Secondary Spectrum Market: To acquire or not to acquire side information?

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Abstract

In a secondary spectrum market primaries set prices for their unused channels to the secondaries. The payoff of a primary depends on the channel state information (CSI) of its competitors. We consider a model where a primary can acquire its competitors CSI at a cost. We formulate a game between two primaries where each primary decides whether to acquire its competitor's CSI or not and then selects its price based on that. Our result shows that no primary will decide to acquire its competitor's CSI with probability 1. When the cost of acquiring the CSI is above a threshold, there is a unique Nash Equilibrium (NE) where both the primaries remain uninformed of their respective competitor's CSI. When the cost is below the threshold, in the unique NE each primary randomizes between its decision to acquire the CSI or not. Our result reveals that irrespective of the cost of acquiring the CSI, the expected payoff of a primary remains the same.

I. Introduction

Secondary access of the spectrum where license holders (primaries) allow unlicensed users (secondaries) to use their channels can enhance the efficiency of the spectrum usage. However, secondary access will only proliferate when it is rendered profitable to the primaries. We investigate a secondary spectrum market where there are *two primaries* which want to lease their unused channels to *a secondary* in lieu of financial remuneration. The secondary seeks to buy a channel with the lowest price.

The transmission rate offered by the channel of a primary evolves randomly because of the fading and the usage statistics of the consumers of the primary. If the transmission rate is below a certain threshold, then the secondary will not buy that channel and the channel is *unavailable for sale*. When its channel is available, the profit or payoff that a primary can obtain will depend in part on if the other primary also has a channel available. Acquiring such an information may be costly as, for example it might require the primary to sense the other primary's spectrum and analyze its traffic patterns. Perceiving how this cost impacts the competition between the primaries is the goal of this paper.

We investigate a setting where each primary decides whether to acquire the channel state information (CSI) of its competitor before selecting a price for its available channel. However, while taking its own decision a primary does not know whether its competitor decides to acquire the CSI or not. Knowledge of the CSI of the other primary has potential advantages. For example, if the primary knows that the channel of its competitor is not available, then, it can select the highest possible price and still can sell its channel because of the lack of competition. However, acquiring this knowledge will make the primary incur a cost, which reduces its profit. On the other hand, it is also not apriori clear whether neither of the primaries acquire the CSI of their competitors. This is because one primary may acquire the CSI of its competitor and take advantage of extra information compared to its competitor.

The inherent uncertainty in the competitor's decision also complicates the pricing strategy of the primary. If one primary (A) knows that the channel of the other primary (B) is available, its pricing decision still depends on if primary B also know that its channel is available; if not then the primary B may randomize among multiple prices, enabling primary A to charge a higher price. On the other hand, if the primary does not know the channel state of its competitor, then it needs to select a lower price which will increase the probability of selling, but fetches a lower payoff in the event of a sale.Thus, it is also not apriori clear how a primary will select its price.

Price competition in economics and cognitive radio network has been extensively studied [1], [2]; however, most of the models do not capture this type of uncertainty about the supply that the sellers will bring to the market. Some recent papers [3]–[5] considered price competition models where a player may not be able to sell its channel. However, the above papers imposed a constraint where the players *can not know* the channel states of their competitors. Thus, the decision of acquiring the CSI of other primary is not a part of the strategy space. In our setting, the decision to acquire the CSI of the other primary is included in the primary's strategy space along with the prices. Thus, in our setting a primary is also unaware whether its CSI is known to the other primary or not, while in [3]–[5] the primary knows that its CSI is unknown to other primary. Thus, a primary now needs to judiciously decide whether to acquire the CSI of its competitor or not and selects a price based on that. We contribute in this space.

We model the setting as a non-cooperative game with the primaries as players. When the channel of a primary is available, it decides i) whether to acquire the CSI of its competitor or not, and ii) a price. Selection of the price depends on the information

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the primary has. Specifically, if the primary acquires the CSI of its competitor it may select different prices depending on whether its competitor's channel is available or not, on the other hand, when the primary does not acquire the CSI of competitor, it has to select a price irrespective of the competitor's channel state.

We completely characterize the NE strategy profile. Our result reveals that there is no NE strategy where a primary decides to know the CSI of its competitor with probability 1 (Theorems 1 and 2). Thus, a primary will never acquire the CSI of the other primary unless the other primary randomizes between its decision to know the channel state of its competitor or not. Our result also reveals that if the cost of acquiring the CSI of its competitor is above a threshold (T, say), then there exists a *unique* NE where both primaries decide *not to acquire* the CSI of their respective competitor with probability 1 (Theorem 3). In the NE strategy, both players select prices from identical continuous distribution.

We also show that when the cost of acquiring the information of the competitor's channel is below the threshold T, then there exists a *unique* NE where both the primaries select statistically identical mixed strategies (Theorem 4). More specifically, in this NE, each primary chooses to acquire (not acquire, resp.) the information of its competitor's channel w.p. p ($1-p$ resp.).We also show that p increases as the cost decreases and thus, the primary is more likely to acquire the information of its competitor's channel as the cost decreases. Intuitively, when a primary knows (does not know, resp.) the channel of its competitor is available, a primary should select prices more conservatively (aggressively, resp.). Our result validates the above intuition and goes beyond. We show that when the primary knows (does not know, resp.) that its competitor's channel is available then it randomizes among the prices in the interval $[\tilde{p}_1, \tilde{p}_2]$ ($[\tilde{p}_2, \tilde{p}_3]$, resp.). We have also fully characterized $\tilde{p}_1, \tilde{p}_2$, and $\tilde{p}_3$.

Our analysis also shows an apparently counter-intuitive result–the expected payoff of a primary is the same irrespective of the cost of acquiring the CSI (Theorems 3 and 4).

II. System Model

We consider a secondary spectrum market with two primaries (players) and one secondary. Each primary has a channel that is available with probability (w.p.) q, where $1 > q > 0$ and q is of common knowledge.

The secondary does not buy an available channel which is priced above v. If the channels of both the primaries are available for sale, then, the secondary will buy the lower priced channel. Ties are broken randomly.

A primary is aware of its own channel state. If its channel is available, the primary decides whether to know the exact state of the channel of its competitor before deciding the price for its available channel. If a primary decides to know the exact state of the channel of its competitor, then it has to incur a cost $c_1 > 0$. We assume that a primary either completely knows (at cost c_1) or does not know whether the channel of its competitor is available [1].

If a primary sets its price at x and it decides to acquire the CSI of its competitor, then, its payoff is

$$\begin{cases} x - c - c_1, & \text{if the primary is able to sell its channel,} \\ -c_1, & \text{otherwise.} \end{cases}$$

where c is a transaction cost which is incurred when the secondary buys the channel.

When a primary does not know the channel state of its competitor, then its payoff at price x is

$$\begin{cases} x - c, & \text{if the primary is able to sell its channel,} \\ 0, & \text{otherwise.} \end{cases}$$

A. Strategy of a Primary

If the channel of a primary is available for sale[2], it will take a decision $D \in \{Y, N\}$ where Y denotes that the primary decides to incur the cost c_1 to acquire the other primary's CSI and N denotes that the primary decides not to acquire this information. Then, primary i also sets a price for its available channel. If a primary selects Y, i) it selects a price from an arbitrary distribution function[3] $F_1(\cdot)$ when its competitor's channel is available, and ii) it selects a price from an arbitrary distribution function $F_2(\cdot)$ when its competitor's channel is not available. If a primary selects N, then it does not know the channel state of its competitor, so it only selects its price using a single distribution function $F(\cdot)$.

