Linear Temporal Type Theory for Event-based Reactive Programming

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Abstract

Inspired by the Curry-Howard correspondence between Functional Reactive Programming (FRP) and Linear-time Temporal Logic (LTL), this paper applies temporal types to describe a functional, event-driven language with an underlying event queue. The result is a compelling abstraction for linear computations with temporal types. We present a toy surface language for event-based GUIs and give some examples of how to program in temporal style. We then define a double-negation translation to adjoint tensor logic, which turns computations into event handlers. Using the structure of the logic we define a step-indexed Kripke logical relation on machine configurations, and prove soundness guarantees about the resulting states.

1. Introduction

Event-driven reactive programming uses two fundamental abstractions: the event and the callback. In GUI programming, for example, events are generated by user behavior when a button is clicked or a key is pressed. When such an event occurs, the event loop triggers callbacks, or event handlers, that execute some code in response to the event. The event-based style of programming is efficient, flexible, and pervasive in real-world GUI programs—see, for instance, the Promises/A+ library for JavaScript or the ReactiveX platform that has been ported to many mainstream languages.

However, the unfettered use of callbacks can lead to erroneous programs. Callback-driven code can be obtuse because event handlers only communicate with the rest of the system using state. Moreover, because implementations need to synchronize these stateful event handlers when processing events in parallel, they can suffer from concurrency errors, including dropped events, data races, and starvation.

Functional Reactive Programming (FRP) is an alternative approach to the event-driven style. FRP programs are purely functional, which alleviates the problems of synchronizing state with behavior. In addition, FRP is known to arise as an instance of Linear-time Temporal Logic (LTL) through the Curry-Howard isomorphism [6, 7]. This strong connection to logic has suggested refinements of the FRP type structure that eliminate common errors such as space leaks and causality violations [8].

In this paper, we analogously extend LTL into a type theory for safer event-driven reactive programming. It is already well-known that event-driven reactive programs can be expressed in a monadic style, which alleviates some of the problems associated with synchronization. The “deferred” type from OCaml’s Async library [11, Chapter 18], the implementation of futures in Scala [5], and the yield operator in Narrative JavaScript [12] are all examples of monadic event-driven programming abstractions. Still, due to the underlying use of callbacks, these libraries are themselves quite difficult to implement and reason about. We present a monadic style of functional programming augmented with a temporal and linear type system, show how it admits an efficient implementation using asynchronous event queues, and prove that this implementation satisfies safety properties. The observations in this work uncover elegant parallels between the event-driven and the functional reactive styles.

After briefly recounting the relevant background about linear temporal logic and its connection to monadic programming, we move on to the main contributions:

**Monadic programming.** Section 3 presents examples of the programming model. Though the ideas in this paper apply to any event-driven system, we make them concrete via a toy language with primitives for GUI programming. We describe the temporal type system in Section 4.

**Synchronization.** We address the question of how to interpret the “linear time” aspect of LTL in a computational setting. We show that it corresponds directly to the ability for programs to safely observe the total ordering on events by “selecting” the first event to occur among several possibilities. We can think of this as allowing “safe” races in a concurrent program, the results of which are resolved nondeterministically, but whose computations are still guaranteed to be well-behaved. While most monadic event libraries support some form of this select primitive, our contribution is to show how it can be derived from the proof-theoretic interpretation of LTL. Section 5 describes our implementation of select in detail.

**Continuation passing style.** Section 5 describes how the surface language event primitives could be implemented by giving a double negation/CPS translation into a stateful target language. We ascribe logical structure to the CPS translation by introducing a novel intermediate adjoint tensor logic. The logic explicitly equates logical negation with continuations, and distinguishes temporal judgments (which are valid at any point in the future) from linear judgments (which are only valid at the current point in time). The relationships among the linear, temporal, and persistent fragments are explained via adjunctions that give rise to the monadic structure. We prove
a cut elimination property for then logic, which in a strong sense says that our select operator correctly and completely captures the reasoning principles justified by LTL.

**Safety.** Using the structure of the adjoint tensor logic, we define a step-indexed Kripke logical relation in Section 7. The target is a stateful, untyped lambda-calculus with an explicit event queue. The behavior of program states is asynchronous, in that events can be handled in any order, but single-threaded, which eliminates issues with race conditions. The realizability proofs imply type safety, which establishes in particular that the proposed implementation of select is free from synchronization bugs.

