Pricing Games under Uncertain Competition

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I. Introduction

The last decade has seen a tremendous growth in wireless networks, resulting in a proportionate increase in demand for spectrum. But spectrum is limited, which has led to the design of techniques such as Cognitive Radio Technology [2], for using the available spectrum more efficiently. In Cognitive Radio Networks (CRNs), there are two types of spectrum users: (i) a *primary* user who leases a certain portion (channel or band) of the spectrum directly from the regulator, and (ii) *secondary* users who can use the channel when it is not used by the primary.

We consider a CRN with multiple primary and secondary users in a region. Time is slotted, and in every slot, each primary has unused bandwidth with some probability, which he would like to sell to secondaries. Now, secondaries would like to buy bandwidth from the primaries that offer it at a low price, which results in *price competition* among the primaries. If a primary quotes a low price, it will attract buyers, but at the cost of reduced revenues. This is a common feature of an *oligopoly* [1], in which multiple firms sell a common good to a pool of buyers. Price competition in an oligopoly is naturally modeled using *game theory* [15], and has been extensively studied in economics using for example the classical *Bertrand game* [1] and its variants.

However, a CRN has several distinguishing features, which makes the price competition very different from oligopolies encountered in economics. First, in every slot, each primary may or may not have unused bandwidth available. So a primary who *has* unused bandwidth is uncertain about the number of primaries from whom he will face competition. A low price will result in unnecessarily low revenues in the event that very few other primaries have unused bandwidth, because even with a higher price the primary's bandwidth would have been bought, and vice versa. Second, spectrum is a commodity that allows *spatial reuse*: the same band can be simultaneously used at far-off locations without interference; on the other hand, simultaneous transmissions at neighboring locations on the same band interfere with each other. To the best of our knowledge, our work is the *first* to consider either of these distinguishing features in context of price competition in wireless networks.

We model the problem using game theory, and our model captures both uncertain bandwidth availability and spatial reuse. In this part of the paper, we focus on the case where all the primaries and secondaries are located in a single location; Part II deals with spatial reuse. We first consider (i) a *one-shot* game, in which bandwidth trading is done only once, and subsequently (ii) a *repeated* game in which there are an infinite number of slots, and bandwidth trading is done every slot. We seek a *Nash equilibrium* [15] (NE) in each case.

In the one-shot game (Section III), we show that there does not exist a *pure-strategy* NE, *i.e.*, one in which each primary deterministically selects a price (Section III-C). This is in sharp contrast with the Bertrand game [1], where each seller always has his ware available– the only equilibrium then is a pure-strategy one in which each seller chooses the lowest possible price [8]. We then explicitly find a *mixed-strategy* NE in which each primary randomly chooses a price from a range, and prove that it is unique in the class of symmetric equilibria (Section III-C). As the probability that a primary has bandwidth available decreases, this range of prices becomes increasingly concentrated at the highest possible price. This confirms the intuition that when spectrum holes are rarely available, whenever a primary has a spectrum hole, he can afford to set a high price in view of the limited competition he anticipates from others. Using the explicit expressions, we quantify the loss of total revenue incurred due to competition under symmetric equilibria (Section III-D). Our numerical computations reveal that this loss, or equivalently, the efficiency of the symmetric equilibria, exhibits interesting threshold behavior, which we also analytically prove in the asymptotic regime (i.e., when the number of primaries is large).

Next, we analyze the repeated game version of the one-shot game (Section IV), and show that there exists an efficient NE in which each primary sets the highest possible price and as a result, the sum of expected revenues of the primaries is maximized. This is achieved through a *threat mechanism*: if any primary lowers his price in a slot, all others retaliate in future slots by playing the one-shot game NE strategy and hence the primary suffers in the long run.

Finally, we consider two generalizations (Section V): (ii) in the first, the probability that a primary has unused bandwidth is different for different primaries (Section V-B) and (i) in the second, the reservation price or valuation of a secondary is not known to the primaries with certainty, but is randomly distributed (Section V-A).

Our main contribution is that we are able to explicitly compute NE in all the games we consider. Since the prices can take real values, the *strategy sets of players are continuous*. Thus, classical results do not establish existence and uniqueness of NE for the games we consider, and there is no standard algorithm for finding a NE, unlike when each player's strategy set is finite [15]. The explicit computations provide valuable insight; in particular, they clearly reveal the effect of the system parameters on equilibrium behavior. *All proofs are deferred to the Appendix.*

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II. Related Work

Pricing related issues have been extensively studied in the context of wired networks and the Internet; see [7] for an overview. Price competition among spectrum providers in wireless networks has been studied in [16], [17], [18], [19], [20], [21]. Specifically, Niyato *et. al.* analyze price competition among multiple primaries in CRNs [20], [21]. However, neither uncertain bandwidth availability, nor spatial reuse is modeled in any of the above papers. Also, most of these papers do not explicitly find a NE (exceptions are [17], [20]). Our model incorporates both uncertain bandwidth availability and spatial reuse, which makes the problem challenging; despite this, we are able to explicitly compute a NE. Zhou *et. al.* [22] have designed double auction based spectrum trades in which an auctioneer chooses an allocation taking into account spatial reuse and bids. However, in the price competition model we consider, each primary independently sells bandwidth, and hence a central entity such as an auctioneer is not required.

In the economics literature, the *Cournot game* and the *Bertand game* are two basic models that have been widely used to study competition among sellers in oligopolies [1]. In a Cournot game, sellers choose the quantity of a good to produce as opposed to prices in a Bertrand game, and hence the latter is more relevant to our model. In a Bertrand game, each seller quotes a price for a good, and the buyers buy from the seller that quotes the lowest price [1] [1]. Several variants of the Bertrand game have been studied, *e.g.*, [4], [5], [6], [25]. Osborne et al [4] consider price competition in a duopoly, when the capacity of each firm is constrained. Chawla *et al.* [25] consider price competition in networks where each seller owns a capacity-constrained link, and decides the price for using it; the consumers choose paths they would use in the networks based on the prices declared and pay the sellers accordingly. The capacities in both cases are deterministic, whereas the availability of bandwidth is random in our model. The work most closely related to ours is the paper by Janssen et al [6], which analyzes the case where each seller may be inactive with some probability. However, none of the above papers [4], [5], [6], [25] consider the spectrum-specific issue of spatial reuse, which introduces a new dimension, that each player not only needs to determine the price of the commodity he owns (as in [4], [5], [6], [25]), but also select an independent set to compete in. The joint decision problem significantly complicates the analysis. Also, the results in [6] are restricted to the case of one buyer; but, a CRN is likely to have multiple secondaries, which our model allows. This again complicates the analysis since multiple primaries can now sell their available bandwidths. Finally, unlike [6], we consider repeated interactions among primaries, unequal probabilities of availability of unused bandwidth and random valuations for secondaries (Sections IV,V-B, V-A).

[1] If two or more sellers quote the lowest price, the demand is equally shared between them.

III. Price Competition in a Single Slot

A. Model

Suppose there are $n \geq 2$ primaries and $k \geq 1$ secondaries in a region. Each secondary may constitute a customer who requires 1 unit of bandwidth, or may simply be a demand for 1 unit of bandwidth. We first consider the case that the primaries know k, and later generalize our results to allow for random, and apriori unknown, k (see Remark 1). Time is divided into slots of equal duration. In every slot, each primary has 1 unit of unused bandwidth with probability q and 0 units with probability $1-q$, where $0 < q < 1$. We initially assume that the bandwidth availability probability q is the same for all primaries, but subsequently allow unequal probabilities in Section V-B. A primary i who has unused bandwidth in a slot can lease it out to a secondary for the duration of the slot, in return for an access fee of p_i. Leasing in a slot incurs a cost of $c \geq 0$. This cost may arise, for example, if the secondary uses the primary's infrastructure to access the Internet. We assume that $p_i \leq v$ for each primary, for some constant $v > c$. This upper bound v may arise as follows:

1) The spectrum regulator may impose this upper bound to ensure that primaries do not excessively overprice bandwidth even when competition is limited owing to bandwidth scarcity or high demands from secondaries, or when the primaries collude.
2) Alternatively, the *valuation* of each secondary for 1 unit of bandwidth may be v, and no secondary will buy bandwidth at a price that exceeds his valuation.

We initially assume that the primaries know this upper limit v, which is likely to be the case for the first interpretation. For the second interpretation, the primaries need not know the secondaries' valuations, – we consider this generalization in Section V-A.

Secondaries buy bandwidth from the primaries that offer the lowest price. More precisely, in a given slot, let Z be the number of primaries who offer unused bandwidth. Then the bandwidth of the $\min(Z, k)$ primaries that offer the lowest prices is bought (ties are resolved at random).

