Optimal Propagation of Security Patches in Mobile Wireless Networks

[Extended Abstract] *

M.H.R. Khouzani
University of Pennsylvania
khouzani@seas.upenn.edu

Saswati Sarkar
University of Pennsylvania
swati@seas.upenn.edu

Eitan Altman
INRIA, France
altman@sophia.inria.fr

ABSTRACT

Reliable security measures against outbreaks of malware is imperative to enable large scale proliferation of wireless technologies. Immunization and healing of the nodes through dissemination of security patches can counter the spread of a malware upon an epidemic outbreak. The distribution of patches however burdens the bandwidth which is scarce in wireless networks. The trade-offs between security risks and resource consumption can be attained by activating at any given time only fractions of dispatchers and dynamically selecting their packet transmission rates. We formulate the above trade-offs as an optimal control problem that seek to minimize the aggregate network costs that depend on security risks and resource consumed by the countermeasures. Using Pontryagin's maximum principle, we prove that the dynamic control strategies have simple structures. When the resource consumption cost is concave, optimal strategy is to use maximum resources for distribution of patches until a threshold time, upon which, the patching should halt. When the resource consumption cost is convex, the above transition is strict but continuous.

Categories and Subject Descriptors: C.2.1 [Network Architecture and Design]: Wireless communication, K.6.5 [Security and Protection (D.4.6, K.4.2)]

General Terms: Security, Theory.

Keywords: Dynamic Patching, Optimal Control, Security-Performance trade-off.

System Model

A **susceptible** node is a mobile wireless device which is not contaminated by the worm, yet is vulnerable to infection. A node is **infective** if it is contaminated by the worm. An infective spreads the worm to a susceptible while transmitting data or control messages to it whenever the two are in *contact*, that is, the infective detects the presence of the susceptible in its transmission range. A functional node that is immune to the worm is referred to as **recovered**. A fraction R_0 of mobile nodes, referred to as *dispatchers*, are preloaded with security patches. The dispatchers are immune to infection themselves, and can transmit the patches to the susceptible and infective nodes and *immunize* the susceptibles and *heal* the infectives to the recovered state. Once a node receives a security patch, it can retransmit it upon contact with other nodes. Thus, all recovered nodes become dispatchers and hence, the fraction of dispatchers grows to $R(t)$ at time t.

Let the fraction of activated dispatchers at time t be $\varepsilon(t)$, and each scans the media at rate $u(t)$ (i.e., $u(t)$ is the rate of transmission of scanning packets). Upon a contact between an activated dispatcher and another node, the security patch is transmitted from the dispatcher to the receiver node. If the receptor is a susceptible node, it installs the security patch, is subsequently immunized, and its state changes to recovered. If however the receptor is an infective, the patch may fail to heal it, or, the worm may prevent its installation. We capture the above possibility, by introducing a coefficient $0 \leq \pi \leq 1$. Let $\vartheta(t) := u(t)\varepsilon(t)$. One can develop the following epidemic model for the spread of the malware [2]:

$$\dot{S}(t) = -\beta I(t)S(t) - \beta\vartheta(t)R(t)S(t) \tag{1a}$$

$$\dot{I}(t) = \beta I(t)S(t) - \pi\beta\vartheta(t)R(t)I(t) \tag{1b}$$

$$\dot{R}(t) = \beta\vartheta(t)R(t)S(t) + \pi\beta\vartheta(t)R(t)I(t) \tag{1c}$$

with initial constraints:

$$I(0) = I_0, \quad R(0) = R_0, \quad S(0) = 1 - I_0 - R_0, \tag{2}$$

where $0 < I_0, R_0,\ I_0 + R_0 < 1$. Also,

$$0 \leq S(t), I(t), R(t), \quad S(t) + I(t) + R(t) = 1. \tag{3}$$

Coefficient $\beta > 0$ in (1) depends on the nodes densities, mobility pattern and average relative velocities of the nodes. It can be readily shown [2] that for any admissible ϑ, the state constraints (3) are satisfied and hence can be ignored. An admissible control should satisfy $0 \leq \vartheta(t) \leq \vartheta_{\max}$ for all $t \in [0, T]$. With appropriate scaling by choice of β, we can assume $\vartheta_{\max} = 1$. Thus,

$$0 \leq \vartheta(t) \leq 1 \text{ for all } t \in [0, T]. \tag{4}$$

The worm may inflict damage over time by attempting to (i) eavesdrop and analyze and/or (ii) alter or destroy the traffic that is generated or relayed by the infected hosts. At each time t, the network incurs a cost of $f(I(t))$ due to the presence of the infectives, where $f(.)$ is a general non-decreasing differentiable function of I, such that $f(0) = 0$ and $f(I) > 0$ for $I > 0$. The resource consumption cost incurred at time t due to the bandwidth consumed in media

