GADTs

1 Definitions

$termvar,\ x,\ y,\ z$
$tyvar,\ X,\ Y,\ Z$
$termcon,\ c$
$tycon,\ C$
$index,\ i,\ j,\ m,\ n$

Δ	$::=$		type contexts: finite maps from tyvars to kinds
Σ	$::=$		signature: finite map from termcons to types and tycons to kinds
Φ	$::=$		substitution: finite map from tyvars to types
$t,\ u$	$::=$		term:
	$\mid$	x	variable
	$\mid$	$\lambda x : S.t$	abstraction
	$\mid$	$t\ t'$	application
	$\mid$	$\lambda X :: K.t$	type abstraction
	$\mid$	$t\,[\,T\,]$	type application
	$\mid$	c	datatype constructor
	$\mid$	$\mathbf{case}[\,S\,]\ t\ \mathbf{of}\ \{\ \overline{c_i \Delta_i x_i \Rightarrow t_i}^{\,i}\ \}$	case analysis
v	$::=$		value:
	$\mid$	$\lambda x : T.t$	abstraction value
	$\mid$	$\lambda X :: K.t$	type abstraction value
	$\mid$	$c\,\overline{[T_i]}^{\,i}\ v$	data constructor application
S	$::=$		(poly)types:
	$\mid$	X	type variable
	$\mid$	$T\ T'$	type application
	$\mid$	$S\ \rightarrow\ S'$	type of function
	$\mid$	C	datatype operator
	$\mid$	$\forall\, X\ ::\ K\ .\ S$	universal type
$T,\ U$	$::=$		monotypes:
	$\mid$	X	type variable
	$\mid$	$T\ T'$	operator application
	$\mid$	$T\ \rightarrow\ T'$	type of function
	$\mid$	C	datatype operator type
Γ	$::=$		term contexts:
	$\mid$	$\emptyset$	empty context
	$\mid$	$\Gamma\ ,\ x\ :\ S$	termvar binding
K	$::=$		kinds:
	$\mid$	$*$	kind of proper types
	$\mid$	$K\ \Rightarrow\ K'$	kind of operators

$\boxed{t \;\rightarrow\; t'}$ Evaluation

$$\frac{t_1 \;\rightarrow\; t_1'}{t_1\; t_2 \;\rightarrow\; t_1'\; t_2} \quad \text{E\_APP1}$$

$$\frac{t_2 \;\rightarrow\; t_2'}{v\; t_2 \;\rightarrow\; v\; t_2'} \quad \text{E\_APP2}$$

$$\frac{}{(\,\lambda x : T_{11} . t_{12}\,)\; v_2 \;\rightarrow\; t_{12}\,\{\, v_2 \,/\, x \,\}} \quad \text{E\_APPABS}$$

$$\frac{t_1 \;\rightarrow\; t_1'}{t_1\,[\, T_2 \,] \;\rightarrow\; t_1'\,[\, T_2 \,]} \quad \text{E\_TAPP}$$

$$\frac{}{(\,\lambda X :: K_{11} . t_{12}\,)\,[\, T_2 \,] \;\rightarrow\; t_{12}\,\{\, T_2 \,/\, X \,\}} \quad \text{E\_TAPPTABS}$$

$$\frac{t_i \;\rightarrow\; t_i'}{\textbf{case}[\, S \,]\; t_1\; \textbf{of}\; \{\, \overline{c_i \Delta_i x_i \Rightarrow t_i}^{\,i} \,\} \;\rightarrow\; \textbf{case}[\, S \,]\; t_1'\; \textbf{of}\; \{\, \overline{c_i \Delta_i x_i \Rightarrow t_i}^{\,i} \,\}} \quad \text{E\_CASE}$$

$$\frac{\Delta_i \;=\; \overline{X_j :: K_j}^{\,j}}{\textbf{case}[\, S \,]\,(\, c_i\; \overline{[T_j]}^{\,j}\; v \,)\; \textbf{of}\; \{\, \overline{c_i \Delta_i x_i \Rightarrow t_i}^{\,i} \,\} \;\rightarrow\; (\, t_i\,\{\, \overline{T_j / X_j}^{\,j} \,\}\,)\,\{\, v \,/\, x_i \,\}} \quad \text{E\_CASECON}$$

$\boxed{\Phi \;:\; \Delta_1 \;\Rightarrow\; \Delta_2}$ Multi-subst well-formed

$$\frac{X :: K \;\in\; \Delta_1 \; \mathit{implies}\; (\Delta_2 \vdash \Phi\,(\, X \,) :: K \wedge X \;\notin\; \mathit{dom}\, \Delta_2)}{\Phi \;:\; \Delta_1 \;\Rightarrow\; \Delta_2} \quad \text{MSWF}$$

