net revenue, unconditional on whether he has unused bandwidth or not, then the expected utilities of primary i in the game analysis would all be scaled by q_i.

[3] Recall that the distribution function [18] of a random variable (r.v.) X is the function $G(x) = P(X \leq x)$.

[4] The choice $v + 1$ is arbitrary. Any other value greater than v also works.

[5] A d.f. $f(x)$ is said to have a *jump* (discontinuity) of size $b > 0$ at $x = a$ if $f(a) - f(a-) = b$ [18].

[6] The support set of a d.f. is the smallest closed set such that its complement has probability zero under the d.f. [18].

Also, let w_i be the probability that k or more primaries out of primaries $\{1,\ldots,n\}\backslash i$ have unused bandwidth, which can be easily computed using the fact that each primary j independently has unused bandwidth w.p. q_j.

Property 2: $L_1 = \ldots L_n = \tilde{p}$, where $\tilde{p} = v - w_1(v-c)$. Also, $u_{i,max} = \tilde{p} - c$, $i = 1,\ldots,n$.

Thus, the lower endpoints of the support sets of the d.f.s $\psi_1(.),\ldots,\psi_n(.)$ of all the primaries are the same, and every primary gets the same expected payoff in the NE.

Theorem 1: The following are necessary conditions for strategies $\phi_1(.),\ldots,\phi_n(.)$ to constitute a NE:
1) $\phi_1(.),\ldots,\phi_n(.)$ satisfy Property 1 and Property 2.
2) There exist numbers $R_j, j = 1,\ldots,n+1$, and a function $\{\phi(x) : x \in [\tilde{p}, v)\}$ such that

$$\tilde{p} = R_{n+1} < R_n \leq R_{n-1} \leq \ldots \leq R_1 \leq v, \tag{4}$$

$$\phi_1(x) = \ldots = \phi_j(x) = \phi(x),\ \tilde{p} \leq x < R_j, \tag{5}$$

for each $j \in \{1,\ldots,n\}$,

$$\text{and} \quad \phi_j(R_j) = q_j,\ j = 1,\ldots,n. \tag{6}$$

Also, every point in $[\tilde{p}, R_j)$ is a best response for primary j and he plays every sub-interval in $[\tilde{p}, R_j)$ with positive probability. Finally, $R_1 = R_2 = v$.

Theorem 1 says that all n primaries play prices in the range $[\tilde{p}, R_n)$, the d.f. $\phi_n(.)$ of primary n stops increasing at R_n, the remaining primaries $1,\ldots,n-1$ play prices in the range $[R_n, R_{n-1})$, the d.f. $\phi_{n-1}(.)$ of primary $n-1$ stops increasing at R_{n-1}, and so on. Also, primary 1's d.f. $\phi_1(.)$ has a jump of height $q_1 - q_2$ at v if $q_1 > q_2$. Fig. 1 illustrates the structure.

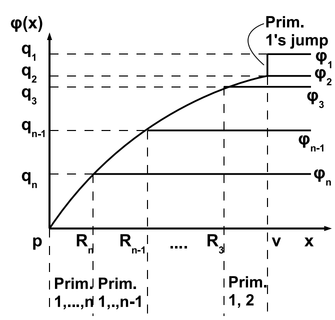

Fig. 1. The figure shows the structure of a NE described in Theorem 1. The horizontal axis shows prices in the range $x \in [\tilde{p}, v]$ and the vertical axis shows the functions $\phi(.)$ and $\phi_1(.),\ldots,\phi_n(.)$.

B. Explicit Computation, Uniqueness and Sufficiency

By Theorem 1, for each $i \in \{1,\ldots,n\}$:

$$\phi_i(x) = \begin{cases} \phi(x), & \tilde{p} \leq x < R_i \\ q_i, & x \geq R_i \end{cases} \tag{7}$$

So the candidate NE strategies $\phi_1(.),\ldots,\phi_n(.)$ are completely determined once the numbers $\tilde{p}, R_1,\ldots,R_n$ and the function $\phi(.)$ are specified. Also note that Property 2 provides the value of $\tilde{p}$, and $R_1 = R_2 = v$ by Theorem 1. First, we will show that there also exist unique $R_3,\ldots,R_n$ and $\phi(.)$ satisfying (4), (5), and (6) and will compute them. Then, we will show that the resulting strategies given by (7) indeed constitute a NE.

Let $p'_{(k),i}$ be the k'th smallest pseudo-price out of the pseudo-prices, $\{p'_l : l \in \{1,\ldots,n\}, l \neq i\}$, of the primaries other than i. Also, let $F_{-i}(x)$ denote the d.f. of $p'_{(k),i}$. Since there are k secondaries, if primary 1 has unused bandwidth and sets $p_1 = x \in [\tilde{p}, v)$, his bandwidth is bought iff [7] $p'_{(k),1} > x$, which happens w.p. $1 - F_{-1}(x)$. Note that primary 1's payoff is $(x-c)$ if his bandwidth is bought and 0 otherwise. So:

$$E\{u_1(x, \psi_{-1})\} = (x-c)(1-F_{-1}(x)) = \tilde{p}-c,\ x \in [\tilde{p}, v) \tag{8}$$

where the second equality follows from the facts that each $x \in [\tilde{p}, v)$ is a best response for primary 1 by Theorem 1, and $u_{1,max} = \tilde{p} - c$ by Property 2. By (8), we get:

$$F_{-1}(x) = g(x),\ x \in [\tilde{p}, v) \tag{9}$$

$$\text{where, } g(x) = \frac{x - \tilde{p}}{x - c},\ x \in [\tilde{p}, v). \tag{10}$$

Next, we calculate $R_i, i = 3,\ldots,n$ and $\phi(.)$ using (9).

1) Computation of $R_i, i = 3,\ldots,n$: For a fixed $k \in \{1,\ldots,n-1\}$ and $0 \leq y \leq 1$, let $f_i(y)$ be the probability of k or more successes out of $n-1$ independent Bernoulli events, $(i-1)$ of which have the same success probability y and the remaining $(n-i)$ have success probabilities $q_{i+1},\ldots,q_n$. An expression for $f_i(y)$ can be easily computed [8].

Now, to compute $R_i, i \in \{3,\ldots,n\}$, we note that by (7) and (4), $\phi_j(R_i) = q_i$, $j = 2,\ldots,i$, and $\phi_j(R_i) = q_j$, $j = i+1,\ldots,n$. So from the preceding paragraph, with the events $\{p'_j \leq R_i\}, j = 2,\ldots,n$ as the $n-1$ Bernoulli events, and by the definition of $F_{-1}(.)$, we get:

$$F_{-1}(R_i) = f_i(q_i). \tag{11}$$

By (9) and (11):

$$g(R_i) = f_i(q_i). \tag{12}$$

By (10) and (12), R_i is unique and is given by:

$$R_i = c + \frac{\tilde{p} - c}{1 - f_i(q_i)}. \tag{13}$$

2) Computation of $\phi(.)$: Now we compute the function $\{\phi(.) : x \in [\tilde{p}, v)\}$ by separately computing it for each interval $[R_{i+1}, R_i)$, $i \in \{2,\ldots,n\}$. If $R_{i+1} = R_i$, then note that the interval $[R_{i+1}, R_i)$ is empty. Now suppose $R_{i+1} < R_i$. For $x \in [R_{i+1}, R_i)$, by (7) and (4):

$$\phi_j(x) = q_j,\ j = i+1,\ldots,n \tag{14}$$

$$\text{and } \phi_1(x) = \ldots = \phi_i(x) = \phi(x). \tag{15}$$

By definition of the function $f_i(.)$, with the events $\{p'_j \leq x\}, j = 2,\ldots,n$ as the $n-1$ Bernoulli events, by definition of $F_{-1}(x)$ and using $P\{p'_j \leq x\} = \phi_j(x)$, (14) and (15):

$$F_{-1}(x) = f_i(\phi(x)),\ R_{i+1} \leq x < R_i. \tag{16}$$

By (9) and (16):

$$f_i(\phi(x)) = g(x),\ R_{i+1} \leq x < R_i. \tag{17}$$

[7] By Property 1, no primary has a jump at any $x \in [\tilde{p}, v)$. So $P(p'_{(k),1} = x) = 0$.