*A full version of this paper is available as *Dispatch then Stop: Optimal Dissemination of Security Patches in Mobile Wireless Networks* at `http://www.seas.upenn.edu/ swati/publication.htm`

SIGMETRICS'10, June 14–18, 2010, New York, New York, USA.
ACM 978-1-4503-0038-4/10/06.

scanning by the dispatchers is $h(R(t)\vartheta(t))$. $h(.)$ is a twice-differentiable and increasing function such that $h(0) = 0$ and $h(x) > 0$ when $x > 0$. Thus, the aggregate network cost is:

$$J(\vartheta) = \int_0^T [f(I(t)) + h(R(t)\vartheta(t))]\, dt, \qquad (5)$$

Note that the assumptions on $f(.), h(.)$ are mild and natural and a large class of functions satisfy them.

Optimal Dynamic Patching

The network seeks to find an admissible $\vartheta(t)$ to minimize the cost function in (5) for the state dynamics (1) and the initial state values (2). We apply Pontryagin's Maximum Principle. Define the Hamiltonian H and costate functions λ_1 to λ_3 as:

$$H = f(I) + h(R\vartheta) + (\lambda_2 - \lambda_1)\beta IS - (\lambda_1 - \lambda_3)\beta\vartheta RS - (\lambda_2 - \lambda_3)\pi\beta\vartheta RI \quad (6)$$

$$\begin{aligned}
\dot{\lambda}_1 &= -\frac{\partial H}{\partial S} = -(\lambda_2 - \lambda_1)\beta I + (\lambda_1 - \lambda_3)\beta\vartheta R \\
\dot{\lambda}_2 &= -\frac{\partial H}{\partial I} = -f'(I) - (\lambda_2 - \lambda_1)\beta S + (\lambda_2 - \lambda_3)\pi\beta_1\vartheta R \\
\dot{\lambda}_3 &= -\frac{\partial H}{\partial R} = (\lambda_1 - \lambda_3)\beta\vartheta S + (\lambda_2 - \lambda_3)\pi\beta\vartheta I - \vartheta h'(R\vartheta)
\end{aligned} \quad (7)$$

and the transversality conditions as:

$$\lambda_1(T) = \lambda_2(T) = \lambda_3(T) = 0. \qquad (8)$$

Then according to Pontryagin's Maximum Principle ([1, P. 109, Theorem 3.14]), there exist continuous and piecewise continuously differentiable co-state functions λ_1, λ_2, λ_3, that (i) satisfy (8), and (ii) at every $t \in [0 \ldots T]$ where ϑ is continuous, satisfy (7), and together with the optimal trajectory S, I, R satisfy

$$\vartheta \in \arg \min_{0 \le \underline{\vartheta} \le 1} H(\vec{\lambda}, (S, I, R), \underline{\vartheta}). \qquad (9)$$

Structure of optimal dynamic patching $\vartheta(t)$:

THEOREM 1. *An optimal immunization rate function $\vartheta(.)$ has the following structure:*

1. *if $h(.)$ is concave, $\vartheta(t) = 1$ for $0 < t < t_1$ and $\vartheta(t) = 0$ for $t_1 < t < T$.*
2. *if $h(.)$ is* strictly *convex, $\exists\ t_0, t_1,\ 0 \le t_0 \le t_1 \le T$: (1) $\vartheta(t) = 1$ on $0 < t \le t_0$; (2) $\vartheta(t)$ strictly and continually decreases on (t_0, t_1); (3) $\vartheta(t) = 0$ on $t_1 \le t \le T$.*

In what follows, we outline the proof of the above theorem for the concave case.

PROOF. We will use the following key properties of the co-state functions, whose proof is omitted due to space limit.

