# A Hierarchical Spatial Game over Licenced Resources

Gaurav S. Kasbekar, Eitan Altman and Saswati Sarkar

***Abstract*— We consider a scenario in which a regulator owns the spectrum in a region. Two service providers lease spectrum from the regulator and set up a base station each to serve mobile subscribers. This leads to a hierarchical game with players at two levels- the mobile subscribers at level 1 and the service providers at level 2. In the game at level 1, each mobile subscriber chooses a service provider to subscribe to and whether to be a primary subscriber of its service provider and receive high-priority service or to be a secondary subscriber and receive low-priority service. In the game at level 2, each service provider chooses the quantity of spectrum to lease from the regulator and the rates at which to charge its mobile subscribers for the throughput they receive. We first analyze the game at level 1 for different models and in each case show the existence of a threshold-type Wardrop equilibrium, in which there exist thresholds that divide the region into sets of mobile subscribers that make the same decision at equilibrium. Next, assuming that the mobile subscribers act so as to give rise to the equilibrium found above, we analyze the game at level 2 and show the existence of a Nash equilibrium.**

## I. Introduction

We consider a scenario in which a regulator owns the spectrum in a certain region. A number of service providers lease spectrum from the regulator. Each service provider sets up a base station to serve mobile subscribers in the region. We examine the interactions between the regulator, service providers and mobile subscribers using a *hierarchical game*. In this hierarchical game, there are games at two levels: the game at level 1 or game 1 in which the mobile subscribers are players and the game at level 2 or game 2 in which the service providers are players.

In game 1, each mobile subscriber chooses the service provider to receive service from and whether to be a *primary user* of its service provider and receive high-priority service or to be a *secondary user* and receive low-priority service. Owing to spectrum limitations, the rate allocated to each mobile subscriber depends on the sets of mobile subscribers that subscribe to different service providers as primary and as secondary users.

Game 2 investigates the interactions among service providers. Each service provider $i$ chooses the quantity $\beta_i$ of spectrum to lease from the regulator. It also chooses at what rate to charge its mobile subscribers for the throughput they receive. The quantity $\beta_i$ chosen by service provider $i$ determines its leasing cost. Now, since there are multiple service providers competing for the same set of mobile subscribers, if a service provider $i$ charges a very high rate to its subscribers or offers poor quality of service (which occurs if it chooses a small $\beta_i$), then only a few mobile subscribers would choose service provider $i$. On the other hand, if it charges a very low rate to its subscribers or chooses a very high $\beta_i$, then its revenues from the subscribers would be low or its leasing cost would be high respectively.

We analyze this setup as a *hierarchical game*. That is, we analyze the games at levels 1 and 2 in that order and find an equilibrium in each case. In particular, first, we fix the amount of spectrum $\beta_i$ purchased by each service provider $i$ and the rate at which the service providers charge the mobile subscribers. For these fixed quantities, we look for a Wardrop equilibrium in game 1. Next, assuming that for a given set of values of $\beta_i$ and usage-based charges, the mobile subscribers will act so as to give rise to the Wardrop equilibrium found above, we analyze game 2 and look for a Nash equilibrium.

### A. Related Work

Hotelling [5] introduced hierarchical games in which competing firms choose their locations and the prices of goods, which determines an equilibrium allocation of customers to the firms. Mazalov and Sakaguchi [4] analyzed a hierarchical game in which two firms have customers in a city. For fixed locations of the firms in the city, a customer chooses the firm from whom to purchase goods based on the prices of the goods set by the firms and its distances from the firms. An equilibrium allocation of the customers to the two firms is obtained. Next, equilibrium locations of the two firms are found that maximize their gains in the induced equilibrium of customer allocations.

A recent paper that is closely related to our paper is by Altman *et. al* [6], which considers a hierarchical game with players at two levels– base stations and mobile subscribers. For given locations of the base stations, each mobile chooses the base station to associate with so as to maximize its Signal to Interference and Noise Ratio (SINR). Then, the locations of base stations are determined so as to maximize the revenues from the associated mobiles.

One of the aspects in which our work differs from [6] is that each base station chooses the amount of spectrum to purchase from the regulator, whereas the base stations in [6] do not make this choice. On the other hand, locations of base stations are fixed in our paper, unlike [6].

### B. Outline of Paper

Throughout the paper, we consider the case in which two base stations serve mobile subscribers located on a straight line. We describe the system model in Section II.

G. Kasbekar and S. Sarkar are with the Department of Electrical and Systems Engineering, University of Pennsylvania, Philadelphia, PA, USA. E. Altman is with INRIA, Sophia-Antipolis, France. Their email addresses are: kgaurav@seas.upenn.edu, Eitan.Altman@sophia.inria.fr and swati@seas.upenn.edu.

Sections III and IV consider the game 1 described above. In Section III, we consider a scenario with two fixed populations of mobile subscribers of the two base stations. Each mobile can choose whether to be a primary or secondary mobile of its service provider. In Section III-A, with rate-fair access (see Section II) for the primary mobiles, we show that a Wardrop equilibrium exists and has a threshold-type structure in which, in the cell of each base station, mobile subscribers on one side of the threshold choose to be primary and on the other side choose to be secondary. In Section III-B, we repeat the process in Section III-A for the case of time-fair access (see Section II) for primaries.

In Section IV, we analyze the case of migration in which a mobile can choose the service provider that it wants to join. In Section IV-A, we consider the simple case in which each mobile has two choices– to be primary of service provider 1 or to be primary of service provider 2. We show that a threshold-type Wardrop equilibrium exists in which mobiles on the left of the threshold become primaries of service provider 1 and mobiles on the right become primaries of service provider 2. In Section IV-B, we analyze the general case in which each mobile can choose to be a primary or secondary of service provider 1 or a primary or secondary of service provider 2. We show that a threshold-type Wardrop equilibrium exists in which there are three thresholds which divide the set of mobile subscribers into secondaries of service provider 1, primaries of service provider 1, primaries of service provider 2 and secondaries of service provider 2.

Section V considers game 2. In Section V-A, we formulate game 2 in a general scenario, which includes as special cases the level 2 games corresponding to all the previous sections. In Section V-B, we find Nash equilibria for the simple case of migration in Section IV-A.

## II. The model

Consider two neighboring cells on the line (that coincides with the horizontal axis $x$) of length $L_1$ and $L_2$ respectively. The cells contain a large number of mobiles continuously distributed with a distribution of $\lambda_1(x)$ and $\lambda_2(x)$ mobiles per meter, respectively. Let $|\lambda_i| := \int_{C_i} \lambda_i(x)dx$. Mobiles in cell $C_1$ and $C_2$ are subscribed to service provider 1 and 2, respectively, and are served by the Base Stations (BS) located at point $(y, R)$ (BS 1) and $(z, R)$ (BS 2) respectively (see Fig. 1).

BS $i$ is assumed to purchase prioritized access for a fraction $\beta_i$ of the time from the regulator. During that fraction of time it is guaranteed to have no interference from the other BS. Mobiles in that cell that communicate with the BS during that time are called primary. The rest of the mobiles are called secondary. In each cell, secondary users communicate during the remaining fraction $1 - \beta_1 - \beta_2$ of the time.

Consider the downlink and assume that whenever BS $i$ transmits, it uses power $P_i$. We assume that at each time BS $i$ transmits with all its power to a single mobile.

The power received at a mobile located at point $(x, 0)$ from BS $i$ is given by $P_i h_i(x)$ where $h_i(x)$ is the gain. We

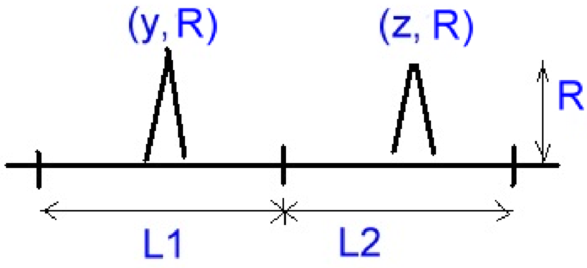


Fig. 1. The cells and base stations

shall assume that it corresponds to the path loss given by

$$h_1(x) = (R^2+(x-y)^2)^{-\xi/2}, \quad h_2(x) = (R^2+(x-z)^2)^{-\xi/2}$$

where $\xi$ is the path loss exponent [7].

The SINR (Signal to Interference and Noise Ratio) at $x \in C_1$ experienced by the secondary users is given by:

$$SINR_1^s(x) = \frac{P_1 h_1(x)}{\sigma^2 + P_2 h_2(x)}. \quad (1)$$

where $\sigma^2$ is the noise power and that experienced by the primary users at $x \in C_1$ is given by:

$$SINR_1^p(x) = \frac{P_1 h_1(x)}{\sigma^2}.$$

Similar expressions hold for primary and secondary users at $x \in C_2$.