## 2. Background

### Modal and temporal logic.

When viewed through the lens of the Curry-Howard correspondence, modal logics give rise to type systems for computations inside of different “worlds.” The interpretation of “world” depends on the setting at hand [15], including stages of compilation [3], security levels [2], and nodes in distributed systems [13], among others.

Linear-time temporal logic (LTL) [16] is a classical modal logic for reasoning over time. The set of worlds can be viewed as snapshots in time arranged in a total (linear) order. The modalities of the logic capture the time-dependent nature of assertions—just because a proposition is true today doesn’t mean it will be true tomorrow. The proposition □ A, read “always A,” indicates that A is true both now and at every point in the future. The proposition ◁ A, read “eventually A,” says that there is some point, now or in the future, at which A is true. In classical temporal logic, □ and ◁ are De Morgan duals, which means that ◁ A ≡ ¬ □ ¬ A.

From a computational point of view, there is some leeway in how to ascribe semantics to these temporal modal operators. One sensible way, adopted in functional reactive programming, is to think of the “always” ◁ A” modality as providing a stream of values over time. This makes “always A” isomorphic to a function of type t → A, where t ranges over times (that is, worlds). This design leads to the abstraction Signal A of time-varying values of type A, and an emphasis on programming by stream transducers. However, in this setting the dual modality, “eventually A” is somewhat inefficient: a computation ◁ A must be “pulled” on every clock tick to see whether it is ready to yield a value.

### Classical linear logic, callbacks, and continuations.

In this paper, we show that another sensible way to ascribe semantics to the modal operators leads to a clean interpretation of event-driven programming, in two parts. First, rather than interpreting “always A” as a stream of values, we instead interpret it as a (non time-varying) value that is stable over time—it is available for use at any point in the future. Second, we interpret it as a resource in the sense of Girard’s linear logic [4]. Linearity means that a temporal object must be used exactly once and cannot be duplicated. The intuition is that each event is a unique occurrence of some stateful action (such as a user’s mouse click), for which duplication makes no sense. Herzclitus said, “You can never step in the same river twice”; in the same way, you can never click on the same button twice.

Linear logic provides other advantages besides managing the inherently stateful nature of events. By using linear logic as the substructural basis for LTL, we can take negation to be a constructive connective that gives types to continuations. This is important because event-driven libraries are usually structured in continuation-passing style (CPS). The connection between events and callbacks (a.k.a. continuations) is usually expressed in monadic libraries via operations like:

```javascript
button.onClick: (clickData → unit) → unit.
```

This `onClick` function takes a callback and registers it with the environment: when a button click event occurs, the corresponding callback will be executed. In temporal logic, the callback must be available at any point in the future, so a more precise type is:

```javascript
button.onClick: (□(clickData → unit)) → unit.
```

The CPS translation can be parametric in the return type [19], so abstracting away the parametric argument gives the type

```javascript
button.onClick: (□ unit) → unit.
```

By the De Morgan duality that ◁ A ≡ ¬ □ ¬ A, we have

```javascript
button.onClick: ◁ unit.
```

Rather than thinking of `button.onClick` as a function that installs a callback (via side effects), we instead think of it as an event that produces some `clickData`. The ◁ modality is a monad in temporal logic, and it is this monad that forms the basis of the event-driven monadic style. Intuitively, the bind operator bind x=t1 in t2 waits for the event t1 to occur, binds the value of the underlying event data to x, and then continues with the computation t2.

### Asynchronous computation and select.

The bind operator waits for a single event and only then proceeds with the result. This paradigm is a good way to structure event-driven programs, but it is not expressive enough to capture concurrent events. Event-driven libraries often provide a select operator, which waits for two or more events in parallel and branches to different alternatives depending on which one happens first.

We show that, remarkably, select arises naturally from a proof-theoretic view of LTL. Consider the following axiom, which defines the total (linear) ordering on timesteps, the worlds of LTL:

\[ ∀a,b, a \neq b \Rightarrow (a < b) \lor (b > a) \]

The select operator corresponds to the propositional variant of the axiom, which encodes the total order on the events of LTL:

\[ ◁ A \rightarrow ◁ B \rightarrow ◁((A \land ◁ B) \lor (◁ A \land B)) \]

The axiom says that if A is eventually true and B is eventually true, then either A comes before B (meaning that B is in A’s future), or B comes before A. The disjunction provides an opportunity for select to branch based on which event comes first.