B. Game Formulation

We formulate the above price competition among primaries as a *game*, which is any situation in which multiple individuals called *players* interact with each other, such that each player's welfare depends on the *actions* of the others [1]. In our model, the primaries are the players, and the action of primary i is the price p_i that he chooses [2]. In Section III, we study the interaction of the primaries in a single slot, which is referred to as the *one-shot game*. In Section IV, we consider a setting where the one-shot game is repeated an infinite number of times, which is referred to as the *repeated game*.

The *utility* or *payoff* of a player in a game is a numerical measure of his satisfaction level [1], which in our context is the corresponding primary's net revenue. In (the one-shot version of) our game, the utility of primary i is 0 if he has

[2] If primary i has no unused bandwidth, it does not matter what price p_i he sets. Yet, for convenience, we speak of p_i as being his action.

no unused bandwidth. Let $u_i(p_1,\ldots,p_n)$ denote his utility if he has unused bandwidth [3] and primary j sets a price of p_j, $j=1,\ldots,n$. Thus,

$$u_i(p_1,\ldots,p_n)=\begin{cases} p_i-c & \text{if primary } i \text{ sells his bandwidth} \\ 0 & \text{otherwise}\end{cases} \quad (1)$$

Recall that the distribution function (d.f.) [24] of a random variable (r.v.) X is the function:

$$G(x)=P(X\le x),\ \ x\in R$$

where R is the set of real numbers. Now, a *strategy* [1] for primary i is a plan for choosing his price p_i. We allow each primary i to choose his price randomly from a set of prices using an arbitrary d.f. $\psi_i(.)$, which is referred to as the strategy of primary i. A d.f. that concentrates its entire mass on a single value allows a primary to deterministically choose this value as his price– such a $\psi(.)$ is referred to as a *pure strategy*. The vector $(\psi_1(.),\ldots,\psi_n(.))$ of strategies of the primaries is called a *strategy profile* [1]. Let $\psi_{-i}=(\psi_1(.),\ldots,\psi_{i-1}(.),\psi_{i+1}(.),\ldots,\psi_n(.))$ denote the vector of strategies of primaries other than i. Let $E\{u_i(\psi_i(.),\psi_{-i})\}$ denote the expected utility of player i when he adopts strategy $\psi_i(.)$ and the other players adopt ψ_{-i}.

A *Nash equilibrium* (NE) is a strategy profile such that no player can improve his expected utility by unilaterally deviating from his strategy [1]. Thus, $(\psi_1^*(.),\ldots,\psi_n^*(.))$ is a NE if for each primary i:

$$E\{u_i(\psi_i^*(.),\psi_{-i}^*)\}\ge E\{u_i(\tilde{\psi}_i(.),\psi_{-i}^*)\},\ \forall\ \tilde{\psi}_i(.). \quad (2)$$

When players other than i play ψ_{-i}^*, $\psi_i^*(.)$ maximizes i's expected utility and is thus his *best-response* [1] to ψ_{-i}.

C. Nash Equilibria

If $k\ge n$, then the number of buyers is always greater than or equal to the number of sellers. So a primary i will sell his unused bandwidth even when he chooses the maximum possible price v. So the strategy profile under which all primaries deterministically choose the price v is the unique NE. So henceforth, we assume that $k\le n-1$.

Theorem 1: There is no pure strategy NE (i.e., one where every primary selects his price deterministically) in the above game.

In contrast, in the Bertrand game, which corresponds to $q=1$ in our model, the pure strategy profile under which each primary deterministically selects c as his price is the unique NE [1]. This strategy profile is not a NE in our context as this provides 0 utility for each primary, whereas by quoting any price above c (and below v) each primary can attain a positive utility since he will sell his unused bandwidth at least when he is the only primary that has unused bandwidth which happens with positive probability (since $q<1$). We have shown that no other deterministic strategy profile is a NE either; the intuition is that if a primary p_i deterministically sets a price $p_i\in(c,v]$, then other primaries can undercut it by a small amount by setting a price $p-\epsilon$ to ensure that their bandwidth is preferred to primary i's. The formal proof follows.

Proof: Recall that an action p_i of player i is said to *strictly dominate* [1] another action p_i' if:

$$E\{u_i(p_i,p_{-i})\}>E\{u_i(p_i',p_{-i})\},\ \ \forall p_{-i}$$

By (1), for every primary i, and any p_{-i}, $u_i(c,p_{-i})=0$. Also, $E\{u_i(p_i,p_{-i})\}>0$ for all $p_i\in(c,v]$ because primary i gets a positive payoff in the event that no other primary has unused bandwidth, which happens with positive probability. Thus, the strategy $p_i=c$ is strictly dominated by each $p_i\in(c,v]$, and hence no primary sets $p_i=c$ in any pure-strategy Nash equilibrium.

Suppose $(p_1,\ldots,p_n)$ is a pure-strategy Nash equilibrium, where $c<p_i\le v$ for $i=1,\ldots,n$. Let $p_{min}=\min(p_1,\ldots,p_n)$, $S_{min}=\{i:p_i=p_{min}\}$, and $n_{min}=|S_{min}|$. Note that S_{min} is the set of primaries who set the lowest price p_{min}, and n_{min} is its cardinality. One of the following two cases must hold:

Case (i): $n_{min}\le k$

Since $k\le n-1$, $n_{min}\le n-1$ and hence at least one primary sets a price above p_{min}. Since $p_i\le v$, $i=1,\ldots,n$, it follows that $p_{min}<v$.

Let $p_j=\min\{p_i:i\notin S_{min}\}$ be the second lowest price. Now, note that $\forall i\in S_{min}$, $u_i(p_{min},p_{-i})=p_{min}-c$ and $u_i(p_i',p_{-i})=p_i'-c\ \forall p_i'\in(p_{min},p_j)$. This is because the bandwidth of primary i always gets sold for any $p_i'<p_j$, since it is among the primaries with the $n_{min}\le k$ lowest prices. So $\forall i\in S_{min}$:

$$u_i(p_{min},p_{-i})<u_i(p_i',p_{-i})\ \ \forall p_i'\in(p_{min},p_j)$$

Hence, $p_i=p_{min}$ is not a best response to p_{-i}, which contradicts the assumption that $(p_1,\ldots,p_n)$ is a Nash equilibrium.

Case (ii): $n_{min}>k$

In this case, for $i\in S_{min}$:

$$E\{u_i(p_{min},p_{-i})\}=(p_{min}-c)P(E_1)$$

where E_1 is the event that primary i's bandwidth is bought by a secondary. Note that $P(E_1)<1$ because with a positive probability, k or more primaries other than i, in S_{min} have unused bandwidth. In this case, k randomly selected primaries, out of the primaries in S_{min} who have unused bandwidth, sell their bandwidth, and with a positive probability, primary i is not among them. Also, note that primary i's bandwidth is always sold if it sets a price less than p_{min} and the vector of prices of primaries other than i is p_{-i}. Hence, for small enough $\epsilon>0$:

$$\begin{aligned} E\{u_i(p_{min}-\epsilon,p_{-i})\} &= (p_{min}-\epsilon-c)\\ &> (p_{min}-c)P(E_1)\\ &= E\{u_i(p_{min},p_{-i})\}\end{aligned}$$

Thus, $p_i=p_{min}$ is not a best response, which contradicts the assumption that $(p_1,\ldots,p_n)$ is a Nash equilibrium. ∎

Next, we focus on a specific class of Nash equilibria, known as *symmetric Nash equilibria*. A NE $(\psi_1^*(.),\ldots,\psi_n^*(.))$ is a symmetric NE if all players play identical strategies under it, i.e., $\psi_1^*(.)=\psi_2^*(.)=\ldots=\psi_n^*(.)$. In practice it is

[3] If instead, $u_i(p_1,\ldots,p_n)$ were defined to be primary i's net revenue, unconditional on whether he has unused bandwidth or not, then the expected utilities in the one-shot game analysis would all be scaled by q.

challenging to implement any other NE– the simple example of two primaries and a NE of $(\psi_1^*(.), \psi_2^*(.))$ elucidates the inherent complications in the current context. If $\psi_1^*(.) \neq \psi_2^*(.)$, then since players have the same action sets, utility functions and probability of having unused bandwidth (such games are referred to as *symmetric games*), $(\psi_2^*(.), \psi_1^*(.))$ also constitutes a NE. If player 1 knows that player 2 is playing $\psi_2^*(.)$ (respectively, $\psi_1^*(.)$), he would choose the best response $\psi_1^*(.)$ (respectively, $\psi_2^*(.)$), but he can not know player 2's choice between the two options without explicitly coordinating with him, which is again ruled out due to the competition between the two. Under symmetric NE, all players play the same strategy, and thus this quandary is somewhat limited– symmetric NE has indeed been advocated for symmetric games by several game theorists [3]. The natural question now is whether there exists at least one symmetric NE, and also whether there is a unique symmetric NE (only uniqueness will eliminate the above quandary). Note that some symmetric games are known to have multiple symmetric NE. For example, consider the simple "Meeting in New York game" [1] with two players, where each player can either be at Grand Central or at Empire State Building, and both receive unit utility if they meet and zero utility otherwise. The strategies where each player is at Grand Central, and where each player is at Empire State Building, both constitute symmetric NE.