Let $\mu_i(x)$ be the density of mobiles at $x \in C_i$ that are primary. We have of course $0 \le \mu_i(x) \le \lambda_i(x)$ for all $x \in C_i$. Let $|\mu_i| := \int_{C_i} \mu_i(x)dx$.

We assume that the instantaneous throughput of both primary and secondary mobiles is given by the following expression, which is based on Shannon's capacity theorem [2]:

$$\theta_i(x) = \log(1 + SINR_i(x))$$

**Rate of secondary mobiles** Assume that each BS devotes an equal fraction of time for transmission to each of its secondary mobile subscribers. We call this the time fair allocation policy.

The number of secondary users in $C_i$ is $|\lambda_i| - |\mu_i|$ and the fraction of time available for their transmission is $1-\beta_1-\beta_2$. Thus the average throughput of a secondary user located at $x \in C_i$ is:

$$\overline{\Theta}_i(x) = \frac{1 - \beta_1 - \beta_2}{|\lambda_i| - |\mu_i|} \log(1 + SINR_i^s(x)) \quad (2)$$

**Rate of primary mobiles** We shall consider two policies for serving primary users: a rate fair and a time fair. Under the time fair policy, each BS devotes an equal fraction of time for transmission to each of its primary mobile subscribers. In the rate fair policy, the fraction of time allotted to a primary mobile is set in a way that the average transmission rate to each primary mobile is the same (within each cell).

Consider the rate fair policy for the primary users. Let $r_i$ be the fixed rate of primary users in cell $C_i$. The fraction of time that a primary mobile at $x \in C_i$ receives positive throughput is

$$\frac{r_i}{\log(1 + SINR_i^p(x))}$$

$r_i$ is then the solution to

$$\beta_i = \int_{C_i} \frac{r_i}{\log(1 + SINR_i^p(x))}\mu_i(x)dx$$

and is thus given by

$$r_i = \beta_i \left( \int_{C_i} \frac{1}{\log(1 + SINR_i^p(x))}\mu_i(x)dx \right)^{-1} \quad (3)$$

## III. Decentralized choice of class

In this section, we assume that each mobile decides whether to become a primary or secondary subscriber of its service provider.

Assume that there is a cost per throughput unit of $\gamma_p$ and $\gamma_s$ for the primary and secondary users. We assume that $0 < \gamma_i < 1$, $i \in \{p, s\}$.

### A. Rate-Fair Policy for Primaries

In this subsection, we assume that the rate-fair policy is used for primary users. The utilities $U_i^p(x)$ and $U_i^s(x)$ for a mobile at $x \in C_i$ of choosing to be primary or secondary respectively, are then given by

$$U_i^p(x) = r_i(1 - \gamma_p), \quad (4)$$

$$U_i^s(x) = \frac{1 - \beta_1 - \beta_2}{|\lambda_i| - |\mu_i|} \log(1 + SINR_i^s(x))(1 - \gamma_s) \quad (5)$$

We seek a *Wardrop equilibrium* [3] in the game in which each mobile chooses whether to be a primary or secondary mobile. A Wardrop equilibrium is the analog of a Nash equilibrium for the case of a large number of small players, as is the case with mobiles in our setup. Now, note that $U_i^p(x)$ and $U_i^s(x)$ are functions of the spatial decision pattern $\{\mu_i(x) : x \in C_i\}$. The pattern $\{\mu_i(x) : x \in C_i\}$ is a Wardrop equilibrium if at each point $x$:

$$\begin{array}{ll} \mu_i(x) = 0 & \text{if } U_i^p(x) < U_i^s(x) \\ \mu_i(x) = \lambda_i(x) & \text{if } U_i^p(x) > U_i^s(x) \\ 0 < \mu_i(x) < \lambda_i(x) & \text{only if } U_i^p(x) = U_i^s(x) \end{array} \quad (6)$$

We now characterize the structure of the Wardrop equilibrium. Assume that $\mu_1(.)$ is an equilibrium. Then by (4), (5) and (6):

$$\mu_1(x) = \begin{cases} 0 & \text{if } SINR_1^s(x) > c, \\ \lambda_1(x) & \text{if } SINR_1^s(x) < c \end{cases}$$

where

$$c = c(\mu_1) = \exp(\frac{r_1(1 - \gamma_p)(|\lambda_1| - |\mu_1|)}{(1 - \gamma_s)(1 - \beta_1 - \beta_2)} - 1$$

From (1):

$$\mu_1(x) = \begin{cases} 0 & \text{if } W(x) > 0 \\ \lambda_1(x) & \text{if } W(x) < 0 \end{cases} \quad (7)$$

where

$$\begin{aligned} W(x) &= P_1(R^2 + (x - y)^2)^{-\xi/2} \\ &\quad -c(\sigma^2 + P_2(R^2 + (x - z)^2)^{-\xi/2}) \end{aligned}$$

*Lemma 3.1:* Assume that for some $x \in C_1$ such that $y < x < z$, $\mu_1(x) = \lambda_1(x)$. Then $\mu_1(x') = \lambda_1(x')$ for all $x' \in [x, 2z - x] \cap C_1$.

*Proof:* The term $(R^2 + (x - y)^2)^{-\xi/2}$ decreases in $x$ for $x > y$. The term $-c(\sigma^2 + (R^2 + (x - z)^2)^{-\xi/2})$ decreases in $x$ at $x < z$ so that

$$\begin{aligned} &-c(\sigma^2 + (R^2 + (x - z)^2)^{-\xi/2}) \\ &> -c(\sigma^2 + (R^2 + (x' - z)^2)^{-\xi/2}) \end{aligned} \quad (8)$$

for $x < x' < z$. Finally, for $z < x' < 2z - x$ we set $s = 2z - x'$ and we note that $x < s < z$ and that

$$-c(\sigma^2 + (R^2 + (x' - z)^2)^{-\xi/2}) = -c(\sigma^2 + (R^2 + (s - z)^2)^{-\xi/2})$$

and hence (8) holds for $x < x' < 2z - x$. ■

Similarly we get

*Lemma 3.2:* Assume that for some $x \in C_1$ such that $y < x < z$, $\mu_1(x) = 0$. Then $\mu_1(x') = 0$ for all $x' \in [2y - x, x] \cap C_1$.

*Lemma 3.3:* Assume that $\mu_1(.)$ is an equilibrium. Let $\zeta := \left( \frac{c(\mu_1)\sigma^2}{P_1} \right)^{-1/\xi}$. Then for all $x \notin [y - \zeta, y + \zeta]$ we have $\mu_1(x) = \lambda_1(x)$.

*Proof:* $x \notin [y - \zeta, y + \zeta]$ implies that:

$$\begin{aligned} (x - y)^2 &> \zeta^2 \\ &\geq \zeta^2 - R^2 \\ &= \left( \frac{c(\mu_1)\sigma^2}{P_1} \right)^{-2/\xi} - R^2 \end{aligned}$$

Hence,

$$\begin{aligned} &P_1[(x - y)^2 + R^2]^{-\xi/2} \\ &< c(\mu_1)\sigma^2 \\ &\leq c(\mu_1)[\sigma^2 + P_2(R^2 + (x - z)^2)^{-\xi/2}] \end{aligned}$$

That is, $W(x) < 0$ and hence $\mu_1(x) = \lambda_1(x)$ by (7). ■

Note that if BS 1 is located at $y \leq \inf(x \in C_1)$ then for $x \in C_1$, $2y - x \leq \inf(x \in C_1)$ and hence Lemma 3.2 implies that $\mu_1(x)$ has a threshold structure: it either equals $\lambda_1(x)$ for all $x \in C_1$ or it is 0 everywhere or there is a threshold $t_1$ such that for $x < t_1$, $\mu_1(x) = 0$ and for $x > t_1$, $\mu_1(x) = \lambda_1(x)$ for $x \in C_1$.

Similar results hold for cell $C_2$. In the next subsection we shall show that the same structure is obtained with the time fair policy for primaries as long as we are in the low SINR regime.

### B. Time Fair Policy for Primaries

We now assume that each base station serves all its primaries for equal amount of time. Thus, the average throughput at $x$ for a primary of base station $i$ is $\beta_i/|\mu_i| \log(1 + S_i(x)/\sigma^2)$ and for a secondary of base station $i$ is $(1 - \beta_1 - \beta_2)/(|\lambda_i| - |\mu_i|) \log(1 + S_i(x)/I_i(x))$ where $S_i(x)$ is the signal power and $I_i(x)$ is the sum of the interference and noise power experienced by a mobile at $x \in C_i$.

Consider cell $C_1$. Let the left endpoint of $C_1$ be the origin and the right endpoint be the point $(L_1, 0)$. Assume that $z > L_1$. We will show that a threshold-type Wardrop equilibrium exists.