[8] The expression is derived in the proof of Lemma 1 in the Appendix.

Lemma 1: For each x, (17) has a unique solution $\phi(x)$. The function $\phi(.)$ is strictly increasing and continuous on $[\tilde{p}, v)$. For $i \in \{2, \dots, n\}$, $\phi(R_i) = q_i$. Also, $\phi(\tilde{p}) = 0$.

Thus, there is a unique function $\phi(.)$, and by (7), unique $\phi_i(.), i = 1, \dots, n$ that satisfy the conditions in Theorem 1.

3) Sufficiency: We have shown that $R_1, \dots, R_n$ and the functions $\phi_1(.), \dots, \phi_n(.)$ computed above are the unique ones that satisfy the necessary conditions for a NE stated in Theorem 1. The following result shows sufficiency:

Theorem 2: The pseudo-price d.f.s $\phi_i(.), i = 1, \dots, n$ in (7), with $R_1 = R_2 = v$, R_i, $i = 3, \dots, n$ given by (13), and $\phi(.)$ being the solution of (17), constitute the unique NE. The corresponding price d.f.s are $\psi_i(x) = \frac{1}{q_i}\phi_i(x)$, $x \in [c, v]$, $i = 1, \dots, n$.

C. Discussion

The structure of the unique NE identified in Theorems 1 and 2 provides several interesting insights:

1) First, from (1), (4) and the fact that the support set of $\psi_i(.)$ is $[\tilde{p}, R_i]$, it follows that primaries with a low bandwidth availability probability (q) do not play high prices, whereas those with a high q do (see Fig. 1). Intuitively this is because all the primaries play low prices (near $\tilde{p}$), so if a primary sets a high price, he is undercut by all the other primaries. But a primary with a high q runs a lower risk of being undercut than one with a low q because of the lower bandwidth availability probabilities of the set of primaries *other than itself*.

2) Second, by Property 1, $\psi_1(.)$ has a jump at v iff $q_1 > q_2$, whereas $\psi_2(.), \dots, \psi_n(.)$ are always continuous on $[c, v]$.

A special case of the model in this paper is the symmetric case $q_1 = \ldots = q_n$, which was studied in prior work [21]; in the NE found in [21], the support set of every d.f. $\psi_i(.), i = 1, \dots, n$ is the same ($[\tilde{p}, v]$) and they are all continuous. Also, in [21], uniqueness of the NE in the class of *symmetric* NE was shown, whereas in the present paper, we have shown uniqueness in the class of *all* NE. Thus, the analysis in this paper is consistent with that in [21] and strengthens that result; also, it reveals the fundamental differences introduced by asymmetric bandwidth availability probabilities in the support sets and continuity of the NE strategies.

IV. Spatial Reuse

We now consider the price competition game with spatial reuse, in which primaries can simultaneously lease bandwidth to secondaries at multiple locations. Each of the n primaries now owns a channel throughout a large region. For $i \in \{1, \dots, n\}$, primary i's own usage of the channel is such that in every slot, he either uses his channel throughout the region (with probability (w.p.) $1 - q_i$), or does not use it anywhere in the region (w.p. q_i). A typical scenario where this happens is when the primary broadcasts the same signal over the entire region, *e.g.*, if the primary is a television broadcaster. Now, the region contains smaller parts (*e.g.*, towns in a state), which we refer to as *locations*. There are k secondaries at each location. As in Section II, each primary quotes a price of at most v, and incurs a cost of c at each location at which it leases bandwidth.

The region can be represented by an undirected graph [20] $\mathcal{G}$, called the *conflict graph*, in which each node represents a location, and there is an edge between two nodes iff transmissions at the corresponding locations interfere with each other. Now, a primary who is not using his channel must offer it at a set of mutually non-interfering locations, or equivalently, at an *independent set* [9] (I.S.) of nodes; otherwise secondaries will not be able to successfully transmit simultaneously using the bandwidth they purchase, owing to mutual interference [10]. Thus, each primary must jointly select an I.S. at which to offer bandwidth, and the prices to set at the nodes in it. *A strategy of a primary now provides a probability mass function (p.m.f.) for selection among the I.S. and the price distribution he uses at each node of the I.S. (both selections contingent on having unused bandwidth).* Note that we allow a primary to use different (and arbitrary) price distributions at different nodes (and therefore allow, but do not require, the selection of different prices at different nodes), and arbitrary p.m.f. (i.e., discrete distributions) for selection among the different I.S.

Now, in presence of spatial reuse, there are multiple NE in general, in contrast to the single location case where we showed that there is a unique NE (Theorem 2). For example, suppose there are two nodes 1 and 2 connected by an edge, two primaries ($n = 2$) and one secondary at each node ($k = 1$). Then it is easy to check that there are three distinct NE– the strategy profiles in which primary 1 offers bandwidth at node 1 and primary 2 at node 2 w.p. 1, or vice versa, and both primaries set a price of v w.p. 1, and a third NE which will be described in Theorem 3 below.

So henceforth, we focus on the class of NE where each primary uses the same I.S. p.m.f. (but possibly different price distributions at individual nodes). We denote this class of NE with symmetric I.S. selection strategies of the primaries by $\mathcal{S}$. We now argue that under any NE in class $\mathcal{S}$, the price distributions at all the nodes are uniquely specified once the I.S. selection strategy is determined. Let there be M I.S. in $\mathcal{G}$, and let each primary select among them as per the p.m.f. $(r_1, \dots, r_M)$. This provides the probabilities with which a primary offers bandwidth at each node when he has unused bandwidth (this probability for a given node equals the sum of the probabilities associated with all the I.S. that contain the node). Let this *selection probability for node* j be denoted α_j. Then, considering that primary i has unused bandwidth w.p. q_i, he offers it at node j w.p. $q_i\alpha_j$. The price selection problem at each node j is now equivalent to that for the single location case analyzed in the preceding two sections, the difference being that primary $i \in \{1, \dots, n\}$ offers unused bandwidth w.p. $q_i\alpha_j$, instead of q_i, at node j. Thus:

Lemma 2: Suppose under a NE in class $\mathcal{S}$, each primary selects node j w.p. α_j if he has unused bandwidth. Then under that NE the price distribution of primary $i \in \{1, \dots, n\}$ at node j is $\psi_i(.)$ in Theorem 2, with $q_1\alpha_j, \dots, q_n\alpha_j$ in place of $q_1, \dots, q_n$ all through.

Thus, the NE strategies of all the primaries are completely specified once the I.S. selection p.m.f. $(r_1, \dots, r_M)$ (which

[9] Recall that an independent set [20] in a graph is a set of nodes such that there is no edge between any pair of nodes in the set.

[10] Note that secondaries are usually customers or local providers, and purchase bandwidth for communication (and not T.V. broadcasts). Thus, two secondaries can not use the same band simultaneously at interfering locations.

will in turn provide the α_js) is obtained. We determine this p.m.f. for a specific conflict graph, $\mathcal{G}_m$, which is a linear arrangement of $m \geq 2$ nodes as shown in Fig. 2, with an edge between each pair of adjacent nodes. As an example, this would be the conflict graph for locations along a highway or a row of roadside shops. Let the nodes be numbered $1, \ldots, m$ from left to right, and $I_o = \{1, 3, \ldots\}$ and $I_e = \{2, 4, \ldots\}$ be the "odd" and "even" I.S (see Fig. 2). Note that I_o and I_e are disjoint I. S., and $I_o \cup I_e$ is the set of all nodes.

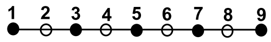

Fig. 2. The figure shows a linear graph $\mathcal{G}_m$ with $m = 9$. The darkened and un-darkened nodes constitute I_o and I_e respectively.