Definition 1. *The strategy S_i of primary $i = 1, 2$ is $\sigma(D, \mathbf{F})$ where $\mathbf{F} = (F_1, F_2)$ when $D = Y$, $\mathbf{F} = (F, F)$ when $D = N$, and $\sigma(D, \mathbf{F})$ is a probability mass function over the strategies $(D, \mathbf{F})$.*

The strategy of the primary other than i is denoted as S_{-i}.

Definition 2. *$E\{u_i(S_i, S_{-i})\}$ denotes the expected payoff of primary i when its channel is available, and it uses strategy S_i and the other primary uses strategy S_{-i}*[4] .

[1] In practice, a primary gets better estimates of the channel by incurring the cost. It seldom know the exact channel state information. The generalization of our setting where the estimation of the channel state of the competitor increases with the cost is a work for the future.

[2] If the channel of the primary is unavailable, then its decision is immaterial.

[3] The distribution functions can be continuous, discrete or mixed.

[4] Note that we consider the expected payoff of a primary as the expected payoff conditioned on the channel of the primary being available. Naturally if the channel of the primary is unavailable, it will attain a payoff of 0.

B. Solution Concept

We consider a non-cooperative game where each primary only wants to maximize its own expected payoff. We use the Nash Equilibrium as a solution concept.

Definition 3. *A* Nash equilibrium *(NE) (S_1, S_2) is a strategy profile such that no primary can improve its expected profit by unilaterally deviating from its strategy [1]. Thus,*

$$E\{u_i(S_i, S_{-i})\} \geq E\{u_i(\tilde{S}_i, S_{-i})\} \; \forall \; \tilde{S}_i. \tag{1}$$

III. Results

We now discuss the main results: leftmargin=*

- Theorem 1 shows that regardless of the cost c_1, there is no NE where both the players have full knowledge of each other's channel states w.p. 1.
- Theorem 2 shows that there is no NE where one primary has the complete knowledge of the channel state of its competitor, but the other does not.
- Theorem 3 shows that when $c_1 \geq q(v-c)(1-q)$, then there exists a unique NE where both the primaries select N w.p. 1. The above theorem entails that if c_1 is high, then in an NE the primaries always prefer to remain uninformed of the channel states of their competitors.
- Theorem 4 reveals that when $c_1 < (v-c)(1-q)$, there exists a unique NE where primaries randomize between Y and N. Specifically, each primary has the complete CSI of other w.p. p $(0 < p < 1)$. The primaries tend to select Y with higher probability (p increases) as c_1 decreases.
- Theorems 3 and 4 show that the expected payoff that a primary attains in any NE strategy profile is $(v-c)(1-q)$. Thus, the provision of selecting Y and the cost c_1 do not impact the expected payoff of a primary.

We now describe the results in details. We first state some price distributions $\phi(\cdot)$ and $\psi(\cdot)$ which we use throughout.

$$\begin{aligned}\phi(x) =& 0 \quad \text{if } x < \tilde{p}\\ & \frac{1}{q}\left(1 - \frac{(v-c)(1-q)}{x-c}\right) \quad \text{if } \tilde{p} \leq x \leq v\\ & 1 \quad \text{if } x > v. \end{aligned} \tag{2}$$

$$\begin{aligned}\psi(x) =& 0 \quad \text{if } x < \tilde{p}\\ & (1 - \frac{(v-c)(1-q)}{x-c}) \quad \text{if } \tilde{p} \leq x < v\\ & 1-q, \quad \text{if } x = v\\ & 1 \quad \text{if } x > v. \end{aligned} \tag{3}$$

$$\text{where } \tilde{p} = (v-c)(1-q) + c. \tag{4}$$

A. Does there exist an NE where both primaries select Y?

Theorem 1. *There is no Nash equilibrium where both the primaries choose Y w.p.* 1.

Outline of the proof: Assume both players choose Y, so that they know each other's channel state. Thus, the competition becomes similar to *Bertrand Competition* [1], i.e. if the channel of its competitor is unavailable, then the primary will set its price at the v, otherwise it will set its price at the lowest value c. Now, the probability with which the channel of a primary is available is q. Thus, the expected payoff of a player is

$$(v-c-c_1)(1-q) + (c-c-c_1)q \tag{5}$$

Now consider the following unilateral deviation for a primary: Primary 1 selects N and sets its price at v w.p. 1. The channel of primary 1 will be bought when the channel of primary 2 is not available for sale. Since primary 1 decides not to incur the cost c_1, thus, its expected payoff is

$$(v-c)(1-q) \tag{6}$$

This is strictly higher than (5). Hence, the strategy profile can not be an NE. □

The above theorem means that there will be at least one primary which will be unaware of its competitor's channel state with a non-zero probability.

B. Does there exist an NE where one selects Y and the other selects N?

Theorem 2. *For positive $c_1 > 0$, there is no NE where a primary selects Y w.p. 1 and the other selects N w.p. 1.*

First, we provide the intuition behind the result. The primary (say, 1) which selects Y tends to select lower prices with higher probability when it knows that the channel of the other primary is available. Thus, in response the primary 2 (which selects N) selects higher prices with higher probabilities in order to gain a high payoff in the event that the channel of primary 1 is unavailable since it knows that its probability of selling is very low in the event that the channel of primary 1 is available. The primary 1 can then gain a higher payoff by selecting N and higher prices as it does not have to incur the cost c_1. Hence, the primary 1 has an incentive to deviate from its own strategy. The detailed proof is given below.

Proof. Without loss of generality, assume that primary 1 selects Y and primary 2 selects N. First, we discuss the pricing strategies of primaries 1 and 2 and calculate the expected payoff of primary 1, subsequently, we show that primary 1 has an incentive to deviate.

When primary 1 knows that the channel of primary 2 is not available, then primary 1 will be able to sell its channel at the highest possible price, thus, it will select v w.p. 1 and its payoff if $(v-c)-c_1$. The above event occurs w.p. $1-q$.

Now, we consider the case when the channel of primary 2 is available. While deciding its price, primary 2 only knows that the channel of primary 1 is available w.p. q. However, while selecting its price primary 2 knows that the primary 1 will know the channel state of primary 2 if the channel of primary 1 is available. Hence, when primary 1 knows that the channel of primary 2 is available, then the pricing decision becomes equivalent to the setting where primary 1 knows that the channel of primary 2 is available w.p. 1 and primary 2 knows that the channel of primary 1 is available w.p. q. The NE pricing strategy in the last setting has been studied in [3] and using Theorem 2 in [3] we have

Lemma 1. *Primary 1 must select its price according to $\phi(\cdot)$ (given in (2)) and primary 2 must select its price according to $\psi(\cdot)$ (given in (3)).*

By Lemma 1 when the channel of primary 2 is available for sale, then expected payoff of primary 1 at any $\tilde{p} \leq x < v$

$$(x-c)(1-\psi(x))-c_1=(v-c)(1-q)-c_1. \tag{7}$$

At $x < \tilde{p}$, the payoff of primary will be strictly less than the expression in (7). On the other hand at v, primary 1 will get strictly a lower payoff compared to the payoff at a price just below v since $\psi(\cdot)$ has a jump at v. Hence, the maximum expected payoff to primary 1 in this case is $(v-c)(1-q)-c_1$.