Suppose the density is of the following form:

$$\mu_1(x) = \begin{cases} \lambda_1(x) & \text{if } x > t_1 \\ 0 & \text{if } x < t_1 \end{cases} \tag{9}$$

where $t_1 \in [0, L_1]$ will be chosen later.

By (6), for $\mu_1(x)$ to be a Wardrop equilibrium, $\mu_1(x) = \lambda_1(x)$ if:

$$(1 + S_1(x)/\sigma^2)^{(1-\gamma_p)\beta_1/|\mu_1|} >$$
$$(1 + S_1(x)/I_1(x))^{(1-\gamma_s)(1-\beta_1-\beta_2)/(|\lambda_1|-|\mu_1|)}.$$

For large $\sigma^2$, since $I_1(x) \geq \sigma^2$, using Taylor series expansion, the relation becomes

$$(1-\gamma_p)\beta_1 S_1(x)/\sigma^2|\mu_1| >$$
$$(1-\gamma_s)(1-\beta_1-\beta_2)S_1(x)/(|\lambda_1|-|\mu_1|)I_1(x),$$

That is,

$$\mu_1(x) = \lambda_1(x) \text{ if } I_1(x) > T_1(t_1) \tag{10}$$

where:

$$T_1(t_1) = \frac{(1-\gamma_s)(1-\beta_1-\beta_2)\sigma^2|\mu_1|}{(1-\gamma_p)\beta_1(|\lambda_1|-|\mu_1|)} \tag{11}$$

Recall that

$$I_1(x) = \sigma^2 + P_2(R^2 + (x-z)^2)^{-\xi/2}$$

Since $I_1(x)$ is a continuous function of $x$, by the intermediate value theorem (Fact 4.1 in Section IV-B), either there exists a value of $x$ such that $I_1(x) = T_1(t_1)$, or $I_1(x) < T_1(t_1) \forall x \in [0, L_1]$ or $I_1(x) > T_1(t_1) \forall x \in [0, L_1]$. Let $f(t_1)$ be the value of $x$ that satisfies $I_1(x) = T_1(t_1)$, if a solution exists, else let $f(t_1) = L_1$ if $I_1(x) < T_1(t_1)\ \forall x \in [0, L_1]$ and $f(t_1) = 0$ if $I_1(x) > T_1(t_1)\ \forall x \in [0, L_1]$. Note that $I_1(x)$ is an increasing function of $x$. Hence, by (10), for $\mu_1(x)$ to be a Wardrop equilibrium, it must satisfy:

$$\mu_1(x) = \begin{cases} \lambda_1(x) & \text{if } x > f(t_1) \\ 0 & \text{if } x < f(t_1) \end{cases} \tag{12}$$

Recall that $|\mu_1| = \int_{t_1}^{L_1} \lambda_1(x)dx$. Hence, as $t_1$ increases, $|\mu_1|$ decreases and $|\lambda_1| - |\mu_1|$ increases. So by (11), $T_1(t_1)$ is a decreasing function of $t_1$. Also, since $I_1(x)$ is a continuous, increasing function of $x$, by definition of $f(.)$, it follows that $f(t_1)$ is a decreasing function of $t_1$. Also, note that $f(t_1)$ is a continuous function of $t_1$. Let $\tilde{f}(t_1) = f(t_1) - t_1$. If $\tilde{f}(0) = f(0) \geq 0$ and $\tilde{f}(L_1) = f(L_1) - L_1 \leq 0$, then by the intermediate value theorem (Fact 4.1 in Section IV-B), there exists a value of $t_1$, say $t_1^*$, such that $\tilde{f}(t_1^*) = 0$, that is $f(t_1^*) = t_1^*$. From (9) and (12), it follows that the density in (9) with $t_1 = t_1^*$ is a Wardrop equilibrium.

If $\tilde{f}(0) < 0$ (respectively, if $\tilde{f}(L_1) > 0$), then since $f(.)$ is decreasing, $\tilde{f}(L_1) < 0$ (respectively, $\tilde{f}(0) > 0$). Also, $f(0) = \tilde{f}(0) < 0$ (respectively, $f(L_1) = \tilde{f}(L_1) + L_1 > L_1$) and hence by (9) and (12), the density in (9) with $t_1 = 0$ (respectively, $t_1 = L_1$) is a Wardrop equilibrium. Thus, a threshold-type Wardrop equilibrium exists in all cases.

A symmetrical argument shows that a threshold-type Wardrop equilibrium exists in cell $C_2$.

## IV. Migration

So far, we have considered two fixed populations of mobile subscribers each connecting to a different operator. We next consider the possibility of choosing to which BS to connect. At each point $x$, a mobile can decide to be primary or secondary of BS1 or primary or secondary of BS2. In Section IV-B, we show the existence of a threshold-type Wardrop equilibrium for this scenario. Before that, in Section IV-A, we consider the simple case in which there are no secondary mobiles and each mobile chooses whether to be a primary of BS 1 or BS 2 and show the existence of a Wardrop equilibrium.

### A. Simple Case

Suppose mobiles are located on the $x$-axis from $x = 0$ to $x = L$ with a density function $\lambda(x)$. Moreover, BS 1 and 2 are located at $(-1, 0)$ and $(L+1, 0)$ respectively. Each mobile decides whether to become a primary user of BS 1 or a primary user of BS 2. Let $\mu(x)$ be the density of mobiles at $x$ that are primaries of BS 1. Suppose both base stations employ rate-fair access. Let $r_1$ and $r_2$ be the rate of each mobile with BS 1 and BS 2 respectively.

Let $SNR_1(x)$ and $SNR_2(x)$ be the signal to noise ratios of the users of BS 1 and BS 2 respectively at $x$. We have:

$$SNR_1(x) = \frac{P_1(x+1)^{-\xi}}{\sigma^2}$$

$$SNR_2(x) = \frac{P_2(L+1-x)^{-\xi}}{\sigma^2}$$

We use the low-SNR approximation of throughput, *i.e.* the throughput of a mobile at $x$ is $\log(1 + SNR_i(x)) \approx SNR_i(x)$. From (3), we get:

$$r_1 = \frac{\beta_1}{\int_0^L \frac{\mu(x)\sigma^2}{P_1}(x+1)^{\xi}dx} \tag{13}$$

$$r_2 = \frac{\beta_2}{\int_0^L \frac{(\lambda(x)-\mu(x))\sigma^2}{P_2}(L+1-x)^{\xi}dx} \tag{14}$$

The utilities of mobiles associated with BS 1 and BS 2 are respectively:

$$U_1^p(x) = r_1(1-\gamma_{1,p}) \tag{15}$$

$$U_2^p(x) = r_2(1-\gamma_{2,p}) \tag{16}$$

where $\gamma_{1,p}$ and $\gamma_{2,p}$ are the costs per throughput unit of primary mobiles associated with BS 1 and BS 2 respectively.

A density $\{\mu(x) : 0 \leq x \leq L\}$ is a Wardrop equilibrium if $\mu(x) = \lambda(x)$ only at points $x$ such that $U_1^p(x) \geq U_2^p(x)$ and $\mu(x) = 0$ only at points $x$ such that $U_1^p(x) \leq U_2^p(x)$.

We show that there exists a threshold-type Wardrop equilibrium. Consider the following density $\mu(x)$:

$$\mu(x) = \begin{cases} \lambda(x) & \text{if } x < s^* \\ 0 & \text{if } x > s^* \end{cases} \tag{17}$$

where $s^*$ is chosen as a solution of $U_1^p(x) = U_2^p(x)$, which, using (13) to (16), can be written as:

$$\frac{1}{\beta_2 P_2}(1-\gamma_{1,p}) \int_s^L \lambda(x)(L+1-x)^\xi dx = \frac{1}{\beta_1 P_1}(1-\gamma_{2,p}) \int_0^s \lambda(x)(x+1)^\xi dx \tag{18}$$

We show that a solution $s = s^*$ to (18) exists. The quantity on the left hand side of (18) decreases from a positive number to 0 as $s$ increases from 0 to $L$. The quantity on the right hand side increases from 0 to a positive number as $s$ increases from 0 to $L$. Also, both these quantities are continuous functions of $s$ since they are integrals and hence differentiable with respect to $s$. From Fact 4.2 in Section IV-B, it follows that for some $s$, say $s = s^*$, the quantity on the left hand side equals the quantity on the right hand side.

Now, to show that (17) is a Wardrop equilibrium, note that with $\mu(x)$ given by (17), since $U_1^p(x) = U_2^p(x)\ \forall x \in [0, L]$, obviously, it is weakly best for a mobile at $x < s^*$ to associate with BS 1 (since $U_1^p(x) \geq U_2^p(x)$) and for a mobile at $x > s^*$ to associate with BS 2 (since $U_2^p(x) \geq U_1^p(x)$).