It is easy to check that in any NE, no primary selects an I.S. that is not maximal [11]. There are however several maximal I.S. in $\mathcal{G}_m$, e.g., $\{1, 4, 6, 8, \ldots\}$. The following lemma allows us to rule out all of them except I_o, I_e under a NE in class $\mathcal{S}$.

Lemma 3: A primary never selects any I.S. other than I_o and I_e under a NE in class $\mathcal{S}$.

By Lemma 3, under a NE in class $\mathcal{S}$, the p.m.f. for I.S. selection is characterized by a single probability t with which a primary selects I_e; each primary selects I_o w.p. $1-t$. Thus,

$$\alpha_j = \begin{cases} t, & j \in I_e \\ 1-t, & j \in I_o \end{cases} \tag{18}$$

We now state a theorem which, for different possible parameter values, provides a value of t that corresponds to a NE in class $\mathcal{S}$ and shows that this value is unique. First, we introduce some notation. Since primary i has unused bandwidth w.p. q_i and offers it at node j w.p. α_j, he offers bandwidth at node j w.p. $q_i\alpha_j$. Let $\overline{w}_i(\alpha_j)$ be the probability that k or more out of primaries $\{1, \ldots, n\}\backslash i$ offer bandwidth at node j. It can be shown that for every fixed i, $\overline{w}_i(\alpha_j)$ is a strictly increasing function of α_j on $[0, 1]$; also, intuitively, $\overline{w}_i(\alpha_j)$ is a measure of the price competition at node j. In particular, if α_j (and hence $\overline{w}_i(\alpha_j)$) is large, then primaries offer bandwidth with a large probability at node j (conditional on having unused bandwidth) and the price competition is intense, and vice versa.

Theorem 3: The strategy profile in which, primary i, if he has unused bandwidth, offers it at I_e and I_o w.p. t and $1-t$ respectively, and at node j selects the price as per the distribution $\psi_i(.)$ in Theorem 2, with $q_1, \ldots, q_n$ replaced by $q_1\alpha_j, \ldots, q_n\alpha_j$ all through, where α_j is as in (18), is the unique NE in class $\mathcal{S}$, where t in different cases is as follows:

- If m is even, $t = \frac{1}{2}$.
- If m is odd and $\overline{w}_1(1) \leq \frac{2}{m+1}$, $t = 0$.
- If m is odd and $\overline{w}_1(1) > \frac{2}{m+1}$, $t \in (0, 1)$ and is the unique solution of:

$$\left(\frac{m+1}{2}\right)(1-\overline{w}_1(1-t)) = \left(\frac{m-1}{2}\right)(1-\overline{w}_1(t)) \tag{19}$$

We now explain Theorem 3. For m even, I_o and I_e are of the same size: $|I_o| = |I_e| = \frac{m}{2}$. So consistent with intuition, in the NE in class $\mathcal{S}$, each primary selects I_o and I_e w.p. $\frac{1}{2}$ each. For odd m, $|I_o| = \frac{m+1}{2}$ and $|I_e| = \frac{m-1}{2}$; so I_o is larger than I_e. Hence, when $\overline{w}_1(1) \leq \frac{2}{m+1}$, every primary strictly prefers and offers bandwidth at the larger I.S. I_o w.p. 1; so $t = 0$. But when $\overline{w}_1(1) > \frac{2}{m+1}$, if each primary were to offer bandwidth w.p. 1 at I_o, the price competition at the nodes in I_o would be intense (recall that $\overline{w}_1(\alpha_j)$ is a measure of the competition at node j), driving down the prices, and a primary would prefer to unilaterally deviate to the smaller I.S. I_e and set the maximum price v at every node in I_e. So $t = 0$ does not constitute a NE in this case. In the NE, each primary offers bandwidth with positive probabilties at both I_o and I_e ($0 < t < 1$) such that the payoffs at I_o and I_e are equalized– the solution of (19) is the value of t at which this happens.

Due to space constraints, we have stated our results only for a linear graph $\mathcal{G}_m$. However, similar to our prior work [22] (which analyzed the symmetric case $q_1 = \ldots = q_n$), a NE can be computed and its uniqueness in class $\mathcal{S}$ can be shown for a large class of graphs, referred to as mean valid graphs*, which includes grid graphs in one, two and three dimensions, and the conflict graph of a cellular network.*

[11] Recall that a maximal I.S. is one that is not a proper subset of any other I.S [20].

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APPENDIX

A. Proofs of results in Section III-A

We first prove a series of lemmas and then deduce Properties 1 and 2 and Theorem 1 from them.

Lemma 4: For $i = 1, \ldots, n$, $\psi_i(.)$ is continuous, except possibly at v. Also, at most one primary has a jump at v.

Proof: Suppose $\psi_i(.)$ has a jump at a point x_0, $c < x_0 < v$. Then for some $\epsilon > 0$, no primary $j \neq i$ chooses a price in $[x_0, x_0 + \epsilon]$ because it can get a strictly higher payoff by choosing a price just below x_0 instead. This in turn implies that primary i gets a strictly higher payoff at the price $x_0 + \epsilon$ than at x_0. So x_0 is not a best response for primary i, which contradicts the assumption that $\psi_i(.)$ has a jump at x_0. Thus, $\psi_i(.)$ is continuous at all $x < v$.

Now, suppose primary i has a jump at v. Then a primary $j \neq i$ gets a higher payoff at a price just below v than at v. So v is not a best response for primary j and he plays it with 0 probability. Thus, at most one primary has a jump at v. ∎

Lemma 5: For every $\epsilon > 0$, there exist primaries m and j, $m \neq j$, such that $\psi_m(v - \epsilon) < 1$ and $\psi_j(v - \epsilon) < 1$.
That is, at least two primaries play prices just below v with positive probability.

Proof: Suppose not. Fix i and let:

$$y = \inf\{x : \psi_l(x) = 1 \ \forall l \neq i\}. \tag{20}$$

By definition of y, $\psi_l(x) = 1 \ \forall l \neq i$ and $x > y$. Also, since $\psi_l(.)$ is a distribution function, it is right continuous [18]. So

$$\psi_l(y) = 1 \ \forall l \neq i. \tag{21}$$

Suppose $y < v$. By (21):

$$P\{p_l \in (y, v]\} = 0, \ \forall l \neq i. \tag{22}$$

So every price $p_i \in (y, v)$ is dominated by $p_i = v$. Hence:

$$P\{p_i \in (y, v)\} = 0 \tag{23}$$

By (22) and (23):

$$P\{p_j \in (y, v)\} = 0, \ j = 1, \ldots, n. \tag{24}$$

By (20), $\forall \epsilon > 0$, $\psi_l(y - \epsilon) < 1$ for at least one primary $l \neq i$; otherwise the infimum in the RHS of (20) would be less than y. So this primary l plays prices just below y with positive probability. Now, if primary l sets a price $p_l < v$, he gets a payoff equal to the revenue, $(p_l - c)$, if bandwidth is sold, times the probability that bandwidth is sold. Also, by Lemma 4, $\psi_j(.), j = 1, \ldots, n$ are continuous at all prices below v. So by (24), a price p_l just below v yields a higher payoff than a price just below y. This is because, $p_l - c$ is lower by approximately $v - y$ for p_l just below y than for p_l just below v, but by (24) and continuity of $\psi_j(.), j = 1, \ldots, n$, the probability that bandwidth is sold for a price p_l just below y can be made arbitrarily close to the probability that bandwidth is sold for a price p_l just below v. This contradicts the assumption that primary l plays prices just below y with positive probability.

Thus, y in (20) equals v and hence at least one primary $j \neq i$ plays prices just below v with positive probability. The above arguments with j in place of i imply that at least one primary other than j plays prices just below v with positive probability. Thus, at least two primaries in $\{1, \ldots, n\}$ play prices just below v with positive probability. ∎

Let $u_{i,max}$ and L_i be as defined in Section III-A.