In our context, we now show that there exists a unique symmetric NE and explicitly compute it. First, we derive some necessary conditions for a strategy profile to be a symmetric NE and then show that they are also sufficient. Suppose in a symmetric NE, each primary $i \in \{1, \ldots, n\}$ sets his price $p_i \in [c, v]$ according to some common distribution $\psi(x)$. Let ψ_{-i} denote the vector $(\psi(.), \ldots, \psi(.))$ of strategies of primaries other than i. To simplify our exposition, we introduce the notion of "pseudo-price" for each primary. The pseudo-price of primary i, p'_i, is the price he selects, p_i, if he has unused bandwidth and $p'_i = v + 1$ otherwise [4]. Let $\phi(x)$ be the d.f. of p'_i. For $x \in [c, v]$, note that $p'_i \leq x$ if and only if primary i has unused bandwidth (which happens w.p. q) and sets a price $p_i \leq x$ (which happens w.p. $\psi(x)$). Thus,

$$\phi(x) = q\psi(x), \;\; x \in [c, v]. \tag{3}$$

Consider primary 1 and let $p'_{(k)}$ denote the k'th smallest pseudo-price among the pseudo-prices $p'_j, j = 2, \ldots, n$ of the rest of the primaries, (which primary 1 will know only after choosing his price or equivalently pseudo-price). Since the primaries choose their prices randomly and since their bandwidth availabilities are random, $p'_{(k)}$ is a random variable; let $F(.)$ be its d.f.

Now, it turns out that in a symmetric NE, primaries do not select any single price in $[c, v]$ with positive probability. The intuition is similar to that behind Theorem 1: if primaries $2, \ldots, n$ set a price p with positive probability, then primary 1 can benefit by setting a price $p - \epsilon$, for a small $\epsilon > 0$, instead of p. Thus, $\psi(.)$, and thereby $\phi(.)$ and $F(.)$, do not have a positive probability mass at any point $x \in [c, v]$:

[4]The choice $v + 1$ is arbitrary. Any other value greater than v would also work.

Lemma 1: $\psi(x)$, $\phi(x)$ and $F(x)$ are continuous on $c \leq x \leq v$.
The proof is deferred to the Appendix.

Now, recall that there are k secondaries who opt for the lowest available prices. So by definition of $p'_{(k)}$, if primary 1 offers a price of x, he sells his bandwidth if and only if either (i) $p'_{(k)} > x$ or (ii) $p'_{(k)} = x$ (in which case there is a tie) and primary 1 is among the primaries selected randomly from those who set the price x. But since $F(x)$ is continuous by Lemma 1, $P(p'_{(k)} = x) = 0$. So primary 1 sells his bandwidth w.p. $P(p'_{(k)} > x) = (1 - F(x))$; the sale fetches a utility of $x - c$. Hence, primary 1's expected utility is:

$$E\{u_1(x, \psi_{-1})\} = (x - c)(1 - F(x)), \;\; x \in [c, v] \tag{4}$$

Now, let B be the set of prices in $[c, v]$ that are best responses of primary 1 to the vector of strategies ψ_{-1} of primaries $2, \ldots, n$ and let u_{max} be the maximum payoff. Thus:

$$E\{u_1(x, \psi_{-1})\} = u_{max}, \;\; \forall x \in B. \tag{5}$$

Using the continuity of $F(.)$ (see Lemma 1), it can be shown that B must be a contiguous and closed set, with upper limit v:

Lemma 2: $B = [\tilde{p}, v]$ for some $\tilde{p} \in (c, v)$.
The proof is deferred to the Appendix.

By (4), (5) and Lemma 2:

$$\begin{aligned} u_{max} &= (x - c)(1 - F(x)) \;\; \forall x \in [\tilde{p}, v] &(6)\\ &= (v - c)(1 - F(v)). &(7) \end{aligned}$$

Now, $F(v)$ is the probability that $p'_{(k)} \leq v$, which happens when k or more primaries have unused bandwidth (among those in $\{2, \ldots, n\}$); so $F(v) = w(q, n)$, where:

$$w(q, n) = \sum_{i=k}^{n-1} \binom{n-1}{i} q^i (1 - q)^{n-1-i}. \tag{8}$$

Hence, by (7), $u_{max} = (v - c)\left(1 - w(q, n)\right)$. Thus, by (6):

$$F(x) = 1 - \frac{(v - c)\left(1 - w(q, n)\right)}{x - c}, \;\; x \in [\tilde{p}, v]. \tag{9}$$

Now, the set of best responses is $B = [\tilde{p}, v]$ and in a NE, every primary plays a best response w.p. 1. So w.p. 1, $p_i \geq \tilde{p}$ for every primary i, and hence $p'_{(k)} \geq \tilde{p}$. Since $F(x) = P(p'_{(k)} \leq x)$, it follows that $F(\tilde{p}) = 0$. So putting $x = \tilde{p}$ in (9), we get:

$$\tilde{p} = v - w(q, n)(v - c). \tag{10}$$

Thus,

$$F(x) = \begin{cases} 0, & x \leq \tilde{p} \\ \frac{x - \tilde{p}}{x - c}, & \tilde{p} < x \leq v. \end{cases} \tag{11}$$

The d.f. $\psi(.)$ for the price of each primary must be such that the above function $F(.)$ is the d.f. of the kth smallest pseudo-price of $n-1$ primaries. Since $F(x)$ is the probability that k or more pseudo-prices out of $n-1$ are $\leq x$ and each pseudo-price is $\leq x$ w.p. $\phi(x)$, $\phi(x)$ must be the solution of:

$$\sum_{i=k}^{n-1} \binom{n-1}{i} [\phi(x)]^i [1 - \phi(x)]^{n-1-i} = F(x). \tag{12}$$

In the Appendix, we prove that:

Lemma 3: With $F(x)$ given by (11), equation (12) has a unique solution $\phi(x) \in [0,1]$. The function $\phi(x)$ is strictly increasing and continuous on $[\tilde{p}, v]$. Also, $\phi(\tilde{p}) = 0$ and $\phi(v) = q$.

Finally, by (3) and Lemma 3:

$$\psi(x) = \begin{cases} 0, & x \leq \tilde{p} \\ \frac{1}{q}\phi(x), & \tilde{p} < x \leq v \\ 1, & x \geq v \end{cases} \tag{13}$$

Note that from the properties of the $\phi(.)$ function obtained in Lemma 3, $\psi(x)$ is a continuous d.f[5].

Theorem 2: The strategy profile in which each primary i chooses his price p_i according to $\psi(.)$, where $\psi(.)$ is defined by (13), (12) and (11) is the unique symmetric NE.

Proof: We have shown above that a necessary condition for a strategy $\psi(.)$ to be a symmetric NE strategy is that it must satisfy (13), (12) and (11). Also, by Lemma 3, these equations have a unique solution $\psi(.)$. It remains to show that the strategy profile in which each primary i plays the strategy $\psi(.)$ is indeed a NE. By (4) and (11), under this strategy profile, each primary i's expected payoff for a price $x \in [\tilde{p}, v]$ is given by:

$$\begin{aligned} E\{u_i(x, \psi_{-i})\} &= (x-c)\left\{1 - \left(\frac{x-\tilde{p}}{x-c}\right)\right\} \\ &= \tilde{p} - c \end{aligned} \tag{14}$$

Also, primary i's expected payoff for a price $p_i < \tilde{p}$ is $p_i - c < \tilde{p} - c$. So each $p_i \in [\tilde{p}, v]$ is a best response. Since $\psi(.)$ randomizes among the prices in $[\tilde{p}, v]$, $\psi(.)$ is a best response for each primary i. Hence, the strategy profile in which each primary i plays $\psi(.)$ is an NE. ■

This random selection of prices as per $\psi(.)$ can be interpreted as follows: each primary i sets a base price v and randomly holds "sales" to attract secondaries by lowering the price to some value $p_i \in [\tilde{p}, v]$[6].

Example: For $n = 2$ and $k = 1$, we have $w(q,n) = q$, $\tilde{p} = v - q(v-c)$, and

$$\psi(x) = \begin{cases} 0 & x \leq \tilde{p} \\ \frac{1}{q}\left(\frac{x-\tilde{p}}{x-c}\right) & \tilde{p} < x \leq v \\ 1 & x \geq v \end{cases} \tag{15}$$

Remark 1: Our results readily generalize to allow for a random number of secondaries (K). Then the primaries apriori know only the probability mass function (p.m.f.) for K, $Pr(K = k) = \gamma_k$, but not the value of K. Unlike in (8), we now define $w(q,n)$ as:

$$w(q,n) = \sum_{k=1}^{n-1} \gamma_k \sum_{i=k}^{n-1} \binom{n-1}{i} q^i (1-q)^{n-1-i} \tag{16}$$

Also, (12) is replaced by:

$$\sum_{k=1}^{n-1} \gamma_k \sum_{i=k}^{n-1} \binom{n-1}{i} [\phi(x)]^i [1-\phi(x)]^{n-1-i} = F(x) \tag{17}$$

Now, $\psi(.)$ computed as before, but with the above modifications in $w(.,.), \phi(.)$, again constitutes the unique symmetric NE strategy of each primary.