Thus, the expected payoff of primary 1 is

$$(v-c)(1-q)+q(v-c)(1-q)-c_1. \tag{8}$$

Now, we show that if primary 1 selects N, then the primary can achieve strictly higher payoff. For $x \in [\tilde{p}, v)$, the expected payoff of primary 1 at N is

$$(x-c)(1-q\psi(x))=(x-c)(1-q)+q(v-c)(1-q) \tag{9}$$

Thus, for every positive c_1 there exists a small enough $\epsilon > 0$ such that at $x=(v-c-\epsilon)$, it will attain strictly higher payoff than (8). Hence, if primary 1 selects N and the price $v-\epsilon$ w.p. 1 then primary 1 attains a strictly higher payoff. The result follows. ☐

C. Does there exist an NE where both primaries select N?

Theorem 3. *Suppose that each primary selects the strategy (N, ϕ) ($\phi(\cdot)$ is given in (2)). The above strategy profile is the unique NE when $c_1 \geq q(v-c)(1-q)$.*
However, the above is not an NE when $c_1 < q(v-c)(1-q)$.

We first provide an intuition. When c_1 is high, if a primary selects Y, then it has to incur high cost compared to the potential gain it will achieve, thus, no primary has any incentive to deviate. When c_1 is low, if a primary deviates and selects Y, then it can gain higher payoff by taking advantage of the CSI of the other primary. Thus, the strategy profile fails to be an NE when c_1 is low. The detailed proof is the following :

Outline of the proof: We only show that the strategy profile is *an* NE when $c_1 \geq q(v-c)(1-q)$ and *not an* NE when $c_1 < q(v-c)(1-q)$. The proof of the uniqueness of the NE is deferred to Section IV.

Note that when both the players select N w.p. 1, then, the setting becomes equivalent to the setting where each primary only knows that the channel of its competitor is available w.p. q. The NE pricing strategy in this setting has been studied in [3]–[5]. From the result, each primary must select its price according to $\phi(\cdot)$. The expected payoff that a primary attains in the strategy profile is $(v-c)(1-q)$.

Now, we show that if $c_1 \geq q(v-c)(1-q)$ then a primary will not have any incentive to deviate to select Y. Suppose primary 1 selects Y and so, knows the channel state of primary 2. Thus, it will select v w.p. 1 when the channel of primary 2 is unavailable. The above event occurs w.p. $1-q$. If the channel of primary 2 is available for sale, then, the payoff of primary 1 at a price x such that $\tilde{p} \leq x \leq v$ is

$$\begin{aligned}(x-c)(1-\phi(x))-c_1 = \\ (x-c)(1-(1/q))+(1/q)(v-c)(1-q)-c_1.\end{aligned} \tag{10}$$

Since $1/q > 1$, the expression in (10) is maximized at $x = \tilde{p}$. Since $\tilde{p} = (v-c)(1-q)+c$ (recall from Lemma 1), hence, the maximum possible expected payoff is $(v-c)(1-q)-c_1$ and this is attained when primary 1 selects price $\tilde{p}$ w.p. 1.

Thus, the maximum expected payoff that primary 1 can attain by selecting Y is

$$(1-q)(v-c)+q(v-c)(1-q)-c_1 \tag{11}$$

However, the expected payoff that primary 1 attains following the strategy profile is $(v-c)(1-q)$. Since $c_1 > q(v-c)(1-q)$, primary 1 will not have any incentive to deviate unilaterally showing that the strategy profile is an NE.

However, if $c_1 < q(v-c)(1-q)$, then the expected payoff in (11) becomes higher than $(v-c)(1-q)$. Thus, the strategy profile is not an NE when $c_1 < q(v-c)(1-q)$. □

Remark: The result shows that when the cost c_1 is high, in an equilibrium both the primaries select N. It is obvious that if $c_1 > (v-c)$, then a primary will never opt for Y. The above theorem shows that even if $c_1 \geq (v-c)q(1-q)$, primaries will select N.

D. Does there exist an NE when c_1 is low?

Note from Theorems 1, 2 and 3 that if c_1 is low, then there is no NE strategy where each primary selects either Y or N w.p. 1. Thus, at least one primary must randomize between Y and N when c_1 is low.

Now, consider the following price distributions

$$\psi_1(x) = \begin{cases} 0, & \text{if } x < \tilde{p}_1, \\ \dfrac{1}{p}(1-\dfrac{\tilde{p}_1-c}{x-c}) & \text{if } \tilde{p}_1 \leq x \leq \tilde{p}_2, \\ 1, & \text{if } x > \tilde{p}_2. \end{cases}$$

and

$$\psi_2(x) = \begin{cases} 0, & \text{if } x < \tilde{p}_2 \\ \dfrac{1}{q(1-p)}(1-\dfrac{(v-c)(1-q)}{x-c}-qp) & \text{if } \tilde{p}_2 \leq x \leq v \\ 1, & \text{if } x > v \end{cases}$$

where $\tilde{p}_1$ and $\tilde{p}_2$ are

$$\tilde{p}_1 = \frac{(v-c)(1-q)(1-p)}{1-qp}+c, \quad \tilde{p}_2 = \frac{(v-c)(1-q)}{1-qp}+c$$

Note that both $\psi_1(\cdot)$ and $\psi_2(\cdot)$ are continuous. In the following, we show that a strategy profile based on these distribution is a NE when c_1 is small enough.

Theorem 4. *Consider the following strategy profile: Each primary selects Y w.p. p and N w.p. $1-p$ where $p = \dfrac{q(v-c)(1-q)-c_1}{q(v-c)(1-q)-c_1 q}$. When choosing Y, the primary selects its price according to $\psi_1(\cdot)$ when it knows that the channel state of the other primary is available, otherwise it selects v w.p. 1. When choosing N, the primary selects price according to $\psi_2(\cdot)$.*
The above strategy profile is the unique NE if $c_1 < q(v-c)(1-q)$. The expected payoff that a primary attains in the NE strategy profile is $(v-c)(1-q)$.

Proof. First, note that $0 < p < 1$, only when $c_1 < q(v-c)(1-q)$. Thus, it is trivial that the above strategy profile can not be an NE when $c_1 < q(v-c)(1-q)$. We defer the proof of the uniqueness of the NE when $c_1 < q(v-c)(1-q)$ to Section IV.

Without loss of generality we only consider the unilateral deviation of primary 1. First in step (i) we show that when primary 1 chooses Y, its maximum expected payoff is $(v-c)(1-q)$. Next, in step (ii) we show that while choosing N, primary 1 can also get a maximum expected payoff of $(v-c)(1-q)$. Finally, in step (iii) we complete the proof by showing that its maximum expected payoff is attained at the given strategy.

i) Suppose that primary 1 selects Y and knows that the channel of primary 2 is available. Note that the support set of $\psi_1(\cdot)$ is $[\tilde{p}_1, \tilde{p}_2]$ and that of $\psi_2(\cdot)$ is $[\tilde{p}_2, v]$. Thus, when $\tilde{p}_1 \leq x \leq \tilde{p}_2$, the probability that primary 2 will select a price less than x is $p\psi_1(x)$. Since $\psi_1(\cdot)$ is continuous, hence, the expected payoff of primary 1 at x is

$$(x-c)(1-p\psi_1(x))-c_1 = \tilde{p}_1-c-c_1. \tag{12}$$

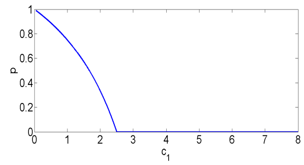

Fig. 1: Variation of p as a function of c_1 in an example setting: $v = 11, c = 1, q = 0.5$. When $c_1 \geq 2.5$, $p = 0$. $p \to 1$ as $c_1 \to 0$.

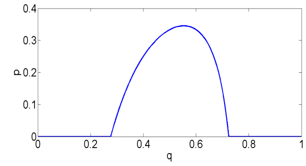

Fig. 2: Variation of p as a function of q in an example setting: $v = 11, c = 1, c_1 = 2$. When either $q \leq 0.28$ or $q \geq 0.72$, $p = 0$. p is maximized at $q^* = 0.55$, maximum value of p is 0.35.