Lemma 6: For $i = 1, \ldots, n$, L_i is a best response for primary i.

Proof: By (3), either primary i has a jump at L_i or plays prices arbitrarily close to L_i and above it with positive probability.

Case (i): If primary i has a jump at L_i, then L_i is a best response for i because in a NE, no primary plays a price other than a best response with positive probability.

Case (ii): If primary i does not have a jump at L_i, then by (3), $\psi_i(L_i) = 0$. Since every primary selects a price in $[c, v]$, $\psi_i(v) = 1$. So $L_i < v$. So by Lemma 4, no primary among $\{1, \ldots, n\} \backslash i$ has a jump at L_i. Hence, primary i's payoff at a price above L_i and close enough to it is arbitrarily close to its payoff *at* L_i. But since primary i does not have a jump at L_i, by (3), he plays prices just above L_i with positive probability and they are best responses for him. So L_i is also a best response for primary i. ∎

Lemma 7: For some $c < \tilde{p} < v$, $L_1 = \ldots L_n = \tilde{p}$. Also, $u_{i,max} = \tilde{p} - c$, $i = 1, \ldots, n$.
That is, the lower endpoint of the support set of the price distribution of every primary is the same.

Proof: Let $L_{min} = \min\{L_m : m = 1, \ldots, n\}$, and $S_{min} = \{m : L_m = L_{min}\}$ be the set of primaries with the lowest endpoint. First, we show by contradiction that:

$$|S_{min}| \geq k + 1. \tag{25}$$

Suppose $|S_{min}| \leq k$. If $L_{min} = v$, then all primaries play the price v w.p. 1, which does not constitute a NE by Lemma 4. So $L_{min} < v$ and again by Lemma 4, no primary has a jump at L_{min}. Also, by Lemma 6, L_{min} is a best response for the primaries in S_{min}. Let $\hat{L} = \min\{L_m : m \notin S_{min}\}$ be the second lowest endpoint. Now, a primary $m \in S_{min}$ who has unused bandwidth can get a higher payoff at a price just below $\hat{L}$ than at L_{min} because in both cases, since $|S_{min}| \leq k$, primary m's bandwidth is sold w.p. 1; however, he gets a higher revenue at a price just below $\hat{L}$ than at L_{min}. This contradicts the fact that L_{min} is a best response for primary m. Thus, (25) must hold.

Now, suppose $L_i < L_j$ for some i, j. By Lemma 6, L_j is a best response for primary j. Now, the expected payoff that primary j gets for $p_j = L_j$ is strictly less than the expected payoff that primary i would get if he set p_i to be just below L_j. This is because, if primaries i or j set a price of approximately L_j, then they see the same price distribution functions of the primaries other than i and j. But primary j may be undercut by primary i, since $L_i < L_j$, whereas primary i may not be undercut by primary j. Also, by (25), primary j's expected payoff is strictly lowered due to this undercutting by primary i. (Note that undercutting by primary i would not lower primary j's probability of winning, and thereby the expected payoff, if a total of $\leq k-1$ primaries played prices below L_j with positive probability. This possibility is ruled out by (25).) Hence, $u_{i,max} > u_{j,max}$.

Now, by Lemma 6, L_i is a best response for primary i. If primary j were to play price L_i, then he would get a payoff of $u_{i,max}$. This is because, when primary i plays price L_i, he gets payoff $u_{i,max}$. Since $L_j > L_i$, primary i is, w.p. 1, not undercut by primary j. If primary j were to set the price L_i, then w.p. 1, he would not be undercut by primary i. Also, the price distributions of the primaries other than i and j are exactly the same from the viewpoints of primaries i and j. Thus, primary j can strictly increase his payoff from $u_{j,max}$ to $u_{i,max}$ by playing price L_i, which contradicts the fact that L_j is a best response for him.

Thus, $L_i < L_j$ is not possible. By symmetry, $L_i > L_j$ is not possible. So $L_i = L_j$. Let $L_1 = \ldots = L_n = \tilde{p}$.

If $\tilde{p} = v$, then every primary plays the price v w.p. 1, which does not constitute a NE. So $\tilde{p} < v$. So by Lemma 4, no primary has a jump at $\tilde{p}$. Thus, since the lower endpoint of the support set of every primary is $\tilde{p}$, by (3), a price of $\tilde{p}$ is a best response for every primary i. Since no primary sets a price lower than $\tilde{p}$, a price of $\tilde{p}$ fetches a payoff of $\tilde{p} - c$. So $u_{i,max} = \tilde{p} - c$, $i = 1, \ldots, n$. ■

Let w_i be as defined in Section III-A. Using (1), it can be easily shown that:

$$w_1 \leq w_2 \leq \ldots \leq w_n. \tag{26}$$

Lemma 8: $\tilde{p} = v - w_1(v - c)$.

Proof: If primary 1 sets the price $p_1 = v$, then he gets an expected payoff of at least $(v - c)(1 - w_1)$ because his bandwidth is sold at least in the event that $k - 1$ or fewer primaries out of $2, \ldots, n$ have unused bandwidth. So $u_{1,max} \geq (v - c)(1 - w_1)$. Since $u_{1,max} = \tilde{p} - c$ by Lemma 7, we get:

$$\tilde{p} \geq v - w_1(v - c). \tag{27}$$

Now, by Lemma 5, at least two primaries, say m and j, play prices just below v with positive probability. By Lemma 4, at most one of them has a jump at v. So assume, WLOG, that no primary other than j has a jump at v. Then a price of $p_j = v$ is a best response for primary j and fetches a payoff of $u_{j,max} = (v - c)(1 - w_j) \leq (v - c)(1 - w_1)$, where the inequality follows from (26). Since $u_{j,max} = \tilde{p} - c$ by Lemma 7, we get:

$$\tilde{p} \leq v - w_1(v - c). \tag{28}$$

The result follows from (27) and (28). ■

Lemma 9: Let $\tilde{p} \leq a < b \leq v$. Then at least two primaries play prices in (a, b) with positive probability.

Proof: If $b = v$, then the claim is true by Lemma 5. If $a = \tilde{p}$, then the claim is true by Lemma 4 and Lemma 7, since $\tilde{p} < v$ is the lower endpoint of the support set of all primaries and no primary has a jump at $\tilde{p}$; hence all primaries play prices just above $\tilde{p}$ with positive probability.

Now, fix any a, b such that $\tilde{p} < a < b < v$. Let:

$$\underline{a} = \inf\{x \leq a : \psi_j(x) = \psi_j(a) \ \forall j = 1, \ldots, n\} \tag{29}$$

By Lemma 7, $\underline{a} > \tilde{p}$. Also, by definition of $\underline{a}$, $P\{p_j \in [\underline{a}, a]\} = 0 \ \forall j = 1, \ldots, n$.

By definition of $\underline{a}$, at least one primary, say primary i, plays prices just below $\underline{a}$ with positive probability. (If not, then the infimum in (29) would be less than $\underline{a}$.) This implies that at least one primary $j \neq i$ plays prices in $(\underline{a}, b)$ with positive probability. (If not, then $p_i = b$ would yield a strictly higher payoff to primary i than prices just below $\underline{a}$.) Now, if primary j is the only primary among primaries $\{1, \ldots, n\}$ who play prices in $(\underline{a}, b)$ with positive probability, then $p_j = b$ yields a strictly higher payoff than $p_j \in (\underline{a}, b)$, which is a contradiction. So at least two primaries play prices in $(\underline{a}, b)$ with positive probability. But $P\{p_l \in [\underline{a}, a]\} = 0 \ \forall l = 1, \ldots, n$ by definition of $\underline{a}$. Hence, at least two primaries play prices in (a, b) with positive probability. ■

Let $F_{-i}(x)$ be as defined in Section III-B.