LEM. 1. *For all $0 \le t < T$, we have $(\lambda_2 - \lambda_1) > 0$, $(\lambda_1 - \lambda_3) > 0$ and $\lambda_3 \le 0$.*

Now, define $\varphi := (\lambda_1 - \lambda_3)\beta RS + (\lambda_2 - \lambda_3)\pi\beta RI$, which is a continuous function of time, and from (8), $\varphi(T) = 0$. The Hamiltonian in (6) can be rewritten as follows:

$$H = f(I) + (\lambda_2 - \lambda_1)\beta IS + h(R\vartheta) - \varphi\vartheta. \qquad (10)$$

From (9), for each admissible control $\underline{\vartheta}$, and $\forall t \in [0, T]$, $h(R(t)\vartheta(t)) - \varphi(t)\vartheta(t) \le h(R(t)\underline{\vartheta}(t)) - \varphi(t)\underline{\vartheta}(t)$, thus

$$\vartheta(t) \in \arg \min_{x \in [0,1]} h(R(t)x) - \varphi(t)x. \qquad (11)$$

Also, since $\underline{\vartheta} = 0$ is an admissible control, $[h(R\vartheta) - \varphi\vartheta] \le 0$ at all t.

When $h(.)$ is concave (i.e., $h'' \le 0$), a minima in (11) is either at $x = 0$ or $x = 1$ at each time t, and this minima is unique unless $h(R) - \varphi = 0$. Then,

$$\vartheta = \begin{cases} 0, & \varphi - h(R) < 0 \\ 1, & \varphi - h(R) > 0 \end{cases} \qquad (12)$$

For the case of $h'' < 0$, whenever $h(R) - \varphi = 0$, $\vartheta \in \{0, 1\}$. Let $\psi(t) := \varphi(t) - h(R(t))$. Because $\varphi(T) = 0$ and from (12) and since $h(R(T)) > 0$, $\psi < 0$ over a subinterval that extends to T. If this sub-interval starts from $t = 0$, the theorem follows from (12) with $t_1 = 0$. Else, from the continuity of ψ, and the Intermediate Value Theorem, $\psi(t) = 0$ for some $t \in [0, T)$. But there can be at most one such t, since (as we will show next) ψ strictly decreases with increasing t. Hence, $\psi(t) > 0$ for $t \in [0, t_1)$, and $\psi(t) < 0$ for $t \in (t_1, T]$. The theorem now follows from (12).

Since ϑ is piecewise continuous and φ, h, R are continuous functions of time, it suffices to show that $\dot{\psi}$ is negative at any $t \in [0, T)$ at which ϑ is continuous. Referring to the definition of ψ, at any such t:

$$\begin{aligned}
\dot{\psi} &= \dot{\varphi} - h'(R)\dot{R} \\
&= (\dot{\lambda}_1 - \dot{\lambda}_3)\beta RS + (\dot{\lambda}_2 - \dot{\lambda}_3)\pi\beta RI + (\lambda_1 - \lambda_3)\beta\dot{R}S \\
&\quad + (\lambda_2 - \lambda_3)\pi\beta\dot{R}I + (\lambda_1 - \lambda_3)\beta R\dot{S} \\
&\quad + (\lambda_2 - \lambda_3)\pi\beta R\dot{I} - h'(R)\dot{R}
\end{aligned}$$

which after replacing from (1) and (7) and simplification yields:

$$\begin{aligned}
&= -\beta^2(1 - \pi)RIS(\lambda_1 - \lambda_3) - \beta^2 RIS(\lambda_2 - \lambda_1) \\
&\quad - \pi\beta f'(I)RI - \dot{R}(h'(R) - h'(R\vartheta))
\end{aligned} \quad (13)$$

We only need to show that the expression in (13) is negative at each $t \in [0, T)$. Note that $\dot{R}(h'(R) - h'(R\vartheta)) \equiv 0$. This follows readily for $h'' \equiv 0$ as then $h'(R) - h'(R\vartheta) \equiv 0$ for any value of ϑ. When $h'' < 0$, as we argued in (12) and after, $\vartheta \in \{0, 1\}$; now for $\vartheta = 1$, $h'(R) - h'(R\vartheta) = 0$ and for $\vartheta = 0$, $\dot{R} = 0$. The negativity follows from positivity of the states (S, I, R) and lem. 1. □

Acknowledgement

The contributions of M. H. R. Khouzani and S. Sarkar have been supported by NSF grants 0621782, 0721308, 0914955, 0915203, 0915697.

1. REFERENCES

[1] D. Grass, J. Caulkins, G. Feichtinger, G. Tragler, and D. Behrens. *Optimal control of nonlinear processes: with applications in drugs, corruption, and terror.* Springer Verlag, 2008.

[2] M. H. R. Khouzani, S. Sarkar, and E. Altman. Dispatch then Stop: Optimal Dissemination of Security Patches in Mobile Wireless Networks. In *http://www.seas.upenn.edu/ swati/publication.htm.*