### B. General Case

Again, suppose mobiles are located on the $x$-axis from $x = 0$ to $x = L$ with a density function $\lambda(x)$. Moreover, BS 1 and 2 are located at $(-1, 0)$ and $(L+1, 0)$ respectively. Assume that $\beta_1 + \beta_2 < 1$. Each mobile decides whether to become a primary user of BS 1, primary user of BS 2, secondary user of BS 1 or secondary user of BS 2.

Let $\mu_1(x)$ (respectively $\mu_2(x)$) be the density at $x$ of primary mobiles associated with BS 1 (respectively BS 2). Let $\alpha_1(x)$ (respectively $\alpha_2(x)$) be the density at $x$ of secondary mobiles associated with BS 1 (respectively BS 2). Assume rate-fair access for primaries. Denote the utility of a primary user of BS 1, secondary user of BS 1, primary user of BS 2 and secondary user of BS 2 at $x$ by $U_1^p(x)$, $U_1^s(x)$, $U_2^p(x)$ and $U_2^s(x)$ respectively.

A set of densities $\{\mu_1(x), \mu_2(x), \alpha_1(x), \alpha_2(x) : x \in [0, L]\}$ constitute a Wardrop equilibrium if at any point $x$:

1) $\mu_1(x) = \lambda(x)$ (respectively, $\mu_2(x) = \lambda(x)$) only if $U_1^p(x)$ (respectively, $U_2^p(x)$) is a maximum among $U_1^p(x), U_2^p(x), U_1^s(x)$ and $U_2^s(x)$ and $\mu_1(x) = 0$ (respectively, $\mu_2(x) = 0$) otherwise; and
2) $\alpha_1(x) = \lambda(x)$ (respectively, $\alpha_2(x) = \lambda(x)$) only if $U_1^s(x)$ (respectively, $U_2^s(x)$) is a maximum among $U_1^p(x), U_2^p(x), U_1^s(x)$ and $U_2^s(x)$ and $\alpha_1(x) = 0$ (respectively, $\alpha_2(x) = 0$) otherwise.

We show that there exists a threshold-type Wardrop equilibrium with thesholds $s_1$, $s_2$ and $s_3$ where $0 < s_1 < s_2 < s_3 < L$ and:

$$\alpha_1(x) = \begin{cases} \lambda(x) & \text{if } 0 \leq x < s_1 \\ 0 & \text{else} \end{cases} \tag{19}$$

$$\mu_1(x) = \begin{cases} \lambda(x) & \text{if } s_1 \leq x < s_2 \\ 0 & \text{else} \end{cases} \tag{20}$$

$$\mu_2(x) = \begin{cases} \lambda(x) & \text{if } s_2 \leq x < s_3 \\ 0 & \text{else} \end{cases} \tag{21}$$

$$\alpha_2(x) = \begin{cases} \lambda(x) & \text{if } s_3 \leq x < L \\ 0 & \text{else} \end{cases} \tag{22}$$

Fig. 2 shows the Wardrop equilibrium.

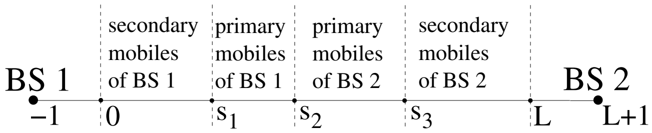


Fig. 2. The threshold-type Wardrop equilibrium in Section IV-B

Let $\gamma_{1,p}$ and $\gamma_{2,p}$ be the costs per throughput unit of primary mobiles associated with BS 1 and BS 2 respectively and $\gamma_{1,s}$ and $\gamma_{2,s}$ be the costs per throughput unit of secondary mobiles associated with BS 1 and BS 2 respectively.

Let $|A_1(s_1)| = \int_0^{s_1} \lambda(u)du$ and $|A_2(s_3)| = \int_{s_3}^L \lambda(u)du$ be the total number of secondary mobiles associated with BS 1 and BS 2 respectively. We use the low-SINR approximation: $\log(1 + SINR_i(x)) \approx SINR_i(x)$. Since the average throughput of a primary mobile of BS $i$ is $r_i$ and that of a secondary mobile is given by a formula similar to (2), we have:

$$U_1^p(x) = r_1(1-\gamma_{1,p}) = \frac{\beta_1(1-\gamma_{1,p})}{\int_{s_1}^{s_2} \frac{\lambda(u)\sigma^2}{P_1}(u+1)^\xi du} \tag{23}$$

$$U_1^s(x) = \frac{1-\beta_1-\beta_2}{|A_1(s_1)|} SINR_1^s(x)(1-\gamma_{1,s}) = \left(\frac{1-\beta_1-\beta_2}{\int_0^{s_1}\lambda(u)du}\right)\left(\frac{P_1(x+1)^{-\xi}(1-\gamma_{1,s})}{\sigma^2 + P_2(L+1-x)^{-\xi}}\right) \tag{24}$$

$$U_2^p(x) = r_2(1-\gamma_{2,p}) = \frac{\beta_2(1-\gamma_{2,p})}{\int_{s_2}^{s_3} \frac{\lambda(u)\sigma^2}{P_2}(L+1-u)^\xi du} \tag{25}$$

$$U_2^s(x) = \frac{1-\beta_1-\beta_2}{|A_2(s_3)|} SINR_2^s(x)(1-\gamma_{2,s}) = \frac{1-\beta_1-\beta_2}{\int_{s_3}^L \lambda(u)du}\left(\frac{P_2(L+1-x)^{-\xi}(1-\gamma_{2,s})}{\sigma^2 + P_1(x+1)^{-\xi}}\right) \tag{26}$$

We assume that there exists a constant $l$ such that

$$\lambda(x) \geq l > 0\ \forall x \in [0, L] \tag{27}$$

*Lemma 4.1:* The equations

$$U_1^s(s_1) = U_1^p(x) = U_2^p(x) = U_2^s(s_3) \tag{28}$$

have a solution $s_1$, $s_2$, $s_3$ where $0 < s_1 < s_2 < s_3 < L$.

We prove Lemma 4.1 later.

Suppose $s_1$, $s_2$ and $s_3$ are chosen as a solution of (28). Now, by (24) and (26), $U_1^s(x)$ (respectively $U_2^s(x)$) is a decreasing (respectively increasing) function of $x$. Also, by (23) and (25), $U_1^p(x)$ and $U_2^p(x)$ are constant functions of $x$. Hence, by (28), it follows that among $U_1^s(x)$, $U_1^p(x)$, $U_2^p(x)$ and $U_2^s(x)$:

1) $U_1^s(x)$ is a maximum for $0 \le x < s_1$
2) $U_1^p(x)$ is a maximum for $s_1 \le x < s_2$
3) $U_2^p(x)$ is a maximum for $s_2 \le x < s_3$
4) $U_2^s(x)$ is a maximum for $s_3 \le x < L$

Hence, we get the following result:

*Theorem 4.1:* The densities given by (19), (20), (21) and (22) constitute a Wardrop equilibrium.

*Proof:* [Proof of Lemma 4.1] Recall the *intermediate value theorem* [1].

*Fact 4.1:* Let $f$ be a continuous real function on the interval $[a, b]$. Suppose $f(a) \neq f(b)$. If $c$ lies between $f(a)$ and $f(b)$, then there is a point $x \in [a, b]$ such that $f(x) = c$. The following fact can be easily proved using Fact 4.1.

*Fact 4.2:* Let $f$ and $h$ be continuous real functions on the interval $[a, b]$. Suppose $f(a) > 0$, $f(b) = 0$, $h(a) = 0$ and $h(b) > 0$. Then there is a point $c \in (a, b)$ such that $f(c) = h(c)$. In addition, if $f$ is strictly decreasing and $h$ is strictly increasing, then there is a unique such point $c$.