Now, we show that at any $v \geq x \geq \tilde{p}_2$, the maximum expected payoff that a primary can get is upper bounded by $\tilde{p}_1 - c - c_1$. The expected payoff of primary 1 at x is now

$$\begin{aligned}&(x-c)(1-p-(1-p)\psi_2(x)) - c_1\\&= (x-c)(1-1/q) + (v-c)(1-q)/q - c_1.\end{aligned} \tag{13}$$

The supremum of the above expression is obtained at $x = \tilde{p}_2$. We can show that at $\tilde{p}_2$, the above expression becomes $\tilde{p}_1 - c - c_1$. When primary 1 selects a price less than $\tilde{p}_1$ then it is trivial that the payoff will be less than $\tilde{p}_1 - c - c_1$.

Thus, when primary 1 knows that the channel of primary 2 is available for sale, then the maximum payoff that primary 1 can attain is $\tilde{p}_1 - c - c_1$ and this is attained at any price in the interval $[\tilde{p}_1, \tilde{p}_2]$.

When primary 1 knows that the channel of primary 2 is not available, then its payoff is $(v-c) - c_1$. Hence, the maximum expected payoff that primary 1 can attain is

$$(1-q)(v-c) + q(\tilde{p}_1 - c) - c_1. \tag{14}$$

Replacing the value of p in $\tilde{p}_1 - c$ it is easy to discern that $q(\tilde{p}_1 - c) = c_1$. Thus from (14), the maximum expected payoff of primary 1 at Y is $(v-c)(1-q)$.

ii) Next, suppose N is selected by primary 1. First, we show that at any x such that $\tilde{p}_2 \leq x \leq v$, the expected payoff is $(v-c)(1-q)$. At x, the expected payoff of primary 1 is

$$(x-c)(1-qp-q(1-p)\psi_2(x)) = (v-c)(1-q). \tag{15}$$

Now, we show that at any x such that $\tilde{p}_1 \leq x \leq \tilde{p}_2$, the expected payoff of primary 1 is at most $(v-c)(1-q)$. The expected payoff of primary 1 at x is

$$\begin{aligned}&(x-c)(1-qp\psi_1(x))\\&= (x-c)(1-q) + q(v-c)(1-q)(1-p)/(1-qp)\end{aligned}$$

The supremum of the above expression is attained at $x = \tilde{p}_2$. The payoff at $\tilde{p}_2$ is $(v-c)(1-q)$. Thus, the maximum expected payoff attained of primary 1 under N is $(v-c)(1-q)$ and this is attained at any $x \in [\tilde{p}_2, v]$.

iii) The maximum expected payoff attained by primary 1 either by selecting Y or N is $(v-c)(1-q)$. Thus, any randomization of Y and N will also yield a maximum expected payoff of $(v-c)(1-q)$. At the strategy $\tilde{S}$, we have shown that the primary can attain an expected payoff of $(v-c)(1-q)$ both at Y and $(v-c)(1-q)$ at N. Thus, under the strategy $\tilde{S}$, the expected payoff of primary 1 is $(v-c)(1-q)$. Hence, the strategy profile $\tilde{S}$ is an NE with the desired expected payoff. $\square$

Discussion: Note from the above theorem that when c_1 is low there exists an NE where *both the primaries randomize* between Y and N. It is also easy to discern that as c_1 decreases, p increases and as $c_1 \to 0$, $p \to 1$ (Fig. 1). Thus, when the cost of obtaining the competitor's CSI decreases, then the primaries will be more likely to acquire that information.

Note also that $q(v-c)(1-q)$ is maximized at $q = 1/2$. Thus, if $c_1 \geq (v-c)/4$, then primaries will never select Y. By differentiating, it is easy to discern that when $c_1 < (v-c)/4$, then p is maximized at $q^* = 1 - \sqrt{c_1/(v-c)}$ (Fig. 2). Thus, the maximum value is *not achieved* at $q = 1/2$ i.e. when the uncertainty of the availability of a channel is the highest (Fig. 2). Since $c_1 < (v-c)/4$, thus, $q^* > 1/2$. Note also that q^* decreases as c_1 increases. Intuitively, when c_1 increases, primaries tend to select Y only when the uncertainty of the availability of channel increases.

The support set of $\psi_1(\cdot)$ is $[\tilde{p}_1, \tilde{p}_2]$ and $\psi_2(\cdot)$ is $[\tilde{p}_2, v]$. Thus, under Y a primary selects lower prices when the primary knows that the channel of its competitor is available compared to the setting where the primary is not aware of the channel state of its competitor. This is because in the former case the uncertainty of the appearance of the competitor is reduced.

Theorems 3 and 4 imply that the expected payoff of a primary is $(v-c)(1-q)$. Note that when the primaries always know each other's channel states, the competition becomes equivalent to the Bertrand competition [1] and the expected payoff is [5] $(v-c)(1-q)$ and when the primaries are constrained to select only N, the expected payoff is again $(v-c)(1-q)$ [3], [4]. Hence, our result also builds the bridge between the two extremes. Specifically, it shows that the cost c_1 or the availability of the competitor's CSI does not impact the expected payoff.

[5] It can also be obtained from (5).

IV. UNIQUENESS

Here, we show that there can not be any NE strategy profile apart from the strategy profiles described in Theorems 3 and 4.

A. Structure of the Pricing strategies

We first investigate the key structure of the NE pricing strategies (if it exists).

Note that under Y, if a primary knows that its competitor's channel is not available then it will choose v w.p. 1. We thus, investigate the structure of $F_1(\cdot)$ and $F(\cdot)$ in an NE strategy. Recall that $F_1(\cdot)$ is the pricing distribution that a primary chooses when it selects Y and knows that the channel of its competitor is available, while $F(\cdot)$ is the pricing distribution that a primary chooses when it selects N.

Theorem 5. *In an NE strategy profile, neither $F(\cdot)$ nor $F_1(\cdot)$ can have a jump at any price which is less than v. Additionally, $F_1(\cdot)$ can not have a jump at v.*

Proof. First, we show that neither $F(\cdot)$ nor $F_1(\cdot)$ can have a jump at any price which is less than v. Subsequently, we show that $F_1(\cdot)$ can not have a jump at v.

Note that a primary can only have a jump at a price if it is a best response. First, note that $F(\cdot)$ can not have a jump at a price less than or equal to c. This is because at a price less than or equal to c will fetch a negative profit, however, if the primary selects v, then it will get an expected payoff of $(v-c)(1-q)$.

Similarly, if $F_1(\cdot)$ has a jump at a price less than or equal to c, then its payoff under $F_1(\cdot)$ is at most $(c-c)-c_1=-c_1$. Note that when the channel of the competitor is unavailable, then the primary will attain the payoff of $(v-c)-c_1$. Hence, the expected payoff under Y is thus, $(v-c-c_1)(1-q)-c_1q=(v-c)(1-q)-c_1$. However, if the primary selects N and the price v which will fetch an expected profit of $(v-c)(1-q)$.

Now if either $F_1(\cdot)$ or $F(\cdot)$ has a jump at $c<x<v$, then the other primary can select a price $x-\epsilon$ and still can gain higher payoff compared to x. Thus, the other primary will not select any price in the interval $(x-\epsilon,x+\epsilon)$ as it will get a strictly higher payoff at $x-\epsilon$ compared to any price in the interval. Hence, the primary itself can gain strictly higher payoff by selecting a price at $y\in(x,x+\epsilon)$ compared to x. It contradicts the fact that either $F_1(\cdot)$ or $F(\cdot)$ will have a jump at $x<v$.