However, adding such an axiom to a proof theory of linear logic is not so straightforward. For one thing, the axiom involves several logical connectives, which means that its semantics is not “modular” with respect to the other types. For another, we must show that it can be effectively implemented in the semantics. Our approach to solving these problems is to structure the semantics in the form of an adjoint tensor logic, extended to account for select.

Before diving into those technical details, however, we first show how monadic-style features (including select) can be conveniently packaged in a surface language that treats ◁ A as the central abstraction. The type theory is supported by a linear sublanguage of “event-driven” computations, with a full “ordinary” intuitionistic lambda calculus for writing pure expressions as well.
3. Examples

To make the ideas of this event-driven programming model more concrete, we explore them in the context of GUI programming. Here, the linear sublanguage is tailored for manipulating GUI objects, though the unrestricted part is a standard simply typed lambda calculus. This toy language would need to be extended in various ways to make it into a full-fledged GUI toolkit, but it suffices to convey the key features of our design.

The components of GUI programs, such as buttons, labels, and text boxes, are called widgets. The type of a widget is written \( W \). The application starts out with a single button of type \( W \), and will return a handle to the widget only after the button has been clicked. The linearity restriction means that the old variable can never be accessed again, so we write over it with the new reference.

Once the button has been clicked, a label will appear with the text “Hello World!” on it. The \texttt{return} construct turns a pure value (a pair of widgets) into an event that immediately returns its value.

A note about the argument to \texttt{newLabel}: Strings are not linear data, and they can be used as many times as necessary. We enclose nonlinear data in bars to indicate that it should be used as an argument to a linear function.

Arguably the hardest part of event-based programming is synchronization—waiting for two events in parallel that depend on each other in some way. The following program uses the \texttt{select} primitive to race to see which button gets clicked first.

The \texttt{race} program constructs two buttons and waits for a click on either one. As soon as the first button is clicked (say, \texttt{b1}), a label will appear with the text “Button 1 won the race!” The program then waits for the second button to be clicked before terminating.

The \texttt{race} program uses a fixpoint specifically for events. The general form of \texttt{select} is as follows:

\[
\texttt{let b1 = newButton () in}
\texttt{let b2 = newButton () in}
\texttt{select onClick b1 | onClick b2 of}
\texttt{| b1, e2 ->}
\texttt{| b2, e1 ->}
\texttt{| e1, e2 ->}
\texttt{| e1, e2 ->}
\texttt{return (b1,b2,1)}
\texttt{end}
\]

The \texttt{race} program compares the buttons’ values:

\[
\texttt{let l = newLabel |"Button 1 won the race!"| in}
\texttt{bind b2 = e2 in}
\texttt{return (b1,b2,l)}
\texttt{end}
\]

The \texttt{race} program then compares the labels:

\[
\texttt{let l = newLabel |"Button 2 won the race!"| in}
\texttt{bind b1 = e1 in}
\texttt{return (b1,b2,l)}
\texttt{end}
\]

The intuition behind linearity in the surface language is that variables of type widget are pure references to impure code. When a variable is consumed, the underlying state changes, so a new variable pointing to the updated state must be provided. The old variable can never again be used because its state no longer exists in the program.

Consider the following “hello world” program:

\[
\texttt{let b1 = newButton () in}
\texttt{let b2 = newButton () in}
\texttt{select onClick b1 | onClick b2 of}
\texttt{| b1, e2 ->}
\texttt{| b2, e1 ->}
\texttt{| e1, e2 ->}
\texttt{return (b1,b2,1)}
\texttt{end}
\]

The race program constructs two buttons and waits for a click on either one. As soon as the first button is clicked (say, \texttt{b1}), a label will appear with the text “Button 1 won the race!” The program then waits for the second button to be clicked before terminating.

Select can be thought of as the \( n \)-ary form of \texttt{bind}. The general form of \texttt{select} is as follows:

\[
\texttt{select c1 | ... | cn as}
\texttt{| \circ x1,e2,...,en \rightarrow t1}
\texttt{| ...}
\texttt{| e1,...,en-1,\circ xn \rightarrow tn}
\texttt{end}
\]

Operationally, \texttt{select} runs the computations \( c1 \) through \( cn \) in parallel. For whichever computation finishes first (say, \texttt{e1}) the result of that computation gets bound to \( x1 \), and the program continues at \( t1 \). For the sake of linearity, the variables \( ej \) are fresh handles to the still ongoing computations.