D. Performance Evaluation under the Unique Symmetric NE

We define the *efficiency*, η, of a NE as $\eta = \frac{R_{\text{NE}}}{R_{\text{OPT}}}$, where R_{NE} is the expected sum of utilities of the n primaries at the NE and R_{OPT} is the maximum possible (optimal) expected sum of utilities. Note that R_{OPT} is attained only when all primaries cooperate and each selects the maximum possible price v so as to ensure that bandwidth is always sold at this price. Clearly, $\eta \leq 1$ quantifies the loss in the total revenue incurred owing to lack of cooperation among primaries. Also, owing to its uniqueness, the efficiency of the symmetric NE we obtain quantifies fundamental limits on the performance of symmetric NE.

Now, $R_{\text{OPT}} = E[\min(Z,k)](v-c)$, where Z is the number of primaries who have unused bandwidth (Z is a Binomial(n,q) r.v. [14]). Also, as discussed in Section III-C, at the unique symmetric NE, whenever a primary has unused bandwidth, he attains an expected utility of $(v-c)(1-F(v)) = (v-c)(1-w(q,n))$ irrespective of the price he offers. Thus, since there are n primaries and each has unused bandwidth w.p. q, $R_{\text{NE}} = nq(1-w(q,n))(v-c)$. Hence,

$$\eta = \frac{nq(1-w(q,n))}{E[\min(Z,k)]}. \tag{18}$$

Fig. 1 plots η of the symmetric NE versus k for three values of q. It is interesting to note that η exhibits a sharp *threshold behavior*: for k below (respectively, above) a threshold the efficiency is close to 0 (respectively, 1). Also, this threshold is around nq, the expected number of primaries who have free bandwidth. Intuitively, this is because, when the supply nq exceeds the demand k for bandwidth (i.e., $k < nq$), there is intense price competition, driving down the equilibrium prices. On the other hand as k increases, $w(q,n)$ decreases and $\tilde{p}$ increases and becomes closer to v (see (8) and (10)). Hence, the d.f. $\psi(.)$ becomes increasingly concentrated at the highest possible price v. Intuitively, this is because, when the demand exceeds the supply, a primary expects to sell even at a high price, and sets his price accordingly. The plots for the density of the unique symmetric NE price distribution for different sets of values of parameters n, k, q reveal the same phenomenon as well (Fig. 2).

In fact, we can analytically establish this threshold behavior for large n:

Lemma 4: Let $q \in (0,1)$ be fixed.

1) If $k \leq (n-1)(q-\epsilon)$ for some $\epsilon > 0$, then $\eta \to 0$ as $n \to \infty$.
2) If $k \geq (n-1)(q+\epsilon)$ for some $\epsilon > 0$, then $\eta \to 1$ as $n \to \infty$.

IV. PRICE COMPETITION UNDER REPEATED INTERACTIONS

We now consider repeated interactions among primaries in multiple slots. We first formulate the problem in Section IV-A and then describe our results in Section IV-B.

[5] A function $f(x)$ is a d.f. iff it is increasing, right continuous, and has limits 0 and 1 as x tends to $-\infty$ and ∞ respectively [24].

[6] This interpretation has been suggested in [10] for random selection of prices in a different context.

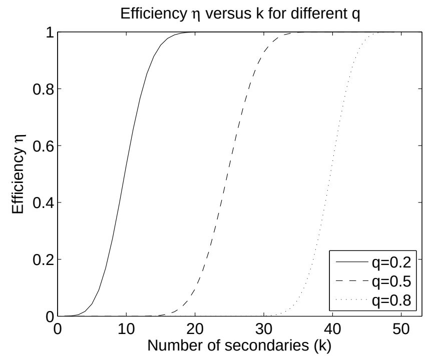

Fig. 1. Efficiency of the Nash equilibrium versus k for three values of q. The other parameters are $n = 50$, $c = 0$, and $v = 100$.

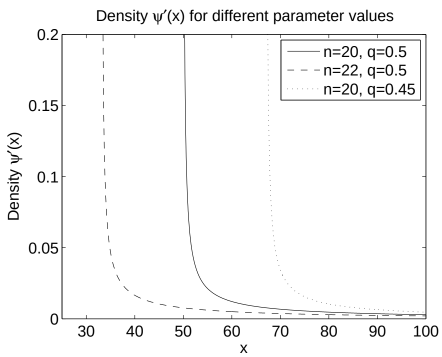

Fig. 2. The density $\psi'(x)$ for three sets of nq: 9, 10 and 11, for the three curves from right to left. The other parameters are $c = 0$, $v = 100$ and $k = 10$.

A. Formulation

We consider a repeated game [1] formulation for the one-shot game at a single location described in Section III, where the one-shot game is repeated an infinite number of times, at $\tau = 1, 2, 3, \ldots$. Each player perfectly recalls the actions of every player in all preceding times. The payoff of player i for the overall repeated game is defined to be $u_i = \sum_{\tau=1}^{\infty} \delta^{\tau-1} u_{i,\tau}$, where $u_{i,\tau}$ is his payoff at time τ and $\delta \in (0, 1)$ is the *discount factor* [1], which is used to discount future payoffs (see [1], [15] for interpretations of the discount factor). The discount factor is usually close to 1 [1].

A strategy of a player in a repeated game is a complete plan for choosing the action in each slot as a function of the actions of all players in all preceding slots [1]. As in a one-shot game (see Section III-B), a Nash equilibrium (NE) in a repeated game is a strategy profile in which no player can improve his payoff by unilateral deviation from his strategy [1]. However, NE constitutes a rather weak notion of equilibria in repeated games [1] and hence we focus on NE with a special property, known as the *Subgame Perfect Nash Equilibria (SPNE)* [1]. A *subgame* [1] of the repeated game is the part of the game starting from some slot $\tau_0 \geq 1$, *i.e.* the stage games in slots $\tau = \tau_0, \tau_0 + 1, \ldots$. An SPNE is an NE of the repeated game that is also an NE of every subgame [1].

B. Results

It is well-known that for any repeated game, the strategy profile under which every player uses the one shot game NE strategy in every time slot is a SPNE [1]. Thus, the symmetric NE we presented for the one-shot game in Section III provides a SPNE in the repeated game version. The efficiency (as defined in the first paragraph of Section III-D) of this SPNE is however low whenever the symmetric NE has low efficiency, which happens for certain ranges of n, k, q (Lemma 4). *Our main contribution is to present an SPNE that is also efficient in the sense that the sum of expected utilities of the n primaries at equilibrium equals the maximum possible sum of utilities, provided the discount factor δ is sufficiently high.*

We consider *Nash reversion* type of strategy profiles [1] in which each player plays a specified strategy (called the pre-deviation strategy) at each time until one of the players deviates from it, and all players play the one-shot game NE strategy thereafter.

Strategy for primary i: *Select a price of v at $\tau = 1$, and also for all other τ so long as all other primaries had chosen v in all previous times. Otherwise, play the one-shot game Nash equilibrium strategy $\psi(.)$ in (13).*

Theorem 3: The strategy profile where every primary uses the above Nash reversion strategy is an SPNE if and only if $\delta \geq \delta_t$, where δ_t is a threshold given by:

$$\delta_t = \frac{w(q,n) - \beta(q,n)}{w(q,n) - \beta(q,n) + q\beta(q,n)}$$

and

$$\beta(q,n) = \sum_{i=k}^{n-1} \left(\frac{k}{i+1}\right) \binom{n-1}{i} q^i (1-q)^{n-1-i}. \quad (19)$$

Note that from (8) and (19), $w(q,n) > \beta(q,n) > 0$ and hence $0 < \delta_t < 1$. Thus, for all values of n, k and q, there exists a threshold such that for values of δ greater than it, the above Nash reversion strategy is an SPNE.

The efficiency of the above SPNE is 1 *because bandwidth is always sold at the highest possible price v.* Thus, an efficient NE can be sustained in the repeated game, unlike in the one-shot game (Lemma 4). This is possible because of the *threat mechanism* inherent in the above SPNE: if a primary tries to undercut the prices of other primaries, then he will gain temporarily, but will suffer in the long run because all primaries will switch to the one-shot game NE strategy immediately afterwards.