$\boxed{\Phi \;\sim\; \Delta}$ Compatibility

$$\frac{\Phi \;:\; \Delta_1 \;\Rightarrow\; \Delta_2 \quad \Delta_1 \;\subseteq\; \Delta \quad \Delta_2 \;\subseteq\; \Delta}{\Phi \;\sim\; \Delta} \quad \text{COMPAT}$$

$\boxed{\Delta \vdash S :: K}$ Kinding

$$\frac{X :: K \;\in\; \Delta}{\Delta \vdash X :: K} \quad \text{K\_TVAR}$$

$$\frac{C :: K \;\in\; \Sigma}{\Delta \vdash C :: K} \quad \text{K\_TCON}$$

$$\frac{\Delta \vdash T_1 :: K_{11} \;\Rightarrow\; K_2 \quad \Delta \vdash T_2 :: K_{11}}{\Delta \vdash T_1\; T_2 :: K_2} \quad \text{K\_APP}$$

$$\frac{\Delta \vdash S_1 :: * \quad \Delta \vdash S_2 :: *}{\Delta \vdash S_1 \;\rightarrow\; S_2 :: *} \quad \text{K\_ARROW}$$

$$\frac{\Delta\,,\, X :: K \vdash S_2 :: *}{\Delta \vdash \forall\, X :: K \,.\, S_2 :: *} \quad \text{K\_ALL}$$

$\boxed{\Delta \;\vdash\; \Gamma}$ Context well-formed

$$\frac{}{\Delta \;\vdash\; \emptyset} \quad \text{C\_EMPTY}$$

$$\frac{\Delta \;\vdash\; \Gamma \quad \Delta \vdash S :: * \quad x \;\notin\; \mathit{dom}\,\Gamma}{\Delta \;\vdash\; \Gamma\,,\, x \;:\; S} \quad \text{C\_VAR}$$

$\boxed{\Delta\,\Gamma \;\vdash\; t \;:\; S}$ Typing

$$\frac{x \;:\; S \;\in\; \Gamma \quad \Delta \;\vdash\; \Gamma}{\Delta\,\Gamma \;\vdash\; x \;:\; S} \quad \text{T\_VAR}$$

$$\frac{\Delta \vdash S_1 :: * \quad \Delta\,\Gamma\,,\, x\,:\,S_1 \,\vdash\, t_2\,:\,S_2}{\Delta\,\Gamma\,\vdash\,\lambda x\!:\!S_1.t_2\,:\,S_1\,\to\,S_2}\quad \text{T\_ABS}$$

$$\frac{\Delta\,\Gamma\,\vdash\,t_1\,:\,S_{11}\,\to\,S_{12} \quad \Delta\,\Gamma\,\vdash\,t_2\,:\,S_{11}}{\Delta\,\Gamma\,\vdash\,t_1\,t_2\,:\,S_{12}}\quad \text{T\_APP}$$

$$\frac{(\,\Delta\,,\,X :: K\,)\,\Gamma\,\vdash\,t\,:\,S}{\Delta\,\Gamma\,\vdash\,\lambda X\!::\!K.t\,:\,\forall\,X :: K\,.\,S}\quad \text{T\_TABS}$$

$$\frac{\Delta\,\Gamma\,\vdash\,t\,:\,\forall\,X :: K\,.\,S \quad \Delta \vdash T :: K}{\Delta\,\Gamma\,\vdash\,t\,[\,T\,]\,:\,S\,\{\,T\,/\,X\,\}}\quad \text{T\_TAPP}$$

$$\frac{c\,:\,\forall\,\Delta'\,.\,S\,\to\,C\;T\,\in\,\Sigma \quad \emptyset \vdash \forall\,\Delta'\,.\,S\,\to\,C\;T :: * \quad \Delta\,\vdash\,\Gamma}{\Delta\,\Gamma\,\vdash\,c\,:\,S}\quad \text{T\_CON}$$

$$\frac{\begin{array}{l}\Delta\,\Gamma\,\vdash\,t\,:\,C\;T \\ \forall\,c\,:\,(\forall\,\Delta_i\,.\,S_i\,\to\,C\;T_i\,)\,\in\,\Sigma\,.\,\mathbf{exists}\;c_i\,.\,c\,=\,c_i \\ \forall\,\Phi\,\sim\,\Delta\,,\,\Delta_i\,.\,(\Phi\,\in\,\mathit{mgu}\,(\,T\,,\,T_i\,)\;\mathit{implies}\;(\Phi\,(\,\Delta\,,\,\Delta_i\,)\,\Phi\,(\,\Gamma\,,\,x_i\,:\,S_i\,)\,\vdash\,\Phi\,(\,t_i\,)\,:\,\Phi\,(\,S\,))) \\ \mathit{dom}\,\Delta_i\,\notin\,\mathit{ftv}\,S\end{array}}{\Delta\,\Gamma\,\vdash\,\mathbf{case}[\,S\,]\,t\,\mathbf{of}\,\{\,\overline{c_i\Delta_i x_i \Rightarrow t_i}^{\,i}\,\}\,:\,S}\quad \text{T\_CASE}$$