Lemma 10: For a fixed $x \in (\tilde{p}, v]$, and primaries i and j, (i) $F_{-i}(x) = F_{-j}(x)$ iff $\phi_i(x) = \phi_j(x)$, (ii) $F_{-i}(x) < F_{-j}(x)$ iff $\phi_i(x) > \phi_j(x)$.

Proof: Let $p'_{(l)}$ be the l'th smallest out of the pseudo-prices of the primaries *other than i and j*. Then conditioning on the event $\{p'_j \leq x\}$ and using the fact that $\{p'_l : l \neq i\}$ are independent, we get:

$$\begin{aligned}
&F_{-i}(x) \\
&= P\{k'\text{th smallest of } \{p'_l : l \neq i\} \leq x\} \\
&= P\{p'_j \leq x\}P\{p'_{(k-1)} \leq x\} + P\{p'_j > x\}P\{p'_{(k)} \leq x\} \\
&= \phi_j(x)P\{p'_{(k-1)} \leq x\} + (1 - \phi_j(x))P\{p'_{(k)} \leq x\} \\
&= \phi_j(x)[P\{p'_{(k-1)} \leq x\} - P\{p'_{(k)} \leq x\}] \\
&\quad + P\{p'_{(k)} \leq x\}
\end{aligned} \tag{30}$$

Similarly,

$$F_{-j}(x) = \phi_i(x)[P\{p'_{(k-1)} \leq x\} - P\{p'_{(k)} \leq x\}] + P\{p'_{(k)} \leq x\} \tag{31}$$

By (30) and (31):

$$\begin{aligned}
&F_{-i}(x) - F_{-j}(x) \\
&= (\phi_j(x) - \phi_i(x))[P\{p'_{(k-1)} \leq x\} - P\{p'_{(k)} \leq x\}] \\
&= (\phi_j(x) - \phi_i(x))\alpha
\end{aligned} \tag{32}$$

where $\alpha = P\{p'_{(k-1)} \leq x\} - P\{p'_{(k)} \leq x\}$. We will next show that $\alpha > 0$. Both parts of the result will then follow from (32).

Note that α equals the probability that exactly $(k-1)$ out of the pseudo-prices of the primaries other than i and j are $\leq x$. Since $x > \tilde{p}$, all primaries play prices in $(\tilde{p}, x)$ with positive probability by Lemma 7. So:

$$\phi_l(x) = P\{p'_l \leq x\} > 0, \ l = 1, \ldots, n. \tag{33}$$

Also,

$$\phi_l(x) \leq \phi_l(v) = q_l < 1, \ l = 1, \ldots, n. \tag{34}$$

By (33) and (34):

$$0 < \phi_l(x) < 1, \ l = 1, \ldots, n. \tag{35}$$

Also, since $1 \leq k \leq n - 1$, we have:

$$0 \leq k - 1 \leq n - 2. \tag{36}$$

Since α equals the probability of exactly $k-1$ successes out of $n-2$ independent Bernoulli events that have success probabilities $\{\phi_l(x) : l = 1, \ldots, n, l \neq i, j\}$, $\alpha > 0$ by (35) and (36). This completes the proof. ■

Lemma 11: (i) $\phi_2(.), \ldots, \phi_n(.)$ are continuous at v. (ii) $\phi_1(.)$ is continuous at v if $q_1 = q_2$ and has a jump of size *at most* $q_1 - q_2$ at v if $q_1 > q_2$. Also,

$$\phi_1(v-) \geq q_2. \tag{37}$$

Proof: If no primary $i > 1$ has a jump at v, then primary 1 gets a payoff of $(v-c)(1-w_1)$, which equals $\tilde{p} - c$ by Lemma 8, for a price p_1 just below v in the limit as $p_1 \to v-$. So if a primary $i \geq 2$ *has* a jump at v, primary 1 can get a payoff strictly greater than $\tilde{p} - c$ by playing a price close enough to v. This contradicts the fact that $u_{1,max} = \tilde{p} - c$ (see Lemma 7). Thus, no primary $i \geq 2$ has a jump at v and $\phi_2(.), \ldots, \phi_n(.)$ are continuous.

First, suppose $q_1 = q_2$. If primary 1 has a jump at v, then similar to the preceding paragraph, primary 2 can get a payoff strictly greater than $\tilde{p} - c$ by playing a price just below v, which contradicts the fact that $u_{2,max} = \tilde{p} - c$. So $\psi_1(.)$ is continuous.

Now suppose $q_1 > q_2$. First, suppose primary 1 has a jump of size *exactly* $q_1 - q_2$ at v. Then if primary 2 sets a price just below v, then the probability of being undercut by primary $j \in \{3, \ldots, n\}$ is approximately q_j. Also, since primary 1 has a jump of size $q_1 - q_2$ at v, the probability of being undercut by primary 1 is approximately $q_1 - (q_1 - q_2) = q_2$. So at a price just below v, primary 2 sees the same set of probabilities of being undercut by primaries other than itself as primary 1 would see if he set a price just below v. Hence, by the first paragraph of this proof, primary 2 gets a payoff of approximately $\tilde{p} - c$ at a price just below v.

Hence, if primary 1 has a jump of size, not equal to, but greater than $q_1 - q_2$ at v, primary 2 gets a payoff of strictly greater than $\tilde{p} - c$ at a price just below v. This contradicts the fact that $u_{2,max} = \tilde{p} - c$.

Thus, primary 1 has a jump of at most size $q_1 - q_2$ at v. So $\phi_1(v) - \phi_1(v-) \leq q_1 - q_2$. This, along with $\phi_1(v) = q_1$, gives (37). ■

Lemma 12: If $\tilde{p} \leq x < y < v$ and $\psi_i(x) = \psi_i(y)$ for some primary i, then $\psi_i(v-) = \psi_i(x)$.

Thus, if $x \geq \tilde{p}$ is the left endpoint of an interval of constancy of $\psi_i(.)$ for some i, then to the right of x, the interval of constancy extends at least until v (there may be a jump at v).

Proof: Suppose not, *i.e.*:

$$\psi_i(v-) > \psi_i(x). \tag{38}$$

Let:

$$\overline{y} = \sup\{z \geq x : \psi_i(z) = \psi_i(x)\} \tag{39}$$

By (38), (39) and the fact that $\psi_i(.)$ is continuous below v (by Lemma 4), we get $\overline{y} < v$. So again by Lemma 4, no primary among $\{1, \ldots, n\}\backslash i$ has a jump at $\overline{y}$. Also, primary i uses prices just above $\overline{y}$ with positive probability (if not, the supremum in the RHS of (39) would be $> \overline{y}$). So $\overline{y}$ is a best response for primary i and hence:

$$E\{u_i(\overline{y}, \psi_{-i})\} = (\overline{y} - c)(1 - F_{-i}(\overline{y})) = u_{i,max} = \tilde{p} - c. \tag{40}$$

where the last equality follows from Lemma 7.

Now, by Lemma 9, there exists a primary $j \neq i$ who plays prices just below $\overline{y}$ with positive probability. Since no primary among $\{1, \ldots, n\}\backslash j$ has a jump at $\overline{y}$, $\overline{y}$ is a best response for primary j. Hence:

$$E\{u_j(\overline{y}, \psi_{-j})\} = (\overline{y} - c)(1 - F_j(\overline{y})) = u_{j,max} = \tilde{p} - c. \tag{41}$$

By (40) and (41), $F_{-i}(\overline{y}) = F_{-j}(\overline{y})$. So by Lemma 10:

$$\phi_i(\overline{y}) = \phi_j(\overline{y}). \tag{42}$$

But since primary j plays prices just below $\overline{y}$ with positive probability, there exists $\epsilon > 0$ such that $x < \overline{y} - \epsilon$ and $\overline{y} - \epsilon$ is a best response for primary j. So

$$\phi_j(\overline{y} - \epsilon) < \phi_j(\overline{y}). \tag{43}$$

But by (39) and the continuity of $\phi_i(.)$ at $\overline{y}$:

$$\phi_i(\overline{y}) = \phi_i(\overline{y} - \epsilon). \tag{44}$$

By (42), (43) and (44), $\phi_i(\overline{y} - \epsilon) > \phi_j(\overline{y} - \epsilon)$. So by Lemma 10:

$$F_{-j}(\overline{y} - \epsilon) > F_{-i}(\overline{y} - \epsilon)$$

This implies:

$$\begin{aligned}
\tilde{p} - c &= E\{u_j(\overline{y} - \epsilon, \psi_{-j})\} \\
&= (\overline{y} - \epsilon - c)(1 - F_{-j}(\overline{y} - \epsilon)) \\
&< (\overline{y} - \epsilon - c)(1 - F_{-i}(\overline{y} - \epsilon)) \\
&= E\{u_i(\overline{y} - \epsilon, \psi_{-i})\}
\end{aligned}$$

which contradicts the fact that every primary gets a payoff of $\tilde{p} - c$ at a best response in the NE. ■

Lemma 13: Part 2 of Theorem 1 holds.