Next, we show that $F_1(\cdot)$ can not have a jump at v. Suppose $F_1(\cdot)$ has a jump at v, then the other primary will never select v with positive probability when its channel is available as it can get strictly higher payoff by selecting a price slightly less than v. Thus, at v, the primary is never going to sell its channel when the channel of other primary is available. Thus, the expected payoff that the primary will get under $F_1(\cdot)$ is $-c_1$. Again, the primary will have an incentive to deviate to select N and select the price v which will fetch a payoff of at least $(v-c)(1-q)$. ☐

The above theorem shows that if the channel of a primary is available then it can not have a jump at any price other than v.

Now, we show an important property of $F_1(\cdot)$ and $F(\cdot)$ when a primary randomizes between Y and N in an NE strategy.

Theorem 6. *Suppose that primary 1 selects Y w.p. p and N w.p. $1-p$ in an NE. Then, the upper end point of the support set of $F_1(\cdot)$ must be lower than or equal to to the lower end-point of the support set of $F(\cdot)$.*

Proof. Note from Theorem 5 that $F_1(\cdot)$ can not have a jump at v. Thus, the lower end point of $F_1(\cdot)$ can never be v. If the lower end-point of the support set of $F(\cdot)$ is v, then the statement is trivially true. So, we consider the setting where the lower end-point of the support set of $F(\cdot)$ is less than v. Suppose the statement is false. Thus, there must exist a $x<y<v$ such that x is in the support set of $F(\cdot)$ and y is in the support set of $F_1(\cdot)$. Now, suppose that the maximum expected payoff of primary 1 when it selects $F_1(\cdot)$ under Y is $\bar{p}_1$. Also let $\bar{p}_2$ be the maximum expected payoff primary 1 gets when it selects $F(\cdot)$ under N.

Since $x<v$, thus, if the channel of competitor is available, it can not have any jump at x. Hence, while choosing N, the probability of winning at x is $(1-q\phi_2(x))$ where $\phi_2(\cdot)$ is the probability that the primary 2 will select a price less than or equal to x when its channel is available. Since $x<v$ and primary 2 does not have a jump at x, thus, x is a best response to primary 1 under N. Thus,

$$(x-c)(1-q\phi_2(x))=\bar{p}_2 \tag{16}$$

Since $\bar{p}_1$ is the maximum expected payoff that primary 1 gets under $F_1(\cdot)$, thus, if primary 1 selects x under $F_1(\cdot)$, then its payoff would be

$$\begin{aligned}(x-c)(1-\phi_2(x)) &\leq \bar{p}_1\\ \frac{1-\phi_2(x)}{1-q\phi_2(x)} &\leq \frac{\bar{p}_1}{\bar{p}_2} \quad \text{from (16)}\end{aligned} \tag{17}$$

Similarly, since $y<v$, thus, primary 2 will not have a jump at y when its channel is available. Thus, primary 1's expected payoff under $F_1(\cdot)$ at the price y is

$$(y-c)(1-\phi_2(y))=\bar{p}_1 \tag{18}$$

If primary 1 selects N and the price y, then its expected payoff is

$$(y-c)(1-q\phi_2(y)) = \bar{p}_1\frac{1-q\phi_2(y)}{1-\phi_2(y)} \quad \text{from (18)}$$
$$\geq \frac{(1-q\phi_2(y))(1-\phi_2(x))}{(1-\phi_2(y))(1-q\phi_2(x))}\bar{p}_2 \text{from (17)} \tag{19}$$

Now, note that $\phi_2(y) \geq \phi_2(x)$ as $y > x$. If $\phi_2(y) = \phi_2(x)$, then the expected payoff at y must be greater than the expected payoff at x, hence, x can not be a best response at N for primary 1. However, if $\phi_2(y) > \phi_2(x)$, then the expected payoff at y at N is strictly higher than $\bar{p}_2$ by (19). Thus, this leads to a contradiction since $\bar{p}_2$ is the maximum expected payoff at N. Hence, the result follows. □

Now, we show that both $F(\cdot)$ and $F_1(\cdot)$ are contiguous. Additionally, if a primary randomizes between Y and N, then there is no "gap" between $F(\cdot)$ and $F_1(\cdot)$.

Theorem 7. *(i) In a NE strategy if a primary selects Y w.p. 1, and it selects $F_1(\cdot)$ when it knows that the channel of other primary is available, then $F_1(\cdot)$ must be contiguous and the upper end-point of $F_1(\cdot)$ must be v.*
(ii) In a NE strategy if a primary selects N w.p. 1, and if it selects $F(\cdot)$ when it knows that channel of other primary is available, then $F(\cdot)$ must be contiguous and the upper end-point of $F(\cdot)$ must be v.
(iii) In a NE strategy if the primary randomizes between Y and N, both $F_1(\cdot)$ and $F(\cdot)$ must be contiguous, there must not be any gap between the support sets of $F_1(\cdot)$ and $F(\cdot)$. Moreover, the upper-end point of $F(\cdot)$ must be v.

Proof. We only show the proof of part (i). The proof of the other parts will be similar.

Part (i): Suppose that primary 1 selects $F_1(\cdot)$ such that $F_1(x) = F_1(y)$ for some $v \geq y > x$ such that both y, x are under the support set of $F_1(\cdot)$. Since $x < v$ thus, primary 2 does not have a jump at x when its channel is available. Hence, x is a best response for primary 1 under $F_1(\cdot)$. By Theorem 6 if a primary randomizes between Y and N, then the lower end-point of $F(\cdot)$ must be greater than or equal to the lower end-point of $F_1(\cdot)$. Thus, $F(x) = F(y) = 0$. Thus, primary 2 will attain a strictly higher payoff at any value $z \in (x, y)$ compared to at x. Thus, there is an $\epsilon > 0$ where primary 2 will never select any price in the interval $[x, x+\epsilon]$, hence, x itself is not a best response for primary 1. But the above contradicts the fact that x is in the support set of $F_1(\cdot)$. Hence, the result follows. □

B. Special Property where primaries randomize

Next theorem shows that in an NE if both the primaries randomize between Y and N. Then both of them should put the same probability mass on Y (and N, resp.).

Theorem 8. *Suppose primary 1 selects Y w.p. $1 > p_1 > 0$ and N w.p. $1-p_1$. Primary 2 selects Y w.p. $1 > p_2 > 1$ and N w.p. $1-p_2$. Then, $p_1 = p_2$ in an NE strategy profile.*

Proof. Suppose that at Y, primary 1 (2, resp.) selects a price using the distribution $F_1(\cdot)$ ($\bar{F}_1(\cdot)$, resp.) when it knows that the channel of primary 2 (1, resp.) is available for sale. At N, suppose that primary 1 (2, resp.) selects a price using the distribution $F(\cdot)$ ($\bar{F}(\cdot)$, resp.).

Let L_1 ($\bar{L}_1$, resp.) and U_1 ($\bar{U}_1$, resp.) be respectively the lower and upper end-points of the support of F_1 ($\bar{F}_1$, resp.). Let L ($\bar{L}$, resp.) and U ($\bar{U}$, resp.) be the lower and upper end-point of the support of $F(\cdot)$ ($\bar{F}$, resp.) respectively. By Theorem 6 $L_1 < L$ and $\bar{L}_1 < \bar{L}$. Note also from Theorem 7 that $U_1 = L$ and $\bar{U}_1 = \bar{L}$.

First, we show that $L_1 = \bar{L}_1$. Suppose not. Without loss of generality assume that $L_1 < \bar{L}_1$. Thus, primary 2 does not select any price in the interval $(L_1, \bar{L}_1)$. Thus, the primary 1 will get a strictly higher payoff at $\bar{L}_1 - \epsilon$ for some $\epsilon > 0$ compared to L_1. Hence, primary 1 must select prices close to L_1 with probability 0 which contradicts that L_1 is the lower end-point of F_1. Thus, $L_1 = \bar{L}_1$.