Let

$$g_1(s_1) = \frac{1}{U_1^s(s_1)} = \left(\frac{\int_0^{s_1} \lambda(u)du}{1-\beta_1-\beta_2}\right)\left(\frac{\sigma^2 + P_2(L+1-s_1)^{-\xi}}{P_1(s_1+1)^{-\xi}(1-\gamma_{1,s})}\right) \quad (29)$$

$$g_2(s_1, s_2) = \frac{1}{U_1^p(x)} = \frac{\int_{s_1}^{s_2} \frac{\lambda(u)\sigma^2}{P_1}(u+1)^{\xi}du}{\beta_1(1-\gamma_{1,p})} \quad (30)$$

$$g_3(s_2, s_3) = \frac{1}{U_2^p(x)} = \frac{\int_{s_2}^{s_3} \frac{\lambda(u)\sigma^2}{P_2}(L+1-u)^{\xi}du}{\beta_2(1-\gamma_{2,p})} \quad (31)$$

$$g_4(s_3) = \frac{1}{U_2^s(s_3)} = \left(\frac{\int_{s_3}^{L} \lambda(u)du}{1-\beta_1-\beta_2}\right)\left(\frac{\sigma^2 + P_1(s_3+1)^{-\xi}}{P_2(L+1-s_3)^{-\xi}(1-\gamma_{2,s})}\right) \quad (32)$$

From (29) and (32), it follows that $g_1(s_1)$ strictly increases from 0 to a positive value as $s_1$ increases from 0 to $L$ and that $g_4(s_3)$ strictly decreases from a positive value to 0 as $s_3$ increases from 0 to $L$. Also, $g_1$ and $g_4$ are continuous functions. From Fact 4.2, there exists a unique value $s^*$ such that $g_1(s^*) = g_4(s^*)$. Also, from continuity and strict monotonicity of $g_1$ and $g_4$, it follows that for every $t$ such that $0 \le t \le L$, there exist unique values $s_1^*(t)$ and $s_3^*(t)$ such that $s_1^*(t) \le s^*$, $s_3^*(t) \ge s^*$, $s_3^*(t) - s_1^*(t) = t$ and $g_1(s_1^*(t)) = g_4(s_3^*(t))$. Moreover, $s_1^*(t)$ and $s_3^*(t)$ are continuous functions of $t$.

Now, from (27), (30) and (31) it follows that for $0 < t \le L$, $g_2(s_1^*(t), s_2)$ strictly increases from 0 to a positive value and that $g_3(s_2, s_3^*(t))$ strictly decreases from a positive value to 0 as $s_2$ increases from $s_1^*(t)$ to $s_3^*(t)$. Moreover, $g_2(s_1^*(t), s_2)$ and $g_3(s_2, s_3^*(t))$ are continuous functions of $s_2$. From Fact 4.2 it follows that there exists a unique value $s_2 = s_2^*(t)$ such that $s_1^*(t) < s_2^*(t) < s_3^*(t)$ and $g_2(s_1^*(t), s_2^*(t)) = g_3(s_2^*(t), s_3^*(t))$.

*Lemma 4.2:* $s_2^*(t)$ is a continuous function of $t$.

We will prove Lemma 4.2 later. By Lemma 4.2, $s_1^*(t)$, $s_2^*(t)$ and $s_3^*(t)$ are continuous functions of $t$, $g_2$ is a continuous function of $s_1$ and $s_2$ and $g_3$ is a continuous function of $s_2$ and $s_3$. $g_1$ and $g_4$ are continuous functions of $s_1$ and $s_3$ respectively. Since the composition of continuous functions is continuous [1], it follows that $g_1$, $g_2$, $g_3$ and $g_4$ are continuous functions of $t$. Denote $g_1(t) = g_1(s_1^*(t))$ etc. We have:

$$g_2(s_1^*(t), s_2^*(t)) = g_3(s_2^*(t), s_3^*(t)) \ \ \forall t \in [0, L]$$

$$g_2(s_1^*(0), s_2^*(0)) = 0 \text{ and } g_2(s_1^*(L), s_2^*(L)) > 0$$

$$g_1(s_1^*(t)) = g_4(s_3^*(t)) \ \ \forall t \in [0, L]$$

$$g_1(s_1^*(0)) > 0 \text{ and } g_1(s_1^*(L)) = 0$$

Using Fact 4.2, from the above equations and continuity of $g_1(t)$ and $g_2(t)$, it follows that there exists a value of $t$, where $0 < t < L$ and:

$$g_1(s_1^*(t)) = g_2(s_1^*(t), s_2^*(t)) = g_3(s_2^*(t), s_3^*(t)) = g_4(s_3^*(t))$$

This completes the proof of Lemma 4.1 ■

*Proof:* [Proof of Lemma 4.2] We show that given any $\epsilon > 0$, there exists a $\delta > 0$ such that:

$$|t' - t| < \delta \Rightarrow |s_2^*(t') - s_2^*(t)| < \epsilon$$

To show this, let $t > 0$ be arbitrary. We have:

$$g_2(s_1^*(t), s_2^*(t)) - g_3(s_2^*(t), s_3^*(t)) = 0 \quad (33)$$

Let $t' > t$. The case $t' < t$ is symmetrical. Since $s_1^*(t')$ and $s_3^*(t')$ are continuous functions of $t'$, $g_2(s_1, s_2^*(t))$ is a continuous function of $s_1$ and $g_3(s_2^*(t), s_3)$ is a continuous function of $s_3$, by the composition rule [1], it follows that for fixed $t$, $g_2(s_1^*(t'), s_2^*(t))$ and $g_3(s_2^*(t), s_3^*(t'))$ are continuous functions of $t'$. So by (33), for any $\epsilon_1 > 0$, there exists a $\delta_1 > 0$ such that

$$0 < t' - t < \delta_1 \Rightarrow |g_2(s_1^*(t'), s_2^*(t)) - g_3(s_2^*(t), s_3^*(t'))| < \epsilon_1$$

Assume that:

$$g_2(s_1^*(t'), s_2^*(t)) - g_3(s_2^*(t), s_3^*(t')) = \Delta(t, t') > 0 \quad (34)$$

The case in which $\Delta(t,t') < 0$ is symmetrical. Let $s_2 < s_2^*(t)$. Then from (27) and (30):

$$\begin{aligned} & g_2(s_1^*(t'), s_2^*(t)) - g_2(s_1^*(t'), s_2) \\ = & \frac{\int_{s_2}^{s_2^*(t)} \frac{\lambda(u)\sigma^2}{P_1}(u+1)^\xi du}{\beta_1(1-\gamma_{1,p})} \\ \geq & \frac{l\sigma^2}{P_1\beta_1(1-\gamma_{1,p})}(s_2^*(t) - s_2) \end{aligned} \quad (35)$$

From (27) and (31):

$$\begin{aligned} & g_3(s_2, s_3^*(t')) - g_3(s_2^*(t), s_3^*(t')) \\ = & \frac{\int_{s_2}^{s_2^*(t)} \frac{\lambda(u)\sigma^2}{P_2}(L+1-u)^\xi du}{\beta_2(1-\gamma_{2,p})} \\ \geq & \frac{l\sigma^2}{P_2\beta_2(1-\gamma_{2,p})}(s_2^*(t) - s_2) \end{aligned} \quad (36)$$

Adding (35) and (36), using (34) and rearranging terms, we get:

$$g_2(s_1^*(t'), s_2) - g_3(s_2, s_3^*(t')) \leq \Delta(t,t')$$
$$-l\sigma^2\left[\frac{1}{P_1\beta_1(1-\gamma_{1,p})} + \frac{1}{P_2\beta_2(1-\gamma_{2,p})}\right](s_2^*(t) - s_2)$$

Hence, if $s_2$ is such that:

$$s_2^*(t) - s_2 \geq \frac{\Delta(t,t')}{l\sigma^2\left[\frac{1}{P_1\beta_1(1-\gamma_{1,p})} + \frac{1}{P_2\beta_2(1-\gamma_{2,p})}\right]}$$

then $g_2(s_1^*(t'), s_2) - g_3(s_2, s_3^*(t')) \leq 0$. Now, $g_2(s_1^*(t'), s_2)$ and $g_3(s_2, s_3^*(t'))$ are continuous in $s_2$. Hence, it follows from (34) and Fact 4.1 that $g_2(s_1^*(t'), s_2) - g_3(s_2, s_3^*(t'))$ vanishes for some $s_2$ in the following interval:

$$\left[s_2^*(t) - \frac{\Delta(t,t')}{l\sigma^2\left[\frac{1}{P_1\beta_1(1-\gamma_{1,p})} + \frac{1}{P_2\beta_2(1-\gamma_{2,p})}\right]}, s_2^*(t)\right]$$

That $s_2$ is $s_2^*(t')$. Hence:

$$|s_2^*(t') - s_2^*(t)| \leq \frac{\Delta(t,t')}{l\sigma^2\left[\frac{1}{P_1\beta_1(1-\gamma_{1,p})} + \frac{1}{P_2\beta_2(1-\gamma_{2,p})}\right]} \quad (37)$$

Recall that given any $\epsilon_1 > 0$, there exists $\delta > 0$ such that

$$|t' - t| < \delta \Rightarrow \Delta(t,t') < \epsilon_1 \quad (38)$$

Choose:

$$\epsilon_1 = \frac{\epsilon}{2} l\sigma^2\left[\frac{1}{P_1\beta_1(1-\gamma_{1,p})} + \frac{1}{P_2\beta_2(1-\gamma_{2,p})}\right]$$

By (37) and (38):

$$|t' - t| < \delta \Rightarrow |s_2^*(t') - s_2^*(t)| < \epsilon$$

Hence, $s_2^*(t)$ is a continuous function of $t$. ■

## V. GAME 2

In this section, we analyze game 2. In Section V-A, we formulate a general version of game 2 and in Section V-B, we find Nash equilibria for the simple version of migration described in Section IV-A.