We plot δ_t versus q in Fig. 3 for different values of n. The plot reveals that δ_t is not close to 1 except when q is close to 0. Thus, since players usually have discount factors δ close to 1 [1], their discount factors would exceed δ_t except for very small q, and hence the above strategy profile will constitute a SPNE unless q is very small. The availability probability of unused bandwidth is rarely close to 0, and even when it is, we have an alternate SPNE strategy profile whose efficiency is very close to 1: when q is very small, the lower limit $\tilde{p}$ in (10), of the symmetric NE price distribution $\psi(.)$ in the one shot game in Section III is close to the upper limit v (refer to (13), (12), (11)), and hence the SPNE that uses this distribution for each player at each time provides prices close to v as well at each time, and thereby attains efficiency close to 1.

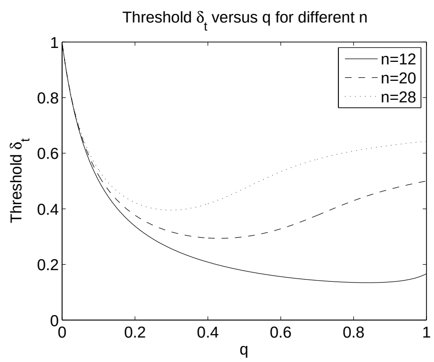

Fig. 3. The threshold δ_t versus q for three values of n. The other parameters are $c = 0$, $v = 100$ and $k = 10$.

V. GENERALIZATIONS

We now generalize our model to allow for random valuations of the secondaries (Section V-A) and asymmetric q values of the primaries (Section V-B).

A. Random Valuations

Recall that in Section III-A, we noted that the parameter v may also be interpreted as the valuation of each buyer. Then the primaries may only know the distribution, but not the exact value of v. We now generalize the results in Section III, which were for a deterministic v, to the case where v is distributed according to some d.f. $G(x) = P(v \le x)$. Assume that $G(x)$ is continuous. Also, let $g(x) = (x - c)P(v \ge x)$. Assume, in addition, that: (i) $g(x)$ has a unique maximizer $v_T > c$, and (ii) $g(x)$ is strictly increasing for $c \le x \le v_T$. Note that a large class of d.f. $G(x)$ satisfy the above technical conditions, *e.g.*, the uniform distribution on some range $[\underline{v}, \overline{v}]$, where $c < \underline{v} < \overline{v}$. Note that by continuity of $G(x)$, $g(x) = (x-c)(1-G(x))$ is continuous. For analytical tractability, we restrict ourselves to the case $k = 1$; n can be arbitrary.

First, note that if there is only one primary, then he sells his unused bandwidth whenever his price does not exceed the secondary's (random) valuation, which happens w.p. $P(v \ge p)$. Thus, his expected utility is $(p-c)P(v \ge p) = g(p)$ when he quotes a price $p \ge c$ for his unused bandwidth. The optimal price that maximizes this expected payoff is v_T, the maximizer of $g(x)$.

Now, suppose there are n primaries and each primary i chooses the price $p_i \in [c, v_T]$ according to a common d.f. $\psi(.)$. As in the constant valuation case in Section III-C, $\phi(.)$ is the distribution of a pseudo-price, p'_i, and $F(x)$ is the distribution of the minimum, $p'_{(1)}$, of $(n - 1)$ pseudo-prices. A primary's pseudo-price, p'_i, is greater than v_T if he has no unused bandwidth, which happens w.p. $1-q$. Also, $p'_{(1)} > v_T$ if and only if all $n - 1$ pseudoprices are greater than v_T. So:

$$1 - F(v_T) = P(p'_{(1)} > v_T) = (1 - q)^{n-1}$$

Hence:

$$F(v_T) = 1 - (1 - q)^{n-1} \tag{20}$$

Now, if primary i sets a price $p_i = x$, then he sells his bandwidth if the minimum of the pseudo-prices of the primaries other than i is greater than x (which happens w.p. $1 - F(x)$) and $v \ge x$. So the expected utility of primary i if he sets a price $p_i = x$ and all other primaries choose the price according to the d.f. $\psi(.)$ is:

$$E\{u_i(x, \psi_{-i})\} = (x - c)(1 - F(x))P(v \ge x) \tag{21}$$

In a NE, this should be a constant over the range $[\tilde{p}, v_T]$ for some $c < \tilde{p} < v_T$ and must equal:

$$\begin{aligned} E\{u_i(\tilde{p}, \psi_{-i})\} &= (\tilde{p} - c)(1 - F(\tilde{p}))P(v \ge \tilde{p}) \\ &= (\tilde{p} - c)P(v \ge \tilde{p}) \end{aligned} \tag{22}$$

since $F(\tilde{p}) = 0$. By (21) and (22):

$$(x - c)(1 - F(x))P(v \ge x) = (\tilde{p} - c)P(v \ge \tilde{p}), \;\; x \in [\tilde{p}, v_T]$$

Thus,

$$F(x) = 1 - \frac{(\tilde{p} - c)P(v \ge \tilde{p})}{(x - c)P(v \ge x)}, \;\; x \in [\tilde{p}, v_T] \tag{23}$$

Note that since $g(x) = (x - c)P(v \ge x)$ is increasing, $0 \le F(x) \le 1$ and $F(x)$ is increasing. Now, $\tilde{p}$ can be found from (20) and (23) to be the solution of:

$$\frac{(\tilde{p} - c)P(v \ge \tilde{p})}{(v_T - c)P(v \ge v_T)} = (1 - q)^{n-1} \tag{24}$$

Now, $\phi(x)$ is the d.f. such that the minimum of $(n - 1)$ i.i.d. random variables, each with d.f. $\phi(.)$, has the d.f. $F(.)$. So similar to (12):

$$F(x) = 1 - (1 - \phi(x))^{n-1} \tag{25}$$

Equations (23) and (25) provide an expression for $\phi(.)$. Also, similar to Theorem 2, we have:

Theorem 4: The strategy profile in which each primary plays $\psi(.)$, where

$$\psi(x) = \begin{cases} 0, & x \le \tilde{p} \\ \frac{1}{q}\phi(x), & \tilde{p} < x \le v_T \\ 1, & x > v_T \end{cases} \tag{26}$$

and $\phi(.)$ is defined by (23) and (25) is a symmetric NE.

The proof is similar to that of Theorem 2.

1) Uniformly Distributed Valuation: Now, we specialize our results to the case in which v is uniformly distributed in $[\underline{v}, \overline{v}]$, where $c < \underline{v} < \overline{v}$, and explicitly compute $F(x)$, which then can be used to compute $\phi(x)$ and $\psi(x)$. Since v is uniformly distributed in $[\underline{v}, \overline{v}]$, it can be checked that $v_T = \max\{\underline{v}, \frac{\overline{v}+c}{2}\}$. If $\underline{v} \ge \frac{\overline{v}+c}{2}$, then $v_T = \underline{v}$ and the results in the constant valuation case go through with v replaced by $\underline{v}$. This is because for every primary i, any price $p_i > \underline{v}$ fetches an expected utility which is lower than that for $p_i = \underline{v}$. Thus, henceforth, we consider the case $\underline{v} < \frac{\overline{v}+c}{2}$. Then, $v_T = \frac{\overline{v}+c}{2}$.

Lemma 5: There exists a unique $\tilde{p}_1$ in (c, v_T) such that:

$$(\tilde{p}_1 - c)(\overline{v} - \tilde{p}_1) = \frac{(1 - q)^{n-1}(\overline{v} - c)^2}{4}. \tag{27}$$

We consider the cases $\tilde{p}_1 \ge \underline{v}$ and $\tilde{p}_1 < \underline{v}$ separately.

2) Case I: $\tilde{p}_1 \geq \underline{v}$: In this case, it can be checked that the $\tilde{p}$ in (24) is equal to $\tilde{p}_1$. Also, $F(x)$ in (23) becomes:

$$F(x) = \begin{cases} 0, & x \leq \tilde{p}_1 \\ 1 - \frac{(\tilde{p}_1 - c)(\overline{v} - \tilde{p}_1)}{(x-c)(\overline{v}-x)}, & \tilde{p}_1 < x \leq v_T \end{cases} \tag{28}$$

3) Case II: $\tilde{p}_1 < \underline{v}$:

Lemma 6: There exists a unique $\tilde{p}_2$ in $(\tilde{p}_1, \underline{v})$ such that:

$$\frac{4(\tilde{p}_2 - c)(\overline{v} - \underline{v})}{(\overline{v} - c)^2} = (1-q)^{n-1}. \tag{29}$$

In this case, it can be checked that the $\tilde{p}$ in (24) is equal to $\tilde{p}_2$. Also, $F(x)$ in (23) becomes:

$$F(x) = \begin{cases} 0, & x \leq \tilde{p}_2 \\ \frac{x - \tilde{p}_2}{x - c}, & \tilde{p}_2 < x \leq \underline{v} \\ 1 - \frac{(\tilde{p}_2 - c)(\overline{v} - \underline{v})}{(x-c)(\overline{v}-x)}, & \underline{v} < x \leq v_T \end{cases}$$

B. Asymmetric q

So far, we have assumed that each primary has unused bandwidth with equal probability, q. Now, we consider that this probability is q_i for primary i, and allow for potentially unequal q_is. This generality leads to some differences in the NE strategies, which we elucidate considering a simple scenario, $n = 2$ and $k = 1$. Without loss of generality, let $q_1 \geq q_2$.