By Theorem 5 L_1 can not be equal to v. Thus, $L_1 = \bar{L}_1 < v$. Thus, both L_1 and $\bar{L}_1$ are best responses to primary 1 and primary 2 respectively at Y. Since $L_1 = \bar{L}_1$, thus, the expected payoff at Y must be the same for both players. Also note that since primaries randomize between Y and N, thus, the payoffs at Y and N must be the same. Hence, the expected payoff of the primaries at N also must be the same. Thus, no primary can have a jump at v under N. This is because if a primary has a jump at N, then the other primary would get a strictly higher payoff at a price just below v which contradicts that both the primaries must have the same payoff under N. Thus, $L, \bar{L} < v$.

Now, we show that $L = \bar{L}$, towards this end, we introduce few more notations. Let $\bar{p}_1 - c$ be the maximum expected payoff of primary 1 (2, resp.) under $F_1(\cdot)$ ($\bar{F}_1(\cdot)$, resp.) and $\bar{p}_2 - c$ be the expected payoff of primary 1 (2, resp.) under $F(\cdot)$ ($\bar{F}(\cdot)$, resp.).

Suppose $L \neq \bar{L}$. Without loss of generality assume that $L > \bar{L}$. Thus, $\bar{L} < v$. Since $\bar{L}$ is the upper end-point of $\bar{F}_1(\cdot)$ and $\bar{L} < v$, thus, the expected payoff of primary 2 at $\bar{L}$ under $\bar{F}_1(\cdot)$ is $\bar{p}_1 - c$. Thus,

$$(\bar{L}-c)(1-p_1F_1(\bar{L})) = \bar{p}_1 - c \tag{20}$$

$\bar{L}$ is also a best response of primary 2 at N, thus,

$$(\bar{L}-c)(1-qp_1F_1(\bar{L})) = \bar{p}_2 - c \tag{21}$$

Since $v > L > \bar{L}$ and L is the upper end-point of $F_1(\cdot)$, thus, L is also a best response of primary 1 under Y.

$$(L-c)(1-p_2-(1-p_2)\bar{F}(L)) = \bar{p}_1 - c \tag{22}$$

Since L is the lower end point of $F(\cdot)$, thus, under N, the expected payoff of primary 1 at L is

$$(L-c)(1-qp_2-q(1-p_2)\bar{F}(L)) = \bar{p}_2 - c \tag{23}$$

Also note that since $L > \bar{L}$, thus, L is in the support of $\bar{F}(\cdot)$, thus, under N, the expected payoff to primary 2 at L is

$$(L-c)(1-qp_1) = \bar{p}_2 - c \tag{24}$$

as $F_1(L) = 1$ and $F(L) = 0$.

Thus, from (24) and (23) $p_1 = p_2 + (1-p_2)\bar{F}(L)$. Now, the expected payoff of primary 2 at L when it selects Y and the channel of primary 1 is available, is

$$\begin{aligned}(L-c)(1-p_1) &= (L-c)(1-p_2-(1-p_2)\bar{F}(L)) \\ &= \bar{p}_1 - c \quad from(22)\end{aligned} \tag{25}$$

Hence, from (20), (25), (21) and (24) that

$$\frac{1-p_1F_1(\bar{L})}{1-p_1} = \frac{1-qp_1F_1(\bar{L})}{1-qp_1} \tag{26}$$

which leads to a contradiction as neither q is not equal to 1 nor $F_1(\bar{L}) = 1$. Hence, we must have $L = \bar{L}$.

Now, at L, the expected payoff of primary 2 at Y is $(L-c)(1-p_1) = \bar{p}_1 - c$. Similarly, at $\bar{L}$, the expected payoff of primary 1 at Y is $(\bar{L}-c)(1-p_2) = \bar{p}_1 - c$. Since $L = \bar{L}$, thus, we must have $p_1 = p_2$. Hence, the result follows. □

Next, we determine the probability with which the primaries must randomize between Y and N in an NE strategy.

Observation 1. *If both the primaries randomize between Y and N, they should do it w.p. p where* $p = \dfrac{q(v-c)(1-q)-c_1}{q(v-c)(1-q)-c_1q}$.

Proof. Suppose that a primary selects its price from $F_1(\cdot)$ under Y and when it knows that the channel of other primary is available. Suppose that under $F_1(\cdot)$ the expected payoff is $\tilde{p}_1 - c$. Thus, the expected payoff of primary 1 under Y is

$$(v-c)(1-q) + q(\tilde{p}_1 - c) - c_1 \tag{27}$$

Suppose that the primary selects its price from $F(\cdot)$ under N. Since no primary has any jump at v when both the primaries randomize between Y and N and v is the upper end-point of $F(\cdot)$ by Theorem 7, thus, the expected payoff under N is $(v-c)(1-q)$. Since the primary randomizes between Y and N, thus, the expected payoff under Y and under N must be the same. Hence, we must have $c_1 = q(\tilde{p}_1 - c)$.

Suppose L be the upper end point of the support of $F_1(\cdot)$ (and thus, also the lower endpoint of $F(\cdot)$). Hence, the expected payoff at L is

$$(L-c)(1-qp) = (v-c)(1-q) \tag{28}$$

Thus, $L = (v-c)(1-q)/(1-qp) + c$. Also note that L is also a best response at $F_1(\cdot)$. Thus,

$$\begin{aligned}(L-c)(1-p) &= \frac{c_1}{q} \\ \frac{(v-c)(1-q)(1-p)}{1-qp} &= \frac{c_1}{q}\end{aligned} \tag{29}$$

Obtaining p from the above expression will give the desired result. □

C. Does there exists an NE where one player selects Y w.p. 1?

Theorem 9. *There is no NE where a primary selects Y w.p. 1 and the other primary selects Y w.p. p and N w.p. $1-p$.*

Proof. Without loss of generality, assume that primary 1 selects Y w.p. 1 and the primary 2 selects Y w.p. p and N w.p. $1-p$.

Now suppose that primary 1 selects a price using the distribution function $F_1(\cdot)$ when it knows that the channel of its competitor is available for sale. Let at Y, primary 2 selects a price using distribution function $F_2(\cdot)$ when it knows that the channel of its competitor is available for sale, and at N, it selects a price using distribution function $\bar{F}_2(\cdot)$.

Let L_1 be the lower end-point of the support of $F_1(\cdot)$ and L_2 ($\bar{L}_2$, resp.) be the lower end-point of $F_2(\cdot)$ ($\bar{F}_2$, resp.).

Note from Theorem 6 that $\bar{L}_2 > L_2$. Now, we show that $L_1 = L_2$. Suppose that $L_1 > L_2$, then, primary 2 can attain strictly higher payoff at any price close to L_1 compared to at L_2 which shows that L_2 can not be a lower end-point of F_2. By symmetry, it also follows that L_1 can not be less than L_2, hence $L_1 = L_2$. Thus, the expected payoff at Y must be equal for both the primaries.

Now, note that $F_1(\cdot)$ can not have a jump at v by Theorem 5. Note that the upper end-point of $\bar{F}_2(\cdot)$ is v by Theorem 7. Since $F_1(\cdot)$ does not have a jump at v, thus, v is a best response of primary 2 under N. Thus, the expected payoff of primary 2 under N is $(v-c)(1-q)$. Since primary 2 randomizes between N and Y, thus, the expected payoff of primary 2 is $(v-c)(1-q)$ under Y. Thus, the expected payoff of primary 1 is also $(v-c)(1-q)$.