Proof: We prove the result by induction. Let:

$$R_n = \inf\{x \geq \tilde{p} : \exists\ y > x \text{ and } i \text{ s.t. } \phi_i(y) = \phi_i(x)\} \tag{45}$$

Note that R_n is the smallest value $\geq \tilde{p}$ that is the left endpoint of an interval of constancy for some $\phi_i(.)$. For this i, $\phi_i(R_n) = \phi_i(y)$ for some $y > R_n$ [12]. We must have $R_n > \tilde{p}$. This is because, if $R_n = \tilde{p}$, then $\phi_i(y) = \phi_i(\tilde{p})$. But $\phi_i(\tilde{p}) = 0$, since $\tilde{p}$ is the lower endpoint of the support set of $\phi_i(.)$ by Lemma 7. So $\phi_i(y) = 0$, which implies that the lower endpoint of the support set of $\phi_i(.)$ is $\geq y > \tilde{p}$. This contradicts Lemma 7. Thus, $R_n > \tilde{p}$.

Now, by definition of R_n, all primaries play every sub-interval in $[\tilde{p}, R_n)$ with positive probability and hence every price $x \in [\tilde{p}, R_n)$ is a best response for every primary. So similar to the derivation of (8), for $j \in \{1, \ldots, n\}$ and $x \in [\tilde{p}, R_n)$, $E\{u_j(x, \psi_{-j})\} = (x - c)(1 - F_{-j}(x)) = \tilde{p} - c$. Hence, $F_{-1}(x) = \ldots = F_{-n}(x)$ and by Lemma 10,

$$\phi_1(x) = \ldots = \phi_n(x) = \phi(x) \text{ (say)},\ \tilde{p} \leq x < R_n. \tag{46}$$

which proves (5) for $j = n$.

Case (i): Suppose $R_n = v$. Then $\phi_l(R_n) = q_l$, $l = 1, \ldots, n$ (since $\psi_l(v) = 1$), which proves (6).

[12]Note that $\phi_i(.)$ is a distribution function and hence is right continuous [18]. So $\phi_i(R_n+) = \phi_i(R_n)$.

Case (ii): Now suppose $R_n < v$. Then $\phi_j(.), j = 1, \ldots, n$ are continuous at R_n by Lemma 4. So by (46):

$$\phi_1(R_n) = \phi_2(R_n) = \ldots = \phi_n(R_n). \tag{47}$$

Since R_n is the left endpoint of an interval of constancy of $\phi_i(.)$, by Lemma 12:

$$\phi_i(R_n) = \phi_i(v-) = \phi_n(R_n) \leq q_n \tag{48}$$

where the second equality follows from (47).

Now, suppose $i = 1$. Then by (37) and (48):

$$\phi_i(R_n) \geq q_2. \tag{49}$$

By (48), (49) and (1), $q_2 = q_3 = \ldots = q_n = \phi_i(R_n)$. Also, by (47), $\phi_j(R_n) = q_j$, $j = 2, \ldots, n$. So $\psi_j(R_n) = 1$, $j = 2, \ldots, n$. This implies, since $R_n < v$ by assumption, that at most one primary (primary 1) plays prices in the interval (R_n, v) with positive probability, which contradicts Lemma 5. Thus, $i \neq 1$.

So by Lemma 11, $\phi_i(.)$ is continuous at v and $\phi_i(v-) = \phi_i(v) = q_i$. So by (48):

$$\phi_i(R_n) = q_i. \tag{50}$$

By (47) and (50), $\phi_n(R_n) = q_i$. If $q_i > q_n$, then $\phi_n(R_n) > q_n$, which is a contradiction because $\phi_n(R_n) = q_n\psi_n(R_n) \leq q_n$. So $q_i \leq q_n$. Also, since $q_i \geq q_n$ by (1), $q_i = q_n$. So:

$$\phi_n(R_n) = q_n. \tag{51}$$

which proves (6) for $j = n$.

Now, as induction hypothesis, suppose there exist thresholds:

$$\tilde{p} < R_n \leq R_{n-1} \leq \ldots \leq R_{i+1} \leq v$$

such that for each $j \in \{i+1, \ldots, n\}$, $\phi_j(R_j) = q_j$,

$$\phi_1(x) = \ldots = \phi_j(x) = \phi(x), \ \tilde{p} \leq x < R_j, \tag{52}$$

and each of primaries $1, \ldots, j$ plays every sub-interval in $[\tilde{p}, R_j)$ with positive probability.

First, suppose $R_{i+1} < v$. Let:

$$\begin{aligned} R_i \quad = \quad & \inf\{x \geq R_{i+1} : \exists\ y > x \text{ and } j \in \{1, \ldots, i\} \\ & \text{s.t. } \phi_j(y) = \phi_j(x)\}. \end{aligned}$$

If $R_i = R_{i+1}$, then clearly by (52):

$$\phi_1(x) = \ldots = \phi_i(x) = \phi(x), \ \tilde{p} \leq x < R_i \tag{53}$$

which proves (5) for $j = i$. Also, similar to (51), it can be shown that $\phi_i(R_i) = q_i$, which proves (6) for $j = i$ and completes the inductive step. Now suppose $R_i > R_{i+1}$. Then similar to the proof of (46), it can be shown that:

$$\phi_1(x) = \ldots = \phi_i(x) = \phi(x), \ R_{i+1} \leq x < R_i. \tag{54}$$

By (52) and (54):

$$\phi_1(x) = \ldots = \phi_i(x) = \phi(x), \ \tilde{p} \leq x < R_i.$$

which proves (5) for $j = i$. Also, similar to the proof of (51), it can be shown that $\phi_i(R_i) = q_i$, which proves (6) for $j = i$. This completes the induction.

If $R_{i+1} = v$, then the induction is completed by simply setting $R_1 = \ldots = R_i = v$.

It remains to show that $R_1 = R_2 = v$. If $R_1 < v$, then no primary plays a price in (R_1, v), which contradicts Lemma 5. So $R_1 = v$. If $R_2 < v$, then only primary 1 plays prices in (R_2, v) with positive probability, which again contradicts Lemma 5. So $R_2 = v$. ■

Lemma 14: If $q_1 > q_2$, then $\phi_1(.)$ has a jump of size $q_1 - q_2$ at v.

Proof: By Lemma 13, $\phi_1(x) = \phi_2(x)$ for all $x < R_2 = v$. So:

$$\begin{aligned} \phi_1(v-) &= \phi_2(v-) \\ &= \phi_2(v) \ \text{ (since } \phi_2(.) \text{ is continuous by Lemma 11)} \\ &= q_2 \end{aligned}$$

Also, $\phi_1(v) = q_1\psi_1(v) = q_1$. So $\phi_1(v) - \phi_1(v-) = q_1 - q_2$. ■

Finally, (i) Property 1 follows from Lemmas 4, 11 and 14; (ii) Property 2 follows from Lemmas 7 and 8; (iii) Theorem 1 follows from Properties 1 and 2 and Lemma 13.