We now describe the equilibrium strategies $\psi_1(.)$ and $\psi_2(.)$ of the two primaries for the one shot game. Define:

$$\tilde{p}_i = v - q_i(v - c), \;\; i = \{1, 2\}$$

Then $\tilde{p}_1 \leq \tilde{p}_2$. Let $\psi_1(x)$ be as in (15) with $\tilde{p}_2$ in place of $\tilde{p}$ and q_1 in place of q. Also, let $\psi_2(x)$ be as in (15) with $\tilde{p}_2$ in place of $\tilde{p}$ and q_2 in place of q.

Theorem 5: The strategy profile in which primary i selects his price using the d.f. $\psi_i(.)$, $i = 1, 2$ is a NE

Note that this NE is not in general symmetric because $\psi_1(.) \neq \psi_2(.)$, which is but expected since q_1, q_2 need not be equal. Also, it can be checked that $\psi_2(.)$ is continuous, whereas $\psi_1(.)$ is not (unless $q_1 = q_2$), and primary 1 chooses price v with a positive probability $1 - \frac{q_2}{q_1}$. This is in contrast to the NE for equal q_is (Theorem 2), where each primary uses a continuous d.f. and hence does not choose any single price with positive probability.

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Appendix

A. Proofs of results in Section III

Proof of Lemma 1: Suppose $\psi(x)$ is not continuous on $[c, v]$. Then $\psi(x)$, (and hence $\phi(x)$ and $F(x)$), has a positive probability mass at a point $x_0 \in [c, v]$, *i.e.* $P\{p_i = x_0\} > 0$. As shown in the proof of Theorem 1, the strategy $p_i = c$ is strictly dominated for each primary i. Hence, primary i plays $p_i = c$ with 0 probability; so $\psi(.)$ does not have a positive probability mass at c. Thus, $x_0 > c$.

We will show that primary 1 gets a higher expected utility by setting p_1 slightly lower than x_0, than by setting $p_1 = x_0$, which will contradict the fact that x_0 is a best response. Let E_2 denote the event that primary 1's bandwidth is sold. If $p_1 = x_0$ and $p'_{(k)} > x_0$, then by definition of $p'_{(k)}$, E_2 occurs. That is:

$$P(E_2 | p_1 = x_0, p'_{(k)} > x_0) = 1. \tag{30}$$

If $p_1 = x_0$ and $p'_{(k)} = x_0$, then two or more primaries including primary 1 have set the k'th lowest price x_0. Since ties are broken at random:

$$P(E_2 | p_1 = x_0, p'_{(k)} = x_0) < 1. \tag{31}$$

Now,

$$\begin{aligned} &P\{E_2|p_1 = x_0\} \\ &= P(E_2|p_1 = x_0, p'_{(k)} > x_0)P\{p'_{(k)} > x_0\} \\ &\quad + P\{E_2|p_1 = x_0, p'_{(k)} = x_0\}P\{p'_{(k)} = x_0\} \\ &= P\{p'_{(k)} > x_0\} + P\{E_2|p_1 = x_0, p'_{(k)} = x_0\}P\{p'_{(k)} = x_0\} \\ &\quad \text{(by (30))} \end{aligned} \tag{32}$$

Similarly, for every $\epsilon > 0$, if $p_1 = x_0 - \epsilon$, then E_2 occurs if $p'_{(k)} \geq x_0$. So:

$$P\{E_2|p_1 = x_0 - \epsilon\} \geq P\{p'_{(k)} > x_0\} + P\{p'_{(k)} = x_0\} \tag{33}$$

By (32) and (33):

$$\begin{aligned} &P\{E_2|p_1 = x_0 - \epsilon\} - P\{E_2|p_1 = x_0\} \geq \\ &P\{p'_{(k)} = x_0\}(1 - P\{E_2|p_1 = x_0, p'_{(k)} = x_0\}) = a \text{ (say)} \end{aligned} \tag{34}$$

By (31), $a > 0$.

Now, primary 1's expected utility if he sets $p_1 = x_0$ is:

$$E\{u_1(x_0, \psi_{-1})\} = (x_0 - c)P\{E_2|p_1 = x_0\} \tag{35}$$

and if he sets $x_0 - \epsilon$ for a small $\epsilon > 0$ is:

$$\begin{aligned} E\{u_1(x_0 - \epsilon, \psi_{-1})\} &= (x_0 - \epsilon - c)P\{E_2|p_1 = x_0 - \epsilon\} \\ &\geq (x_0 - \epsilon - c)(P\{E_2|p_1 = x_0\} + a) \end{aligned}$$

by (34). Taking limits:

$$\begin{aligned} \lim_{\epsilon \to 0+} E\{u_1(x_0 - \epsilon, \psi_{-1})\} &\geq (x_0 - c)(P\{E_2|p_1 = x_0\} + a) \\ &> E\{u_1(x_0, \psi_{-1})\} \text{ (by (35))} \end{aligned}$$

Thus, for small enough ϵ, $p_1 = x_0 - \epsilon$ yields a higher expected payoff than $p_1 = x_0$. So $p_1 = x_0$ is not a best response, which is a contradiction. Thus, $\psi(.)$ cannot have a positive probability mass at any point and is continuous. ■

Proof of Lemma 2: We showed in the proof of Theorem 1 that $c \notin B$. Let $z_l, z_r \in B$, where $c < z_l < z_r \leq v$. To show that B is contiguous, we need to show that $z \in B$ $\forall z \in (z_l, z_r)$. Suppose not. Then there exists $z_0 \in (z_l, z_r)$ such that $z_0 \notin B$. One of the following cases must hold:

Case (i): For every $\epsilon > 0$, there exists a $z \in (z_0 - \epsilon, z_0 + \epsilon)$ such that $z \in B$. Then we can find a sequence $z_1, z_2, z_3, \ldots$ such that $z_i \in B, i = 1, 2, 3, \ldots$ and $\lim_{i\to\infty} z_i = z_0$ [12]. By (4):

$$\begin{aligned} E\{u_1(z_0, \psi_{-1})\} &= (z_0 - c)(1 - F(z_0)) \\ &= \lim_{i\to\infty} (z_i - c)(1 - F(z_i)) \\ &\quad \text{(by continuity of } F(.)) \\ &= \lim_{i\to\infty} u_{max} \text{ (by (5))} \\ &= u_{max} \end{aligned} \tag{36}$$

Thus, $z_0 \in B$, which is a contradiction.

Case (ii): There exists an interval $(z_0 - \epsilon, z_0 + \epsilon)$ such that $z \notin B$ $\forall z \in (z_0 - \epsilon, z_0 + \epsilon)$. Let:

$$\begin{aligned} \tilde{z}_l &= \sup\{z \leq z_0 - \epsilon : z \in B\} \\ \tilde{z}_r &= \inf\{z \geq z_0 + \epsilon : z \in B\} \end{aligned}$$

Note that the supremum and infimum exist because $z_l, z_r \in B$ and hence the sets over which the supremum and infimum are taken are non-empty. Now, similar to (36), it can be shown using continuity of $F(.)$ that $\tilde{z}_l, \tilde{z}_r \in B$. Putting $x = \tilde{z}_l$ and $x = \tilde{z}_r$ in (5) and using (4), we get:

$$\begin{aligned} F(\tilde{z}_l) &= 1 - \frac{u_{max}}{\tilde{z}_l - c} \\ F(\tilde{z}_r) &= 1 - \frac{u_{max}}{\tilde{z}_r - c} \end{aligned}$$

By the above two equations, since $u_{max} > 0$ and $\tilde{z}_l < \tilde{z}_r$:

$$F(\tilde{z}_l) < F(\tilde{z}_r) \tag{37}$$

But by definition of $\tilde{z}_l$ and $\tilde{z}_r$, $z \notin B$ $\forall z \in (\tilde{z}_l, \tilde{z}_r)$. So for every primary i, $P(p_i \in (\tilde{z}_l, \tilde{z}_r)) = 0$. Hence, $P(p'_{(k)} \in (\tilde{z}_l, \tilde{z}_r)) = 0$. That is, $F(\tilde{z}_r-) - F(\tilde{z}_l) = 0$, where $F(x-) = \lim_{y\uparrow x} F(y)$. By continuity of $F(.)$, $F(\tilde{z}_r) = F(\tilde{z}_r-)$. So $F(\tilde{z}_l) = F(\tilde{z}_r)$, which contradicts (37).

Thus, B is contiguous, and hence is an interval. Also, by continuity of $F(x)$, it can be shown similar to (36) that the endpoints of B are best responses, *i.e.* B is closed. Let $B = [\tilde{p}, z_r]$ for some $c < \tilde{p} < z_r \leq v$.