At any $x \in [\bar{L}_2, v)$ primary 2 does not have any jump, thus, x is a best response for primary 1. Thus, at any $x \in [\bar{L}_2, v)$ the expected payoff of primary 1 is

$$
\begin{aligned}
(x-c)(1-p-(1-p)\bar{F}_2(x)) &= \tilde{p}_1 - c \\
\bar{F}_2(x) &= \frac{1}{1-p}(1-p-\frac{1}{1-p}\frac{\tilde{p}_1-c}{x-c})
\end{aligned} \tag{30}
$$

Note that under Y, the expected payoff of primary 1 is $(v-c)(1-q) + q(\tilde{p}_1 - c) - c_1$.

Now, if primary 1 selects N and a price $x \in [\bar{L}_2, v)$, then its expected payoff is

$$
\begin{aligned}
&(x-c)(1-qp-q(1-p)\bar{F}_2(x)) \\
&= (x-c)(1-qp-q(1-p-\frac{\tilde{p}_1-c}{x-c})) \\
&= (x-c)(1-q) + (\tilde{p}_1-c)q
\end{aligned} \tag{31}
$$

Thus, for any small enough $\epsilon > 0$, we have $(v-c-\epsilon)(1-q) + (\tilde{p}_1-c)q > (v-c)(1-q)(1-q) + q(\tilde{p}_1-c) - c_1$¿ Hence, primary 1 has profitable unilateral deviation. Hence, such a strategy profile can never be an NE. ☐

Note that we have already ruled out the possibility of the NE strategy profile where a primary selects Y w.p. 1 and the other selects either N or Y w.p. 1. Hence, there is no NE where a primary selects Y w.p. 1.

D. Does there exist a NE where one player selects N w.p. 1?

Theorem 10. *There is no NE where a primary selects N w.p. 1 and the other primary randomizes between Y and N.*

Proof. Without loss of generality assume that primary 1 selects N w.p. 1 and primary 2 randomizes between Y and N.

Suppose that primary 1 selects its price using $F(\cdot)$. Let L be the lower end-point of the support of $F(\cdot)$. Let $\tilde{p}_1 - c$ be the expected payoff of primary 1. Let primary 2 selects $F_2(\cdot)$ when it selects Y and it knows that the channel of primary 1 is available. Let L_2 be the lower end-point of $F_2(\cdot)$. First, note that L_1 must be equal to the L_2. Since $L_2 < v$ by Theorem **??** and $L_1 = L_2$, thus, L_2 is a best response for both primary 1 and 2. The expected payoff of primary 2 under Y when the channel of primary 1 is $L_2 - c - c_1$. Similarly, the expected payoff of primary 1 is $L_2 - c$. Thus, $\tilde{p}_1 - c = L_2 - c$. Expected payoff of primary 2 under Y is, $q(L_2-c) + (v-c)(1-q) - c_1$.

Also let L be the lower end point of $\bar{F}_2$ where $\bar{F}_2$ be the pricing strategy that primary 2 uses when it selects N. From Theorem 7 the upper end-point of the support of $F_2(\cdot)$ is also L. From Theorem7 also note that the upper end-point of $\bar{F}_2(\cdot)$ is v.

First, note that under N the expected payoff of primary 2 must be at least $(v-c)(1-q)$ as this is the payoff that primary 2 can at least get when it selects v. Now, we show that under N, the expected payoff of primary 2 must be equal to $(v-c)(1-q)$. Suppose not, i.e. primary 2 attains an expected payoff of larger than $(v-c)(1-q)$. Since the upper end-point of $\bar{F}_2$ is v, thus, primary 1 must have a jump at v. Since primary 1 has a jump at v, thus, v is a best response for primary 1. Thus, primary 1 attains an expected payoff of $(v-c)(1-q)$ under N. Thus, $\tilde{p}_1 - c = (v-c)(1-q)$. Since primary 2 is randomizing between Y and N, thus, the primary 2's expected payoff is also greater than $(v-c)(1-q)$ when it selects Y. Thus, if the primary 1 select Y and price L_2 w.p. 1 when the channel of primary 2 is available and v w.p. 1 otherwise; then it will also get an expected payoff of $q(L_2-c) + (v-c)(1-q) - c_1$ which is higher compared to $(v-c)(1-q)$. Hence, this is not possible.

Thus, the expected payoff of primary 2 must be equal to $(v-c)(1-q)$. Since primary 1 gets an expected payoff of at least of $(v-c)(1-q)$, thus, $L_2 - c \geq (v-c)(1-q)$. Since L is the upper end-point of the support of $F_2(\cdot)$ and L is also the lower end-point of the support of $\bar{F}_2$, thus,

$$\begin{aligned}(L-c)(1-F_1(L)) - c_1 &\geq (v-c)(1-q) - c_1\\ (L-c)(1-qF_1(L)) &= (v-c)(1-q)\end{aligned} \tag{32}$$

both can not be true simultaneously since $q \neq 1$. Hence, the result follows. □

V. Can a primary have an advantage of Extra Information?

Theorem 4 shows that both the primaries attain the same expected payoff irrespective of the cost of acquiring the CSI. We now investigate if only one of the primaries (primary 1, say) can acquire the CSI of its competitor whether it can gain more compared to the other primary (primary 2, say).

We first introduce some pricing distributions for $c_1 < q(v-c)(1-q)$

$$\psi_{1,Y}(x) = \begin{cases} 0, & \text{if } x < \tilde{p}\\ \dfrac{1}{qp_1}(1-\dfrac{(v-c)(1-q)}{x-c}), & \text{if } \tilde{p} \leq x \leq \tilde{p}_1\\ 1, & \text{if } x > \tilde{p}_1\end{cases}$$

$$\psi_{1,N}(x) = \begin{cases} 0, & \text{if } x < \tilde{p}_1\\ \dfrac{1}{q(1-p_1)}(1-\dfrac{(v-c)(1-q)}{x-c} - qp_1) & \text{if } \tilde{p}_1 \leq x \leq v\\ 1, & \text{if } x > v\end{cases}$$

$$\psi_2(x) = \begin{cases} 0, & \text{if } x < \tilde{p}\\ (1-\dfrac{(v-c)(1-q)}{x-c}), & \text{if } \tilde{p} \leq x \leq \tilde{p}_1\\ \dfrac{1}{q}(1-\dfrac{(v-c)(1-q)+q(v-c)(1-q)-c_1}{x-c}), & \text{if } \tilde{p}_1 \leq x < v\\ 1-\dfrac{1}{q}(1-\dfrac{(v-c)(1-q)+q(v-c)(1-q)-c_1}{v-c}), & \text{if } x = v\\ 1, & \text{if } x > v.\end{cases}$$

where

$$\tilde{p} = (v-c)(1-q)+c. \quad \tilde{p}_1 = \frac{(v-c)(1-q)}{1-qp_1} + c \tag{33}$$

and

$$p_1 = \frac{1}{q}(1-\frac{(v-c)(1-q)^2}{(v-c)(1-q)-c_1}) \tag{34}$$

Note that

$$\begin{aligned}\tilde{p}_1 - c &= \frac{(v-c)(1-q)[(v-c)(1-q)-c_1]}{(v-c)(1-q)^2}\\ &= \frac{(v-c)(1-q)-c_1}{1-q}\end{aligned} \tag{35}$$

From the expression of $\psi_2(\cdot)$, it seems that the function is not defined at $\tilde{p}_1$ as it may have different values. We, first, rule out the above possibility.