### A. General Problem Formulation

In all the sections above, we obtained a threshold-type Wardrop equilibrium of the mobiles. We now describe a scenario, which captures the distributions arising in the above sections as special cases. We formulate the level 2 game for this general scenario.

Let $c_1(\beta_1)$ and $c_2(\beta_2)$ be the amounts paid by BS 1 and BS 2 to purchase fractions $\beta_1$ and $\beta_2$. Also, let $\Theta_1^p$ (respectively, $\Theta_2^p$) be the total throughput of primary mobiles of BS 1 (respectively, BS 2) and $\Theta_1^s$ (respectively, $\Theta_2^s$) be the total throughput of secondary mobiles of BS 1 (respectively, BS 2).

The utilities of BS 1 and BS 2 are given by (39) and (40) respectively:

$$U_1^{BS} = \Theta_1^p\gamma_{1,p} + \Theta_1^s\gamma_{1,s} - c_1(\beta_1) \quad (39)$$
$$U_2^{BS} = \Theta_2^p\gamma_{2,p} + \Theta_2^s\gamma_{2,s} - c_2(\beta_2) \quad (40)$$

This is a game in which BS 1 and BS 2 are the players. BS 1 chooses $\beta_1$, $\gamma_{1,p}$ and $\gamma_{1,s}$. BS 2 chooses $\beta_2$, $\gamma_{2,p}$ and $\gamma_{2,s}$.

Suppose $s_L < s_R$ and that mobiles are located in the interval $[s_L, s_R]$. Let $\lambda(x)$ be the density of mobiles at $x$. Let $s_1$, $s_2$ and $s_3$ be thresholds such that $s_L \leq s_1 \leq s_2 \leq s_3 \leq s_R$. Suppose all mobiles in the intervals $[s_L, s_1)$, $[s_1, s_2)$, $[s_2, s_3)$ and $[s_3, s_R]$ are secondary users of BS 1, primary users of BS 1, primary users of BS 2 and secondary users of BS 2 respectively.

We now explain how the level 2 games corresponding to the level 1 games in the previous sections are special cases of the above general game. In Sections III-A and III-B, $s_L = \inf(x \in C_1)$, $s_2 = \sup(x \in C_1) = \inf(x \in C_2)$, $s_R = \sup(x \in C_2)$. It was shown that in each cell $C_i$, there is a threshold such that a user of BS $i$ is primary when it is on the same side of the threshold as the other BS and a secondary otherwise. $s_1$ and $s_3$ are these thresholds. In Section IV, $s_L = 0$ and $s_R = L$. In Section IV-A, $s_1 = 0$, $s_3 = L$, $s_2 = s^*$. In Section IV-B, $s_1$, $s_2$ and $s_3$ are the same variables as in the general game above.

Let $|\mu_1|$ and $|\mu_2|$ be the total number of primary users of BS 1 and BS 2 respectively. Let $|\alpha_1|$ and $|\alpha_2|$ be the total number of secondary users of BS 1 and BS 2 respectively. Next, we derive expressions for $\Theta_1^p$ and $\Theta_2^p$ under rate-fair and time-fair access for primaries and expressions for $\Theta_1^s$ and $\Theta_2^s$.

Suppose rate-fair access is used for primaries. Similar to the derivation of (3), if $|\mu_1| > 0$ (respectively $|\mu_2| > 0$), the rate of primary mobiles of BS 1 (respectively BS 2) is given by (41) (respectively (42)).

$$r_1 = \frac{\beta_1}{\int_{s_1}^{s_2} \frac{1}{\log(1+SINR_1^p(x))}\lambda(x)dx} \quad (41)$$

$$r_2 = \frac{\beta_2}{\int_{s_2}^{s_3} \frac{1}{\log(1+SINR_2^p(x))}\lambda(x)dx} \quad (42)$$

If $|\mu_1| > 0$ (respectively $|\mu_2| > 0$), then $\Theta_1^p$ (respectively $\Theta_2^p$) is given by (43) (respectively (44)), else it is 0.

$$\Theta_1^p = r_1 \int_{s_1}^{s_2} \lambda(x)dx \quad (43)$$

$$\Theta_2^p = r_2 \int_{s_2}^{s_3} \lambda(x)dx \quad (44)$$

Now suppose time-fair access is used by primaries. Since the average throughput of a primary user is given by an expression similar to (2), if $|\mu_1| > 0$ (respectively $|\mu_2| > 0$), then $\Theta_1^p$ (respectively $\Theta_2^p$) is given by (45) (respectively (46)), else it is 0.

$$\Theta_1^p = \frac{\beta_1}{\int_{s_1}^{s_2} \lambda(x)dx} \int_{s_1}^{s_2} \log(1 + SINR_1^p(x))\lambda(x)dx \quad (45)$$

$$\Theta_2^p = \frac{\beta_2}{\int_{s_2}^{s_3} \lambda(x)dx} \int_{s_2}^{s_3} \log(1 + SINR_2^p(x))\lambda(x)dx \quad (46)$$

By (2), if $|\alpha_1| > 0$ (respectively $|\alpha_2| > 0$), $\Theta_1^s$ (respectively $\Theta_2^s$) is given by (47) (respectively (48)), else it is 0.

$$\Theta_1^s = \frac{1-\beta_1-\beta_2}{\int_{s_L}^{s_1} \lambda(x)dx} \int_{s_L}^{s_1} \log(1 + SINR_1^s(x))\lambda(x)dx \quad (47)$$

$$\Theta_2^s = \frac{1-\beta_1-\beta_2}{\int_{s_3}^{s_R} \lambda(x)dx} \int_{s_3}^{s_R} \log(1 + SINR_2^s(x))\lambda(x)dx \quad (48)$$

Note that $r_1$, $r_2$, $\Theta_1^p$, $\Theta_2^p$, $\Theta_1^s$, $\Theta_2^s$, $U_1^{BS}$ and $U_2^{BS}$ are all functions of $\beta_1$, $\beta_2$, $\gamma_{1,p}$, $\gamma_{2,p}$, $\gamma_{1,s}$ and $\gamma_{2,s}$. $s_1$, $s_2$ and $s_3$ may also be functions of these parameters. We drop the arguments for brevity.

### *B. Nash Equilibria for Migration*

In this subsection, we specialize the formulation in Section V-A to the simple version of migration described in Section IV-A and find Nash equilibria in the game.

In the game in Section IV-A, let $s^*(\beta_1, \beta_2)$ denote the threshold $s^*$ and $r_1(\beta_1, \beta_2)$ and $r_2(\beta_1, \beta_2)$ denote the rates $r_1$ and $r_2$ with fractions $\beta_1$ and $\beta_2$. The total throughputs of mobiles associated with BS 1 and BS 2 are respectively given by:

$$\Theta_1(\beta_1, \beta_2) = r_1(\beta_1, \beta_2) \int_0^{s^*(\beta_1,\beta_2)} \lambda(x)dx \quad (49)$$

$$\Theta_2(\beta_1, \beta_2) = r_2(\beta_1, \beta_2) \int_{s^*(\beta_1,\beta_2)}^{L} \lambda(x)dx \quad (50)$$

The net profits of BS 1 and BS 2 are:

$$U_1^{BS}(\beta_1, \beta_2) = \Theta_1(\beta_1, \beta_2)\gamma_{1,p} - c_1(\beta_1) \quad (51)$$

$$U_2^{BS}(\beta_1, \beta_2) = \Theta_2(\beta_1, \beta_2)\gamma_{2,p} - c_2(\beta_2) \quad (52)$$

Substituting $\Theta_i(\beta_1, \beta_2)$ and $r_i(\beta_1, \beta_2)$ from (13), (14), (49) and (50) into the above equations and using (17), we get:

$$U_1^{BS}(\beta_1, \beta_2) = \frac{\gamma_{1,p}\beta_1 \int_0^{s^*(\beta_1,\beta_2)} \lambda(x)dx}{\frac{\sigma^2}{P_1} \int_0^{s^*(\beta_1,\beta_2)} \lambda(x)(x+1)^\xi dx} - c_1(\beta_1) \quad (53)$$

$$U_2^{BS}(\beta_1, \beta_2) = \frac{\gamma_{2,p}\beta_2 \int_{s^*(\beta_1,\beta_2)}^{L} \lambda(x)dx}{\frac{\sigma^2}{P_2} \int_{s^*(\beta_1,\beta_2)}^{L} \lambda(x)(L+1-x)^\xi dx} - c_2(\beta_2) \quad (54)$$

For brevity, in the sequel, we denote $s^*(\beta_1, \beta_2)$ simply by $s$.