Next, we show, using contradiction, that $z_r = v$. Suppose $z_r < v$. Then $z \notin B$ $\forall z \in (z_r, v]$. So $P(p_i \in (z_r, v]) = 0$ and hence $P(p'_{(k)} \in (z_r, v]) = 0$. Thus:

$$F(z_r) = F(v) \tag{38}$$

By (4) and (38):

$$E\{u_1(z_r, \psi_{-1})\} = (z_r - c)(1 - F(v)). \tag{39}$$

Also,

$$\begin{aligned} E\{u_1(v, \psi_{-1})\} &= (v - c)(1 - F(v)) \\ &> E\{u_1(z_r, \psi_{-1})\} \text{ (by (39))} \end{aligned} \tag{40}$$

which contradicts the fact that $z_r \in B$. So $z_r \not< v$ and hence $z_r = v$, which completes the proof. ■

Proof of Lemma 3: Let

$$\mathcal{F}(y) = \sum_{i=k}^{n-1} \binom{n-1}{i} y^i (1-y)^{n-1-i}, y \in [0, 1] \tag{41}$$

$\mathcal{F}(y)$ is a continuous and strictly increasing function and $\mathcal{F}(0) = 0$, $\mathcal{F}(1) = 1$ [9]. So $\mathcal{F}(.)$ is invertible. By (12) and (41), $F(x) = \mathcal{F}(\phi(x))$; so $\phi(.)$ is unique and given by:

$$\phi(x) = \mathcal{F}^{-1}(F(x)) \tag{42}$$

Also, since $\mathcal{F}$ is a continuous one-to-one map from the compact set $[0, 1]$ onto $[0, 1]$, $\mathcal{F}^{-1}$ is continuous (see Theorem 4.17 in [12]). Also, $F(x)$ in (11) is continuous. So from (42), $\phi(x)$ is a continuous function of x, since it is the composition of continuous functions $\mathcal{F}^{-1}$ and F (see Theorem 4.7 in [12]). Now, by (11), $F(\tilde{p}) = 0$ and $F(v) = w(q, n)$. Also, by (41), $\mathcal{F}(0) = 0$ and by (8) and (41), $\mathcal{F}(q) = w(q, n)$. So by (42), $\phi(\tilde{p}) = \mathcal{F}^{-1}(F(\tilde{p})) = \mathcal{F}^{-1}(0) = 0$ and $\phi(v) = \mathcal{F}^{-1}(F(v)) = \mathcal{F}^{-1}(w(q, n)) = q$. The result follows. ■

Proof of Lemma 4: Since Z, the number of primaries who have unused bandwidth, is a Binomial(n, q) r.v., its mean and variance are $E(Z) = nq$ and $var(Z) = nq(1 - q)$

respectively [14]. First, suppose $k \leq (n-1)(q-\epsilon)$ for some $\epsilon > 0$. Let:

$$Y = \begin{cases} k, & \text{if } Z \geq k \\ 0, & \text{else} \end{cases}$$

Then:

$$\begin{aligned} & E\{\min(Z,k)\} \\ \geq \; & E(Y) \\ = \; & kP(Z \geq k) \\ = \; & k(1 - P(Z < k)) \\ \geq \; & k(1 - P(Z \leq (n-1)(q-\epsilon))) \\ & (\text{since } k \leq (n-1)(q-\epsilon)) \\ \geq \; & k(1 - P(|Z - nq| \geq (n-1)\epsilon)) \\ \geq \; & k\left(1 - \frac{nq(1-q)}{(n-1)^2\epsilon^2}\right) \\ & (\text{by Chebyshev's inequality [14]}) \end{aligned} \tag{43}$$

Now, let Z_1 be a Binomial$(n-1,q)$ random variable. Note that $E(Z_1) = (n-1)q$ and $var(Z_1) = (n-1)q(1-q)$. By (8):

$$\begin{aligned} 1 - w(q,n) &= P(Z_1 < k) \\ &\leq P(Z_1 \leq (n-1)(q-\epsilon)) \\ &\quad (\text{since } k \leq (n-1)(q-\epsilon)) \\ &\leq P(|Z_1 - (n-1)q| \geq (n-1)\epsilon) \\ &\leq 2\exp\left(\frac{-2(n-1)^2\epsilon^2}{n-1}\right) \\ &\quad (\text{by Hoeffding's inequality [26]}) \\ &= 2\exp\left(-2(n-1)\epsilon^2\right) \end{aligned} \tag{44}$$

By (18), (43) and (44):

$$\eta \leq \frac{2nq\exp\left(-2(n-1)\epsilon^2\right)}{k\left(1 - \frac{nq(1-q)}{(n-1)^2\epsilon^2}\right)} \to 0 \text{ as } n \to \infty$$

which proves the first part.

Now, suppose $k \geq (n-1)(q+\epsilon)$ for some $\epsilon > 0$. Since $E\{\min(Z,k)\} \leq E(Z) = nq$, by (18):

$$\begin{aligned} \eta &\geq \frac{nq(1-w(q,n))}{nq} \\ &= 1 - w(q,n) \\ &= 1 - P(Z_1 \geq k) \\ &\geq 1 - P(Z_1 \geq (n-1)(q+\epsilon)) \\ &\quad (\text{since } k \geq (n-1)(q+\epsilon)) \\ &\geq 1 - P(|Z_1 - (n-1)q| \geq (n-1)\epsilon) \\ &\geq 1 - \frac{(n-1)q(1-q)}{(n-1)^2\epsilon^2} \text{ (by Chebyshev's inequality)} \\ &\to 1 \text{ as } n \to \infty \end{aligned}$$

which proves the second part. ■

B. Proofs of Results in Section IV

Proof of Theorem 3: Let u_i^{OS} be the expected payoff that primary i receives in the one-shot game symmetric NE, in which each primary plays the strategy $\psi(.)$ in (13). Let u_i^{PD} be his expected payoff in each stage game of the repeated game when all primaries play the pre-deviation strategy in the above Nash reversion strategy. Also, let u_i^{sup} be the supremum over the possible expected payoffs that primary i can get in a single stage game by using any strategy, when all primaries played the pre-deviation strategy in all slots until the previous stage game, and primaries other than i play the pre-deviation strategy in the current stage game.

It can be shown that a necessary and sufficient condition for the above Nash reversion strategy to be a SPNE is (the proof is similar to that of Lemma 12.AA.1 in [1]):

$$u_i^{sup} + \frac{q\delta}{1-\delta}u_i^{OS} \leq u_i^{PD} + \frac{q\delta}{1-\delta}u_i^{PD} \tag{45}$$

Note that the left-hand side is primary i's maximum (discounted) payoff starting from a given slot if he deviates from the pre-deviation strategy, and the right-hand side is the payoff if he does not deviate. The factor q appears in the second term on either side to account for the fact that primary i would have free bandwidth in each future slot with probability q. So if condition (45) is met, primary i would not deviate from its pre-deviation strategy.

Next, we compute u_i^{PD}, u_i^{OS} and u_i^{sup}. Let $E(n,q)$ be the event that when primary 1 has bandwidth available, and sets $p_1 = v$, and each primary $i = 2,\ldots,n$ sets $p_i = v$ (provided he has bandwidth available), then primary 1 is among the primaries from whom bandwidth is bought by secondaries. Let the random variable A be the number of primaries from $2,\ldots,n$ who have bandwidth available. Note that A has a Binomial$(n-1,q)$ distribution. So:

$$P(A=i) = \binom{n-1}{i} q^i(1-q)^{n-1-i} \tag{46}$$

By definition of A, $A+1$ primaries have unused bandwidth, and if $A+1 > k$, then bandwidth is bought from k of them selected uniformly at random. So:

$$P(E(n,q)|A=i) = \begin{cases} 1, & i < k \\ \left(\frac{k}{i+1}\right), & i \geq k \end{cases} \tag{47}$$

Now,

$$\begin{aligned} P(E(n,q)) &= \sum_{i=0}^{n-1} P(E(n,q)|A=i)P(A=i) \\ &= \sum_{i=0}^{k-1} \binom{n-1}{i} q^i(1-q)^{n-1-i} \\ &\quad + \sum_{i=k}^{n-1}\left(\frac{k}{i+1}\right)\binom{n-1}{i} q^i(1-q)^{n-1-i} \\ &\quad (\text{by (46) and (47)}) \\ &= 1 - w(q,n) + \beta(q,n) \text{ (by (8) and (19))} \end{aligned} \tag{48}$$

Now, $u_i^{OS} = (v-c)(1-w(q,n))$ as derived in Section III. Also, by definition of the event $E(n,q)$, $u_i^{PD} = (v-c)P(E(n,q))$. Finally, since the strategy $p_i = v - \epsilon$ yields the payoff $v - \epsilon - c$ for every $\epsilon > 0$, we have $u_i^{sup} = \sup_{\epsilon>0}\{v-\epsilon-c\} = (v-c)$. Substituting these expressions into (45), putting the value of $P(E(n,q))$ from