Note from (35 that

$$1-\frac{(v-c)(1-q)}{\tilde{p}_1 - c} = 1-\frac{(v-c)(1-q)^2}{(v-c)(1-q)-c_1} \tag{36}$$

Again from (35) we have–

$$\begin{aligned}&\frac{1}{q}(1-\frac{(v-c)(1-q)+q(v-c)(1-q)-c_1}{\tilde{p}_1-c})\\ &= \frac{1}{q}(1-\frac{(1-q)[(v-c)(1-q)+q(v-c)(1-q)-c_1]}{(v-c)(1-q)-c_1}\\ &= 1-\frac{(v-c)(1-q)^2}{(v-c)(1-q)-c_1}\end{aligned} \tag{37}$$

Hence, $\psi_2(\cdot)$ is a valid function.

Also note that when $c_1 < q(v-c)(1-q)$, $\psi_2(\cdot)$ has a jump at v. Now, we are ready to state the main result of this section.

Theorem 11. *Consider the following strategy profile: Primary* 1 *selects* Y *w.p.* p_1 *and* N *w.p.* $1-p_1$ *(*p_1 *is given in (34)) and primary* 2 *selects* N*. While selecting* Y*, if the channel of primary* 2 *is available, then primary* 1 *selects its price according to* $\psi_{1,Y}(\cdot)$*, otherwise it selects* v *w.p.* 1*. While selecting* N*, primary* 1 *selects its price according to* $\psi_{1,N}(\cdot)$*. Primary* 2 *selects its price according to* $\psi_2(\cdot)$*.*

The above strategy profile is an NE when $c_1 < q(v-c)(1-q)$*. The expected payoff that primary* 1 *attains is* $(v-c)(1-q)+q(v-c)(1-q)-c_1$ *and the expected payoff of primary* 2 *is* $(v-c)(1-q)$*.*

Proof. First, we show that there is no profitable deviation for primary 1 (Case I), subsequently, we show that there is also no profitable deviation for primary 2 (Case II).

Case I: In the first step (i), we show that primary 1 can attain a maximum expected payoff of $(v-c)(1-q)+q(v-c)(1-q)-c_1$ under Y. Next in step (ii), we show that primary 1 can attain a maximum expected payoff of $(v-c)(1-q)+q(v-c)(1-q)-c_1$ under N. Finally in step (iii), we show that primary 1 attains the maximum expected payoff following the strategy which will show that primary 1 does not have any profitable unilateral deviation.

(i) Suppose that the channel of primary 2 is available. At any x such that $\tilde{p}_c \leq x \leq \tilde{p}_1$ the primary 1 gets under Y is

$$(x-c)(1-\psi_2(x)) = (v-c)(1-q) \tag{38}$$

Now, at any $v > x \geq \tilde{p}_1$, the expected payoff of primary 1 in this setting is

$$\begin{aligned}
&(x-c)(1-\psi_2(x))\\
&=(x-c)(1-\frac{1}{q}(1-\frac{(v-c)(1-q)+q(v-c)(1-q)-c_1}{x-c}))\\
&=(x-c)(1-\frac{1}{q})+\frac{(v-c)(1-q)+q(v-c)(1-q)-c_1}{q}
\end{aligned} \tag{39}$$

Since $1/q > 1$, thus, the supremum is attained at $x = \tilde{p}_1$. Evaluating the above expression at $x = \tilde{p}_1$ we obtain

$$(v-c)(1-q) \tag{40}$$

Hence, when the channel of primary 2 is available, then the maximum expected payoff that primary 1 can attain at Y is $(v-c)(1-q)-c_1$ and it is attained at any price in the interval $[\tilde{p}, \tilde{p}_1]$.

Now, when the channel of primary 2 is unavailable, the expected payoff of primary 1 is $(v-c)-c_1$. Hence, the maximum expected payoff that primary 1 attains in Y is

$$\begin{aligned}
&(v-c-c_1)(1-q)+q[(v-c)(1-q)-c_1]\\
&=(v-c)(1-q)+q(v-c)(1-q)-c_1
\end{aligned} \tag{41}$$

(ii) Now when primary 1 selects N, then at any x such that $\tilde{p}_1 \leq x < v$, the expected payoff of primary 1 is

$$(x-c)(1-q\psi_2(x)) = (v-c)(1-q)+q(v-c)(1-q)-c_1 \tag{42}$$

Now, at any x such that $\tilde{p} \leq x \leq \tilde{p}_1$, the expected payoff of primary 1 under N is

$$\begin{aligned}
(x-c)(1-q\psi_2(x)) &= (x-c)(1-q(1-\frac{(v-c)(1-q)}{x-c}))\\
&=(x-c)(1-q)+q(v-c)(1-q)
\end{aligned} \tag{43}$$

The supremum is attained at $x = \tilde{p}_1$. Putting the value of $\tilde{p}_1$ we obtain

$$(v-c)(1-q)-c_1+q(v-c)(1-q) \tag{44}$$

Thus, primary 1 can attain at most an expected payoff of $(v-c)(1-q)+q(v-c)(1-q)-c_1$. The maximum expected payoff is attained at any price in the interval $[\tilde{p}_1, v)$.

(iii) Hence, we show that the primary 1 can attain an expected payoff of $(v-c)(1-q)+q(v-c)(1-q)-c_1$ under either Y or N. Thus, any randomization between Y and N will also give an expected payoff of $(v-c)(1-q)+q(v-c)(1-q)-c_1$. Now, under the strategy profile the expected payoff is also $(v-c)(1-q)+q(v-c)(1-q)-c_1$, hence, primary 1 does not have any profitable deviation.

Case II: Now, we show that primary 2 does not have any profitable deviation. Towards this end, we first show that (i) at any price in the interval $[\tilde{p}, \tilde{p}_1]$ will give an expected payoff of $(v-c)(1-q)$ to primary 2, subsequently, we show that at any price in the interval $[\tilde{p}_1, \tilde{p}_2]$ will also provide an expected payoff of $(v-c)(1-q)$ to primary 2. Finally in step (iii), we show that at any price $x < \tilde{p}$ will give a strictly lower payoff compared to $(v-c)(1-q)$. Note that now, primary 2 can not select Y, thus, the above will prove that primary 2 does not have any profitable deviation.

(i) Note that when $x \in [\tilde{p}, \tilde{p}_1]$ primary 1 can select a price less than x only when the channel of primary 1 is available and primary 1 selects Y, thus, at any x such that $x \in [\tilde{p}, \tilde{p}_1]$, the expected payoff of primary 2 is

$$(x - c)(1 - qp_1\psi_{1,Y}(x)) = (v - c)(1 - q) \tag{45}$$

(ii) Similarly, when $x \in [\tilde{p}_1, v]$, then the expected payoff of primary 2 is

$$(x - c)(1 - qp_1 - q(1 - p_1)\psi_{1,N}(x)) = (v - c)(1 - q) \tag{46}$$

(iii) At any price less than $\tilde{p}$ will fetch a payoff which is strictly less than $\tilde{p} - c$. However, $\tilde{p} - c = (v - c)(1 - q)$. Thus, the expected payoff of primary 2 is strictly less than $(v - c)(1 - q)$ at any price less than $\tilde{p}$. Hence, primary 2 also does not have any profitable deviation. $\square$

Note that when $c_1 < q(v - c)(1 - q)$, the payoff of primary 1 is higher compared to the primary 2. Intuitively, when c_1 is low, then primary 1 takes advantage of the acquired CSI and gains more compared to primary 2 which can not acquire the CSI of primary 1. On the other hand if primary 2 would can acquire the CSI of primary 1, then it would also do that which will bring down the expected payoff of primary 1 as we have seen in Theorem 4.

VI. FUTURE WORK

In this paper we consider a simple model in which primaries have the option of acquiring CSI of their competitors. There are many ways this model could be enhanced such as considering more than two primaries (and, multiple secondaries) as well as allowing the channels of the primaries to have different availability probabilities.

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