We assume that $\lambda(x) = 1, \forall x \in [0, L]$ and that $\xi$ is an integer. Also, we consider the case $c_i(\beta_i) = c_i\beta_i, i \in \{1, 2\}$ for non-negative constants $c_1, c_2$. Finally, we assume that $\gamma_{1,p}$ and $\gamma_{2,p}$ are constants and BS $i$ chooses $\beta_i$ for $i = 1, 2$ such that $\beta_1, \beta_2 \geq 0$ and $\beta_1 + \beta_2 \leq 1$.

Let

$$K(L, \xi) = \frac{[(L+1)^{\xi+1} - 1]^2}{L(L+1)^\xi(\xi+1)^3}. \\ (1 + 2(L+1) + 3(L+1)^2 + \ldots + \xi(L+1)^{\xi-1}) \quad (55)$$

The following theorem identifies a set of conditions under which $(\beta_1^*, \beta_2^*)$ is a Nash equilibrium.

*Theorem 5.1:* Suppose:

1) $\beta_1^*, \beta_2^* \geq 0$, $\beta_1^* + \beta_2^* = 1$
2)

$$c_1 \leq \max\left(0, \frac{\gamma_{1,p}P_1(\xi+1)}{\sigma^2}. \left[\frac{L}{(L+1)^{\xi+1} - 1} - \frac{P_1(1-\gamma_{1,p})}{P_2(1-\gamma_{2,p})} K(L,\xi) \frac{\beta_1^*}{\beta_2^*}\right]\right) \quad (56)$$

and

3)

$$c_2 \leq \max\left(0, \frac{\gamma_{2,p}P_2(\xi+1)}{\sigma^2}. \left[\frac{L}{(L+1)^{\xi+1} - 1} - \frac{P_2(1-\gamma_{2,p})}{P_1(1-\gamma_{1,p})} K(L,\xi) \frac{\beta_2^*}{\beta_1^*}\right]\right) \quad (57)$$

Then $(\beta_1^*, \beta_2^*)$ is a Nash equilibrium.

Intuitively, if $\beta_1^* + \beta_2^* = 1$, and the costs $c_1, c_2$ for leasing spectrum are small enough, then $U_1^{BS}(\beta_1, \beta_2^*)$ (respectively, $U_2^{BS}(\beta_1^*, \beta_2)$) is an increasing function of $\beta_1$ (respectively, $\beta_2$). So when BS 2 chooses $\beta_2^*$, the best response for BS 1 is $\beta_1^* = 1 - \beta_2^*$ since $\beta_1 + \beta_2^* \leq 1$. Similarly, when BS 1 chooses $\beta_1^*$, the best response for BS 2 is $\beta_2^* = 1 - \beta_1^*$ since $\beta_1^* + \beta_2 \leq 1$. Hence, $(\beta_1^*, \beta_2^*)$ is a Nash equilibrium.

*Proof:* Substituting $\lambda(x) = 1$ and evaluating the integrals in (53) we get:

$$\begin{aligned} U_1^{BS}&(\beta_1, \beta_2) \\ &= \frac{\gamma_{1,p}P_1(\xi+1)}{\sigma^2}\beta_1 \left[\frac{s}{(s+1)^{\xi+1} - 1}\right] - c_1\beta_1 \\ &= \frac{\gamma_{1,p}P_1(\xi+1)}{\sigma^2}\beta_1 h(s) - c_1\beta_1 \end{aligned} \quad (58)$$

where:

$$h(s) = \frac{s}{(s+1)^{\xi+1} - 1} \quad (59)$$

Note that $h(.)$ is a function of $s = s^*(\beta_1, \beta_2)$, which in turn is a function of $\beta_1$ for fixed $\beta_2$. For brevity, in the sequel, we denote $h(.)$ by $h(s)$ or $h(\beta_1)$ or simply by $h$.

We next find $\frac{dh(\beta_1)}{d\beta_1}$ by evaluating $\frac{dh(s)}{ds}$ and $\frac{ds}{d\beta_1}$ and using the chain rule [1]:

$$\frac{dh(\beta_1)}{d\beta_1} = \frac{dh(s)}{ds}\frac{ds}{d\beta_1} \quad (60)$$

Evaluating the integrals in (18) using $\lambda(x) = 1$, we find that $s$ is the solution of the equation:

$$\frac{(s+1)^{\xi+1} - 1}{\beta_1 P_1(1-\gamma_{1,p})} = \frac{(L-s+1)^{\xi+1} - 1}{\beta_2 P_2(1-\gamma_{2,p})} \quad (61)$$

Solving for $\beta_1$:

$$\beta_1 = \frac{\beta_2 P_2(1-\gamma_{2,p})}{P_1(1-\gamma_{1,p})}\left[\frac{(s+1)^{\xi+1} - 1}{(L-s+1)^{\xi+1} - 1}\right] \quad (62)$$

We differentiate this expression with respect to $s$ to get $\frac{d\beta_1}{ds}$ and use the fact that $\frac{ds}{d\beta_1} = \frac{1}{\frac{d\beta_1}{ds}}$ (see [1]) to give:

$$\frac{ds}{d\beta_1} = \left(\frac{P_1(1-\gamma_{1,p})}{\beta_2 P_2(1-\gamma_{2,p})}\frac{1}{\xi+1}\right). \frac{\left[(L-s+1)^{\xi+1} - 1\right]^2}{(L+2)(L-s+1)^{\xi}(s+1)^{\xi} - (s+1)^{\xi} - (L-s+1)^{\xi}} \quad (63)$$

Note that the denominator of the second factor in the right hand side is positive for all $s \in [0, L]$ (see (71)) and hence $\frac{ds}{d\beta_1} \geq 0$.

Now, differentiating (59) with respect to $s$, we get:

$$\frac{dh(s)}{ds} = \frac{(s+1)^{\xi}(1-\xi s) - 1}{[(s+1)^{\xi+1} - 1]^2} \quad (64)$$

Let $t = s+1$. Since $s \in [0, L]$, we have $t \in [1, L+1]$. Then:

$$\frac{dh(s)}{ds} = \frac{-\xi t^{\xi+1} + t^{\xi}(1+\xi) - 1}{(t^{\xi+1} - 1)^2} \quad (65)$$

Now, by factorization, we get:

$$\begin{aligned} & -\xi t^{\xi+1} + t^{\xi}(1+\xi) - 1 \\ = \;& -(t-1)^2(\xi t^{\xi-1} + (\xi-1)t^{\xi-2} + \ldots + 2t + 1) \end{aligned} \quad (66)$$

and

$$(t^{\xi+1} - 1)^2 = (t-1)^2(t^{\xi} + t^{\xi-1} + \ldots + t + 1)^2 \quad (67)$$

By (65), (66) and (67), we get:

$$\frac{dh(s)}{ds} = \frac{-(\xi t^{\xi-1} + (\xi-1)t^{\xi-2} + \ldots + 2t + 1)}{(t^{\xi} + t^{\xi-1} + \ldots + t + 1)^2} \quad (68)$$

Now, by (60), (63) and (68), we get:

$$\frac{dh}{d\beta_1} = -\left(\frac{P_1(1-\gamma_{1,p})}{\beta_2 P_2(1-\gamma_{2,p})}\frac{1}{\xi+1}\right). \frac{\left[(L-s+1)^{\xi+1} - 1\right]^2}{(L+2)(L-s+1)^{\xi}(s+1)^{\xi} - (s+1)^{\xi} - (L-s+1)^{\xi}}. \frac{(\xi t^{\xi-1} + (\xi-1)t^{\xi-2} + \ldots + 2t + 1)}{(t^{\xi} + t^{\xi-1} + \ldots + t + 1)^2} \quad (69)$$

Next, we upper bound $\left|\frac{dh}{d\beta_1}\right|$ by bounding the different factors in the above expression.

$$\left[(L-s+1)^{\xi+1} - 1\right]^2 \leq \left[(L+1)^{\xi+1} - 1\right]^2 \quad (70)$$

since $s \in [0, L]$.

Now,

$$\begin{aligned} & (L+2)(L-s+1)^{\xi}(s+1)^{\xi} - (s+1)^{\xi} - (L-s+1)^{\xi} \\ = \;& L(L-s+1)^{\xi}(s+1)^{\xi} + (L-s+1)^{\xi}[(s+1)^{\xi} - 1] \\ & +(s+1)^{\xi}[(L-s+1)^{\xi} - 1] \\ \geq \;& L(L+1)^{\xi} \end{aligned} \quad (71)$$

since the second and third terms are nonnegative and the first term attains a minimum value of $L(L+1)^{\xi}$ on $s \in [0, L]$ at $s = 0$ and $s = L$.