2 Additional Definitions

- Φ is a substitution: an unordered finite map from type variables to types. The empty map is $\emptyset$. The composition of two maps is written $\Phi_1 \circ \Phi_2$.
- If $\Phi\,:\,\Delta_1\,\Rightarrow\,\Delta_2$ and $\Phi\,\sim\,\Delta$ then $\Phi\,(\,\Delta\,) = \Delta - \Delta_1$.
- If $\Delta_1 \vdash T_1 :: *$ and $\Delta_2 \vdash T_2 :: *$ then $\Phi\,\in\,\mathit{mgu}\,(\,T_1\,,\,T_2\,)$ when
  1. $\Phi\,\sim\,\Delta_1\,,\,\Delta_2$
  2. $\Phi\,(\,T_1\,) = \Phi\,(\,T_2\,)$
  3. For all $\Phi'\,\sim\,\Delta_1\,,\,\Delta_2$ and $\Phi'\,(\,T_1\,) = \Phi'\,(\,T_2\,)$ there exists a $\Phi''\,\sim\,\Phi\,(\,\Delta_1\,,\,\Delta_2\,)$ s.t.$\Phi' = \Phi'' \circ \Phi$.

3 Notes

- This language has explicit type abstraction and application.
- Like Haskell, this language has predicative polymorphism. Type variables may only be instantiated with monotypes.
- For simplicity, data constructor take exactly one term argument and their indexed types take exactly one type argument.
- However, data constructors can be polymorphic over several type variables.

4 Metatheory

Lemma 1 (Regularity)

1. *If* $\Delta\,\Gamma\,\vdash\,t\,:\,T$ *then* $\Delta \vdash T :: *$ *and* $\Delta\,\vdash\,\Gamma$.

Lemma 2 (Type substitution) *Suppose* $\Phi\,\sim\,\Delta$*:*

1. *If* $\Delta \vdash S :: K$ *then* $\Phi\,(\,\Delta\,) \vdash \Phi\,(\,S\,) :: K$
2. *If* $\Delta\,\vdash\,\Gamma$ *then* $\Phi\,(\,\Delta\,)\,\vdash\,\Phi\,(\,\Gamma\,)$

3. *If* $\Delta\,\Gamma \vdash t : S$ *then* $\Phi(\Delta)\,\Phi(\Gamma) \vdash \Phi(t) : \Phi(T)$

Lemma 3 (Term substitution in terms) *If* $\Delta\,(\Gamma_1, x : T_1, \Gamma_2) \vdash t : T_2$ *and* $\Delta\,\Gamma_1 \vdash u : T_2$ *then* $\Delta\,(\Gamma_1, \Gamma_2) \vdash t\{u/x\} : T_2$.

Lemma 4 (MGU existance) *If* $\Delta_1 \vdash T_1 :: *$ *and* $\emptyset \vdash T_2 :: *$ *and* $\Phi : \Delta_1 \Rightarrow \emptyset$ *and* $\Phi(T_1) = T_2$ *then* $\Phi \in mgu(T_1, T_2)$.

Lemma 5 (MGU renaming) *If* $\Phi_1 \in mgu(T_1, T_2)$ *and* $\Phi_2 \in mgu(T_1, T_2)$*, then exists* $\Phi_3 = X1/Y1..Xn/Yn$ *such that* $\Phi_1 = \Phi_3 \circ \Phi_2$.

Theorem 6 (Preservation) *If* $\emptyset\,\emptyset \vdash t : T$ *and* $t \rightarrow t'$ *then* $\emptyset\,\emptyset \vdash t' : T$

Lemma 7 (Canonical Forms) 1. *If* $\emptyset\,\emptyset \vdash v : S_1 \rightarrow S_2$ *then* v *is* $\lambda x\!:\!S.t$.

2. *If* $\emptyset\,\emptyset \vdash v : \forall X :: K_1 . S$ *then* v *is* $\lambda X :: K_2.t$.

3. *If* $\emptyset\,\emptyset \vdash v : C\ T$ *then* v *is* $c\,\overline{[T_i]}^{\,i}\,v'$.

Theorem 8 (Progress) *If* $\emptyset\,\emptyset \vdash t : T$ *then either* t *is a value or there exists a* t' *such that* $t \rightarrow t'$.