B. Proofs of results in Section III-B

We verify that with R_i as in (13), $R_i \geq R_{i+1}$ as required by (4) in Theorem 1. Recall from Section III-B1 that $f_i(q_i)$ is the probability of k or more successes out of $n-1$ independent Bernoulli events, $i-1$ with success probability q_i and $n-i$ with $q_{i+1}, \ldots, q_n$. Also, $f_{i+1}(q_{i+1})$ is the probability of k or more successes out of $n-1$ Bernoulli events, $i-1$ with success probability q_{i+1} and $n-i$ with $q_{i+1}, \ldots, q_n$. Since $q_i \geq q_{i+1}$ by (1), it is easy to check that $f_i(q_i) \geq f_{i+1}(q_{i+1})$. So by (13), $R_i \geq R_{i+1}$, which is consistent with (4).

Proof of Lemma 1: First, let $f_i(.)$ be as defined in Section III-B1. To compute $f_i(y)$, for $i \in \{3, \ldots, n\}$ and $l \in \{0, \ldots, n-i\}$, let $v_l^i(q_{i+1}, \ldots, q_n)$ be the probability of exactly l successes out of $n-i$ independent Bernoulli trials with success probabilities $q_{i+1}, \ldots, q_n$. Recall the definition of $f_i(y)$. Conditioning on the number of successes, say l, out of the $n-i$ trials with success probabilities $q_{i+1}, \ldots, q_n$, we get:

$$\begin{aligned} f_i(y) \quad = \quad & \sum_{l=k}^{n-i} v_l^i(q_{i+1}, \ldots, q_n) \\ & + \sum_{l=0}^{\min(k-1, n-i)} v_l^i(q_{i+1}, \ldots, q_n)h(y), \end{aligned} \tag{55}$$

where $h(y) = \sum_{m=k-l}^{i-1} \begin{pmatrix} i-1 \\ m \end{pmatrix} y^m(1-y)^{i-1-m}$. Now, for l satisfying:

$$1 \leq k - l \leq i - 1, \tag{56}$$

$h(y)$ is a strictly increasing function of y [3]. Also, it can be checked that $l = \min(k-1, n-i)$, which is one of the indices in the expression in (55), satisfies (56). So $f_i(y)$ is a strictly increasing function of y. Also, note that $f_i(.)$ is a continuous function.

Now, it can be checked from the definition of the function $f_i(.)$ that:

$$f_i(q_{i+1}) = f_{i+1}(q_{i+1}). \tag{57}$$

Also, replacing i with $i+1$ in (12), we get:

$$f_{i+1}(q_{i+1}) = g(R_{i+1}). \tag{58}$$

By (57) and (58), we get:

$$f_i(q_{i+1}) = g(R_{i+1}). \tag{59}$$

Now, as shown above, $f_i(y)$ is a continuous and strictly increasing function of y. So $f_i(.)$ is invertible. By (17), $\phi(.)$ is unique and is given by:

$$\phi(x) = f_i^{-1}(g(x)), \; R_{i+1} \leq x < R_i. \tag{60}$$

Also, by (59) and (12), $f_i(q_{i+1}) = g(R_{i+1})$ and $f_i(q_i) = g(R_i)$. So $f_i(.)$ is a continuous one-to-one map from the compact set $[q_{i+1}, q_i]$ onto $[g(R_{i+1}), g(R_i)]$, and hence $f_i^{-1}(.)$ is continuous (see Theorem 4.17 in [4]). Also, $g(x)$ in (10) is continuous for all $x \in [\tilde{p}, v)$ since $x \geq \tilde{p} > c$. So from (60), $\phi(.)$ is a continuous function on $[R_{i+1}, R_i]$, since it is the composition of continuous functions f_i^{-1} and g (see Theorem 4.7 in [4]). Also, as shown above, $f_i(.)$ is strictly increasing; so $f_i^{-1}(.)$ is strictly increasing. Also, using $x \geq \tilde{p} > c$, it can be checked from (10) that $g'(x) > 0$; so $g(.)$ is strictly increasing. By (60), $\phi(.)$ is the composition of the strictly increasing functions $f_i^{-1}(.)$ and $g(.)$ and hence is strictly increasing on $[R_{i+1}, R_i]$. Also, by (12) and (60), $\phi(R_i) = f_i^{-1}(g(R_i)) = q_i$.

Thus, the function $\phi(.)$ is strictly increasing and continuous within each individual interval $[R_{i+1}, R_i]$; also, $\phi(R_i) = q_i$, $i = 2, \ldots, n$, and hence $\phi(.)$ is continuous at the endpoints $R_i, i = 2, \ldots, n$ of these intervals. So $\phi(.)$ is strictly increasing and continuous on $[\tilde{p}, v)$.

It remains to show that $\phi(\tilde{p}) = 0$. By definition of the function $f_i(.)$, $f_n(0) = 0$. As shown above, $f_n(.)$ is one-to-one. So $f_n^{-1}(0) = 0$. Also, by (10), $g(\tilde{p}) = 0$ and by (4), $R_{n+1} = \tilde{p}$. Putting $i = n$ and $x = R_{n+1} = \tilde{p}$ in (60), we get $\phi(\tilde{p}) = f_n^{-1}(g(\tilde{p})) = f_n^{-1}(0) = 0$. ■

Proof of Theorem 2: By Lemma 1 and equation (7), the functions $\phi_i(.)$, $i = 1, \ldots, n$ computed in Section III-B are continuous and non-decreasing on $[\tilde{p}, v]$; also, $\phi_i(\tilde{p}) = 0$ and $\phi_i(v) = q_i$. This is consistent with the fact that $\phi_i(.)$ is the d.f. of the pseudo-price p_i' and hence should be non-decreasing and right continuous [18], and $\phi_i(v) = q_i \psi_i(v) = q_i$ (see the beginning of Section III).

Now, we have shown in Sections III-A and III-B that (7) is a necessary condition for the functions $\phi_i(.), i = 1, \ldots, n$ to constitute a NE. We now show sufficiency. Suppose for each $i \in \{1, \ldots, n\}$, primary i uses the strategy $\phi_i(.)$ in (7). Similar to the derivation of (8), the expected payoff that primary i gets at a price $x \in [\tilde{p}, v)$ is:

$$E\{u_i(x, \psi_{-i})\} = (x - c)(1 - F_{-i}(x)). \tag{61}$$

Now, for $x \in [\tilde{p}, R_i)$, by (4) and (7), $\phi_i(x) = \phi_1(x) = \phi(x)$, and hence by Lemma 10, $F_{-i}(x) = F_{-1}(x)$. Also note that $\phi(.)$ is the solution of (8), (16) and (17). By (8), (61) and the fact that $F_{-i}(x) = F_{-1}(x)$, for primary i, prices $x \in [\tilde{p}, R_i)$ fetch an expected payoff of $\tilde{p} - c$.

Now let $x \in [R_i, v)$. Note that $R_i \leq x < v = R_1$. So by (7), $\phi_i(x) = q_i$ and $\phi_1(x) = \phi(x) \geq \phi(R_i) = q_i$. So $\phi_1(x) \geq \phi_i(x)$. Hence, by Lemma 10, $F_{-1}(x) \leq F_{-i}(x)$, which by (8) and (61) implies $E\{u_i(x, \psi_{-i})\} \leq \tilde{p} - c$.

Finally, note that a price below $\tilde{p}$ fetches a payoff of less than $\tilde{p} - c$ for primary i. So each price in $[\tilde{p}, R_i)$ is a best response for primary i; also, by (7), he randomizes over prices only in this range under $\phi_i(.)$. So $\phi_i(.)$ is a best response. Thus, the functions $\phi_i(.), i = 1, \ldots, n$ constitute a NE. ■

C. Proofs of results in Section IV

Proof of Lemma 3: The proof is similar to that in the symmetric case $q_1 = \ldots = q_n$ (see Lemma 4 in [21]) and is omitted. ■

Let $\overline{w}_1(.)$ be as defined in Section IV. The proof of Theorem 3 uses the following lemma:

Lemma 15: $\overline{w}_1(\alpha)$ is a strictly increasing and continuous function of α on $[0, 1]$. Also, $\overline{w}_1(0) = 0$.