Also, since $t \in [1, L+1]$,

$$\begin{aligned} & \xi t^{\xi-1} + (\xi-1)t^{\xi-2} + \ldots + 2t + 1 \\ \leq \;& \xi(L+1)^{\xi-1} + (\xi-1)(L+1)^{\xi-2} + \ldots + 2(L+1) + 1 \end{aligned} \quad (72)$$

and

$$(t^{\xi} + t^{\xi-1} + \ldots + t + 1)^2 \geq (\xi+1)^2 \quad (73)$$

From (69), (70), (71), (72), (73) and (55), we get:

$$\left|\frac{dh}{d\beta_1}\right| \leq \frac{P_1(1-\gamma_{1,p})}{\beta_2 P_2(1-\gamma_{2,p})} K(L, \xi) \quad (74)$$

From (68), it follows that $\frac{dh(s)}{ds} < 0$ and hence $h(s)$ decreases with increase in $s$. So $h(s)$ takes its minimum value on $s \in [0, L]$ at $s = L$. Therefore,

$$h(s) \geq h(L) = \frac{L}{(L+1)^{\xi+1} - 1}, \;\; \forall s \in [0, L] \quad (75)$$

Now, suppose $\beta_1^*$ and $\beta_2^*$ satisfy $\beta_1^* + \beta_2^* = 1$. Suppose BS 2 selects $\beta_2 = \beta_2^*$. Then BS 1 must select $\beta_1 \leq \beta_1^* = 1 - \beta_2^*$. We show that $U_1^{BS}(\beta_1', \beta_2^*) \leq U_1^{BS}(\beta_1^*, \beta_2^*)$ for all $\beta_1' < \beta_1^*$. From this it will follow that $\beta_1^*$ is a best response to $\beta_2^*$.

Let $\beta_1' < \beta_1^*$. From (58), we get:

$$\begin{aligned} & U_1^{BS}(\beta_1^*, \beta_2^*) - U_1^{BS}(\beta_1', \beta_2^*) \\ = \;& \frac{\gamma_{1,p} P_1(\xi+1)}{\sigma^2}[\beta_1^* h(\beta_1^*) - \beta_1' h(\beta_1')] \\ & -c_1(\beta_1^* - \beta_1') \quad (76) \\ = \;& \frac{\gamma_{1,p} P_1(\xi+1)}{\sigma^2}[\beta_1^*(h(\beta_1^*) - h(\beta_1')) \\ & +h(\beta_1')(\beta_1^* - \beta_1')] - c_1(\beta_1^* - \beta_1') \quad (77) \\ = \;& \frac{\gamma_{1,p} P_1(\xi+1)}{\sigma^2}[\beta_1^*(\beta_1^* - \beta_1')h'(\alpha) \\ & +h(\beta_1')(\beta_1^* - \beta_1')] - c_1(\beta_1^* - \beta_1') \quad (78) \\ \geq \;& (\beta_1^* - \beta_1')\left\{\frac{\gamma_{1,p} P_1(\xi+1)}{\sigma^2}\left[\frac{-\beta_1^* P_1(1-\gamma_{1,p})}{\beta_2^* P_2(1-\gamma_{2,p})} K(L, \xi)\right.\right. \\ & \left.\left. + \frac{L}{(L+1)^{\xi+1} - 1}\right] - c_1\right\} \quad (79) \end{aligned}$$

In (78), we used the fact that:

$$h(\beta_1^*) - h(\beta_1') = h'(\alpha)(\beta_1^* - \beta_1') \quad (80)$$

for some $\alpha \in (\beta_1', \beta_1^*)$, which is a consequence of the *mean value theorem* [1]. Equation (79) follows from (74) and (75).

From (79), we get the following result:

*Lemma 5.1:* If

$$c_1 \leq \frac{\gamma_{1,p} P_1(\xi+1)}{\sigma^2}.$$
$$\left[\frac{L}{(L+1)^{\xi+1}-1} - \frac{P_1(1-\gamma_{1,p})}{P_2(1-\gamma_{2,p})} K(L,\xi) \frac{\beta_1^*}{\beta_2^*}\right]$$

then $U_1^{BS}(\beta_1^*, \beta_2^*) \geq U_1^{BS}(\beta_1', \beta_2^*)$.

Now, if $c_1 = 0$, then

$$U_1^{BS}(\beta_1, \beta_2^*) = \frac{\gamma_{1,p} P_1(\xi+1)}{\sigma^2} \beta_1 h(\beta_1) \qquad (81)$$

*Lemma 5.2:* $U_1^{BS}(\beta_1, \beta_2^*)$ in (81) is an increasing function of $\beta_1$.

We prove Lemma 5.2 later. Now, from Lemma 5.2, it follows that if $c_1 = 0$, then $\beta_1 = \beta_1^*$ is a best response to $\beta_2^*$. Combining this fact with Lemma 5.1, we conclude that if the condition (56) is true, then $\beta_1^*$ is a best response of BS 1 to the strategy $\beta_2^*$ of BS 2.

By symmetry, it follows that if the condition (57) is true, then $\beta_2^*$ is a best response of BS 2 to the strategy $\beta_1^*$ of BS 1.

Hence, under the conditions in Theorem 5.1, $(\beta_1^*, \beta_2^*)$ is a Nash equilibrium. This concludes the proof of Theorem 5.1. ∎

Note that for small enough $\frac{\beta_1^*}{\beta_2^*}$, the second entry in the $\max$ function in (56) is positive and hence we get the non-trivial result that if $c_1$ is positive and less than this number and the other conditions in Theorem 5.1 are satisfied, then $(\beta_1^*, \beta_2^*)$ is a Nash equilibrium.

*Proof:* [Proof of Lemma 5.2] Putting (59) in (81), we get:

$$U_1^{BS}(\beta_1, \beta_2^*) = \frac{\gamma_{1,p} P_1(\xi+1)}{\sigma^2} s \frac{\beta_1}{(s+1)^{\xi+1}-1} \qquad (82)$$

By the remark after (63), $s$ is an increasing function of $\beta_1$. So it is sufficient to prove that $\frac{\beta_1}{(s+1)^{\xi+1}-1}$ is an increasing function of $\beta_1$.

By (62):

$$\frac{\beta_1}{(s+1)^{\xi+1}-1} = \frac{\beta_2^* P_2(1-\gamma_{2,p})}{P_1(1-\gamma_{1,p})} \left[\frac{1}{(L-s+1)^{\xi+1}-1}\right] \qquad (83)$$

The expression on the right hand side is an increasing function of $\beta_1$ because $s$ is an increasing function of $\beta_1$. This concludes the proof. ∎

## VI. Conclusions and Future Work

We analyzed a hierarchical game with players at two levels– mobile subscribers and service providers. Under different models, we showed the existence of a threshold-type Wardrop equilibrium for the game at level 1. We also formulated a general game at level 2 that includes as special cases the level 2 games corresponding to the different level 1 games we considered. Also, we found a Nash equilibrium corresponding to the simple version of the migration model.

As part of future work, the existence of Nash equilibria can be shown for the level 2 games corresponding to the other level 1 games we described. Also, we assumed fixed functions $c_i(\beta_i)$ for the leasing costs charged by the regulator to the service providers. As future work, these functions can be considered as variables to be optimized. Assuming that for given functions $c_i(\beta_i)$, the base stations and mobile subscribers act so as to give rise to the Nash equilibria found at level 2, optimal functions $c_i(\beta_i)$ can be found that achieve socially beneficial equilibria in some suitably defined sense.


## VII. Acknowledgments

The contributions of G. Kasbekar and S. Sarkar have been supported by NSF grants NCR-0238340, CNS-0721308, ECS-0622176.


## References


[1] W. Rudin, *"Principles of Mathematical Analysis"*, Mc-Graw Hill, Third Edition, 1976.

[2] T. Cover, J. Thomas *"Elements of Information Theory"*, Wiley, 1991.

[3] M. Patriksson *"The Traffic Assignment Problem– Models and Methods'*, VSP, 1994.

[4] V. Mazalov, M. Sakaguchi "Location Game on the Plane" *International Game Theory Review*, 2003.

[5] H. Hotelling "Stability in Competition" *Economic Journal*, 39:41-57, 1929.

[6] E. Altman, A. Kumar, C. Singh, R. Sundaresan "Spatial SINR Games Combining Base Station Placement and Mobile Association" To appear in *Proceedings of Infocom*, Rio De Janeiro, Brazil, 2009.

[7] T. Rappaport *"Wireless Communications: Principles and Practice"*, 2nd edition, Prentice Hall, 2002.