(48), and some algebraic simplification, yields that the above Nash reversion strategy is an SPNE if and only if $\delta \geq \delta_t$. ■

C. Proofs of Results in Section V

Proof of Lemma 5: Let

$$f_2(x) = (x-c)(\overline{v}-x) - \frac{(1-q)^{n-1}(\overline{v}-c)^2}{4}$$

Note that $f_2(x)$ is a quadratic; so it has at most two distinct roots. Hence, it is sufficient to show that $f_2(x)$ has exactly one root in (c, v_T), the other root being in $(v_T, \overline{v})$. Now,

$$f_2(c) = -\frac{(1-q)^{n-1}(\overline{v}-c)^2}{4} < 0$$

$$\begin{aligned} f_2(v_T) &= (v_T - c)(\overline{v} - v_T) - \frac{(1-q)^{n-1}(\overline{v}-c)^2}{4} \\ &= \frac{(\overline{v}-c)^2}{4}\left\{1-(1-q)^{n-1}\right\} > 0 \end{aligned}$$

So, since $f_2(x)$ is continuous, by the intermediate value theorem [12], it has a root in (c, v_T). Also,

$$f_2(\overline{v}) = -\frac{(1-q)^{n-1}(\overline{v}-c)^2}{4} < 0$$

So again, by the intermediate value theorem, $f_2(x)$ has a root in $(v_T, \overline{v})$, and the result follows. ■

Proof of Lemma 6: Let

$$f_3(x) = \frac{4(x-c)(\overline{v}-\underline{v})}{(1-q)^{n-1}(\overline{v}-c)^2}$$

Then:

$$\begin{aligned} f_3(\tilde{p}_1) &= \frac{4(\tilde{p}_1-c)(\overline{v}-\underline{v})}{(1-q)^{n-1}(\overline{v}-c)^2} \\ &= \frac{\overline{v}-\underline{v}}{\overline{v}-\tilde{p}_1} \quad (\text{by}(27)) \\ &< 1 \quad (\text{since } \tilde{p}_1 < \underline{v} < \overline{v}) \end{aligned}$$

and

$$\begin{aligned} f_3(\underline{v}) &= \frac{4(\underline{v}-c)(\overline{v}-\underline{v})}{(1-q)^{n-1}(\overline{v}-c)^2} \\ &= \frac{(\underline{v}-c)(\overline{v}-\underline{v})}{(\tilde{p}_1-c)(\overline{v}-\tilde{p}_1)} \quad (\text{by}(27)) \end{aligned}$$

So $f_3(\underline{v}) > 1$ iff:

$$\begin{aligned} &(\underline{v}-c)(\overline{v}-\underline{v}) > (\tilde{p}_1-c)(\overline{v}-\tilde{p}_1) \\ &\Leftrightarrow (\underline{v}-\tilde{p}_1)(\overline{v}+c-\underline{v}-\tilde{p}_1) > 0 \end{aligned}$$

which is true because $\underline{v}-\tilde{p}_1 > 0$ and $\overline{v}+c > 2\underline{v} > \underline{v}+\tilde{p}_1$. Thus, $f_3(\tilde{p}_1) < 1$ and $f_3(\underline{v}) > 1$. Since $f_3(x)$ is continuous, by the intermediate value theorem [12], there exists $\tilde{p}_2$, where $\tilde{p}_1 < \tilde{p}_2 < \underline{v}$, such that $f_3(\tilde{p}_2) = 1$, *i.e.* (29) is satisfied.

Also, since $f_3(.)$ is of the form $f_3(x) = a(x-c)$, where $a > 0$, f_3 is one-to-one and hence $\tilde{p}_2$ is unique. ■

Proof of Theorem 5: Let p_1, $\tilde{p}_2 \leq p_1 \leq v$, be fixed. If $p_1 < p_2$, then primary 1's bandwidth is sold, and if $p_1 > p_2$, then primary 1's bandwidth is sold iff primary 2 has no unused bandwidth, which happens w.p. $1-q_2$. So primary 1's utility is given by:

$$u_1(p_1,p_2) = \begin{cases} p_1 - c & \text{if } p_1 < p_2 \\ (p_1-c)(1-q_2) & \text{if } p_1 > p_2 \end{cases} \tag{49}$$

Suppose primary 2 uses the mixed strategy $\psi_2(.)$. Note that for fixed p_1, since $\psi_2(.)$ is continuous, the event $p_2 = p_1$ has zero probability. Hence,

$$\begin{aligned} &E_{p_2}[u_1(p_1,p_2)] \\ &= (p_1-c)P(p_2>p_1) + (p_1-c)(1-q_2)P(p_2<p_1) \\ &= (p_1-c)[1-\psi_2(p_1)] + (p_1-c)(1-q_2)\psi_2(p_1) \\ &= (p_1-c)(1-q_2\psi_2(p_1)) \\ &= (p_1-c)\left(1-q_2\frac{1}{q_2}\left(\frac{p_1-\tilde{p}_2}{p_1-c}\right)\right) \\ &= \tilde{p}_2 - c \end{aligned} \tag{50}$$

Thus, primary 1's expected payoff is constant, equal to $\tilde{p}_2 - c$, for $\tilde{p}_2 \leq p_1 \leq v$. For $p_1 < \tilde{p}_2$, primary 1's payoff is at most $p_1 - c < \tilde{p}_2 - c$. So p_1 is a best response to $\psi_2(.)$ for each $p_1 \in [\tilde{p}_2, v]$. Since $\psi_1(.)$ randomizes over $p_1 \in [\tilde{p}_2, v]$, $\psi_1(.)$ is a best response to $\psi_2(.)$.

Now, let p_2, $\tilde{p}_2 \leq p_2 < v$, be fixed. Similar to (49):

$$u_2(p_1,p_2) = \begin{cases} p_2 - c & \text{if } p_2 < p_1 \\ (p_2-c)(1-q_1) & \text{if } p_2 > p_1 \end{cases}$$

Suppose primary 1 uses the mixed strategy $\psi_1(.)$. Since $\psi_1(p_1)$ is continuous on $\tilde{p}_2 \leq p_1 < v$, $P(p_1 = p_2) = 0$ for $\tilde{p}_2 \leq p_2 < v$. So for $\tilde{p}_2 \leq p_2 < v$, similar to the derivation of (50):

$$\begin{aligned} E_{p_1}(u_2(p_1,p_2)) &= (p_2-c)P(p_1>p_2) \\ &\quad +(p_2-c)(1-q_1)P(p_1<p_2) \\ &= \tilde{p}_2 - c \end{aligned} \tag{51}$$

For $p_2 < \tilde{p}_2$:

$$u_2(p_1,p_2) \leq p_2 - c < \tilde{p}_2 - c \tag{52}$$

Now, let $p_2 = v$. If $p_1 = v$, then primary 2's payoff is $(v-c)$ if primary 1 has no unused bandwidth or if primary 1 has unused bandwidth and the secondary (randomly) selects primary 2 to buy bandwidth from, and 0 otherwise. Thus, if $p_1 = v$, then the probability that primary 2's payoff is $(v-c)$ is $1-q_1+\frac{q_1}{2} = 1-\frac{q_1}{2}$. Thus,

$$u_2(p_1,v) = \begin{cases} (v-c)(1-q_1) & \text{if } p_1 < v \\ (v-c)\left(1-\frac{q_1}{2}\right) & \text{if } p_1 = v \end{cases}$$

Now, $P(p_1 < v) = \psi_1(v-) = \frac{1}{q_1}\left(\frac{v-\tilde{p}_2}{v-c}\right)$ and $P(p_1 = v) = 1 - \frac{q_2}{q_1} > 0$. Hence:

$$\begin{aligned} E_{p_1}(u_2(p_1,v)) &= (v-c)(1-q_1)P(p_1<v) \\ &\quad +(v-c)\left(1-\frac{q_1}{2}\right)P(p_1=v) \\ &= (v-c)(1-q_1)\frac{1}{q_1}\left(\frac{v-\tilde{p}_2}{q_1(v-c)}\right) \\ &\quad +(v-c)\left(1-\frac{q_1}{2}\right)\left(1-\frac{q_2}{q_1}\right) \\ &\leq \frac{1-q_1}{q_1}(v-\tilde{p}_2) + (v-c)\left(1-\frac{q_2}{q_1}\right) \\ &= \tilde{p}_2 - c \ \ (\text{using } \tilde{p}_2 = v - q_2(v-c)) \end{aligned} \tag{53}$$

By (51), (52) and (53), each p_2, $\tilde{p}_2 \leq p_2 < v$, is a best response. Since $\psi_2(.)$ randomizes over $p_2 \in [\tilde{p}_2, v)$, $\psi_2(.)$ is a best response to $\psi_1(.)$. Thus, $(\psi_1(.), \psi_2(.))$ is a Nash equilibrium. ■