Proof: From the definition of $\overline{w}_1(\alpha)$, it follows that it is a polynomial function of α and hence continuous, and $\overline{w}_1(0) = 0$.

To show that $\overline{w}_1(.)$ is strictly increasing, let $0 \leq \alpha < \alpha' \leq 1$. It suffices to show that $\overline{w}_1(\alpha) < \overline{w}_1(\alpha')$.

Let $Y_i, i = 2, \ldots, n$ be independent Bernoulli random variables and let Y_i have mean $q_i \alpha$. Also, let $Z_i, i = 2, \ldots, n$ be independent Bernoulli random variables that are independent of $Y_i, i = 2, \ldots, n$ and let Z_i have mean $\frac{q_i \alpha' - q_i \alpha}{1 - q_i \alpha}$.

For $i = 2, \ldots, n$, let:

$$X_i = \begin{cases} 1, & \text{if } Y_i = 1 \text{ or } Z_i = 1 \text{ (or both)} \\ 0, & \text{else} \end{cases} \tag{62}$$

$$\begin{aligned} P(X_i = 1) &= P(\{Y_i = 1\} \cup (Z_i = 1)\}) \\ &= P(Y_i = 1) + P(Z_i = 1) \\ &\quad - P(\{Y_i = 1\} \cap \{Z_i = 1\}) \\ &= P(Y_i = 1) + P(Z_i = 1) - P(Y_i = 1)P(Z_i = 1) \\ &\quad \text{(since } Y_i \text{ and } Z_i \text{ are independent)} \\ &= q_i \alpha + \frac{q_i \alpha' - q_i \alpha}{1 - q_i \alpha} - (q_i \alpha) \left(\frac{q_i \alpha' - q_i \alpha}{1 - q_i \alpha} \right) \\ &= q_i \alpha' \end{aligned}$$

So X_i is Bernoulli with mean $q_i \alpha'$. Also, since $Y_i, i = 2, \ldots, n$ and $Z_i, i = 2, \ldots, n$ are independent, $X_i, i = 2, \ldots, n$ are independent.

But by (62),

$$\{Y_i = 1\} \subset \{X_i = 1\}, \; i = 1, \ldots, n \tag{63}$$

Also,

$$\begin{aligned} P\{X_i = 1, Y_i = 0\} &= P(Z_i = 1, Y_i = 0) \\ &= P(Z_i = 1)P(Y_i = 0) \\ &= \left(\frac{q_i \alpha' - q_i \alpha}{1 - q_i \alpha} \right) (1 - q_i \alpha) \\ &= q_i \alpha' - q_i \alpha \\ &> 0 \end{aligned} \tag{64}$$

By (63) and (64):

$$P(X_i = 1) > P(Y_i = 1). \tag{65}$$

Now, let $X = \sum_{i=2}^n X_i$ and $Y = \sum_{i=2}^n Y_i$. We interpret X_i (respectively, Y_i) as the indicator of the event that primary

i offers bandwidth at a node v with node probability $\alpha_v = \alpha'$ (respectively, $\alpha_v = \alpha$). So X(respectively, Y) is the number of primaries who offer bandwidth at node v when $\alpha_v = \alpha'$ (respectively, $\alpha_v = \alpha$). By definition of the function $\overline{w}_1(.)$:

$$\overline{w}_1(\alpha') = P(X \geq k) \tag{66}$$

and

$$\overline{w}_1(\alpha) = P(Y \geq k). \tag{67}$$

By (65), (66), (67) and the facts $X = \sum_{i=2}^{n} X_i$ and $Y = \sum_{i=2}^{n} Y_i$, it follows that $\overline{w}_1(\alpha) < \overline{w}_1(\alpha')$. This completes the proof. ■

Proof of Theorem 3: Recall from Section III that when there is a single location, primary $i \in \{1, \ldots, n\}$ offers bandwidth (at that location) w.p. q_i and in the NE, by Property 2, every primary gets the same payoff, equal to $(v - c)(1 - w_1)$. Suppose in the case of the linear graph $\mathcal{G}_m$, under a NE in class $\mathcal{S}$, each primary offers bandwidth at node j w.p. α_j if he has unused bandwidth. Then by Lemma 2, at node j, primary $i \in \{1, \ldots, n\}$ selects the price as per the distribution $\psi_i(.)$ in Theorem 2, with $q_1, \ldots, q_n$ replaced by $q_1\alpha_j, \ldots, q_n\alpha_j$. Now, as shown in Section IV after Lemma 3, α_j is given by (18) for some t. It remains to show that the value of t stated in Theorem 3 in different cases is the unique value corresponding to a NE in class $\mathcal{S}$.

By the definition of $\overline{w}_1(.)$, at node j, every primary gets an expected payoff of $(v - c)(1 - \overline{w}_1(\alpha_j))$. Hence, if a primary offers bandwidth at an I.S. I, its total expected payoff at the nodes in I is

$$U(I) = \sum_{j \in I} (v - c)(1 - \overline{w}_1(\alpha_j)). \tag{68}$$

By (68) and (18):

$$U(I_e) = (v - c)(1 - \overline{w}_1(t))|I_e| \tag{69}$$

$$\text{and} \quad U(I_o) = (v - c)(1 - \overline{w}_1(1 - t))|I_o|. \tag{70}$$

Now, recall that a strategy profile is a NE iff every primary plays only best responses with positive probability. Since in a NE in class $\mathcal{S}$, every primary offers bandwidth w.p. t (respectively, $1 - t$) at I_e (respectively, I_o), each one of the following cases provides necessary and sufficient conditions for a NE in class $\mathcal{S}$ (i) $t = 1$ and $U(I_e) \geq U(I_o)$, (i) $t = 0$ and $U(I_o) \geq U(I_e)$, and (iii) $0 < t < 1$ and $U(I_e) = U(I_o)$.

Now, when m is even, $|I_o| = |I_e| = \frac{m}{2}$. In this case, it can be checked using (69), (70) and Lemma 15, that if $t = 1$ (respectively, $t = 0$), then $U(I_e) < U(I_o)$ (respectively, $U(I_o) < U(I_e)$); also, $t = \frac{1}{2}$ is the unique value for which $U(I_e) = U(I_o)$. So the first two cases above do not correspond to a NE in class $\mathcal{S}$, and the third case does iff $t = \frac{1}{2}$. Hence, $t = \frac{1}{2}$ corresponds to the unique NE in class $\mathcal{S}$.

Now, let m be odd. Then $|I_o| = \frac{m+1}{2}$ and $|I_e| = \frac{m-1}{2}$. If $\overline{w}_1(1) \leq \frac{2}{m+1}$, then it can be checked using (69), (70) and Lemma 15 that if $t > 0$, then $U(I_e) < U(I_o)$; also, if $t = 0$, then $U(I_o) \geq U(I_e)$. So the second and third cases above do not correspond to a NE in class $\mathcal{S}$ and $t = 0$ (the first case) corresponds to the unique NE in class $\mathcal{S}$.

Finally, if $\overline{w}_1(1) > \frac{2}{m+1}$, then it can be checked using (69), (70) and Lemma 15 that if $t = 0$ (respectively, $t = 1$), then $U(I_o) < U(I_e)$ (respectively, $U(I_e) < U(I_o)$). So the first two cases above do not correspond to a NE. Also, substituting (69) and (70) into the equation $U(I_o) = U(I_e)$ we get (19). By the continuity and strictly increasing property of $\overline{w}_1(.)$ (Lemma 15) and the intermediate value theorem (Theorem 4.23 in [4]), it follows that (19) has a unique solution $t \in (0, 1)$. So by the third case above, this t corresponds to the unique NE in $\mathcal{S}$. ■