The \texttt{helloWorld} and \texttt{race} programs only last for a finite amount of time. In general we will want to write nonterminating programs, for which we use a fixpoint specifically for events. The following program increments a counter every time its button is clicked.

\[
\texttt{let b = newButton () in}
\texttt{let l = newLabel |toString 0| in}
\texttt{let rec loop b l \[n]\ =}
\texttt{bind b = onClick b in}
\texttt{let l = updateLabel l |toString \( n+1 \)\| in}
\texttt{force loop b l \[n+1]\ in loop b l \[0]\}
\]

Figure 1. Typing rules for linear types.
The counter program initializes two widgets: a button and a label. It then calls loop with the buttons and an initial value for the counter. The recursive function loop waits for a click on the button, updates the label, and then passes the updated value n+1 to the recursive call. The recursive call to loop is actually a thunk of an event, so we invoke it using force.

The `letrec` construct does not provide general recursion because it applies only to functions whose result is a monadic event. As a consequence, only linear computations, and not pure code, can cause it applies only to functions whose result is a monadic event. force

loop
recursive call. The recursive call to
updates the label, and then passes the updated value n+1 to the recursive function

The syntax of terms is as follows:

$$t ::= x \mid \lambda x.t \mid t_1.t_2 \mid \text{let } (\cdot) = t \mid \text{let } (\cdot) = t \mid \text{let } (\cdot) = t \mid \text{let } (\cdot) = t \mid \text{let } (\cdot) = t \mid \text{let } (\cdot) = t$$

There are two typing rules for variables, one for each fragment:

$$Γ; Δ ⊢ t : A \quad Γ ⊢ t : τ$$

$$Γ; Δ ⊢ t : A \quad Γ ⊢ t : τ$$

4. Surface Language

In this section we present a linear/non-linear type system for event-based monadic programming. In the linear/non-linear paradigm [1] there are two kinds of types: linear and unrestricted. Linear types are denoted A, and unrestricted types, also called persistent, are denoted τ.

$$A ::= 1 \mid A \otimes A \mid 0 \mid A \otimes A \mid A \rightarrow A \mid \diamond A \mid \tau$$

$$τ ::= \text{Unit} \mid t_1 \times t_2 \mid \text{Void} \mid t_1 + t_2 \mid t_1 \rightarrow t_2 \mid \tau$$

Similarly, there are two kinds of typing judgments—one for linear types and one for persistent types. Linear typing judgments can use variables both from persistent contexts Γ, as well as from linear contexts Δ. Persistent typing judgments can only use variables from persistent contexts, because they themselves might be treated non-linearly. The typing judgments have the form

$$Γ; Δ ⊢ t : A \quad Γ ⊢ t : τ$$

where

$$Δ ::= · \mid Δ, x : A \quad Γ ::= · \mid Γ, x : τ$$

Figure 1 shows the rules for linear types. The linearity constraints are enforced by how the linear contexts are managed. In the 1-introduction rule for example, the unit (1) only has type 1 if there are no other linear variables in scope. This ensures that all linear resources get used. The △-introduction rule, on the other hand, enforces the fact that linear variables cannot be duplicated. In order to construct a pair (t_1, t_2), we must be able to partition the context into two non-overlapping contexts of resources, Δ_1 and Δ_2, such that t_1 only uses resources from Δ_1 and t_2 only uses resources from Δ_2.

The rules are fairly standard for intuitionistic linear type systems, with the exception of the monadic △ operator from LTL. The return operator constructs an event that immediately yields the pure term t. The select operator runs n computations, t_1 through t_n, in parallel. Depending on which computation finishes first, it proceeds with one of the results, t_1 through t_n. In particular, if t_i finishes first then the result of that computation will be bound to the variable x_i, and the program will proceed as t_i. The variables x_j for j ≠ i will be bound to the ongoing events t_j.
The unary form of \texttt{select} is just the monadic bind operator. It is distinguished with its own syntactic sugar:

\[
\frac{\Gamma; \Delta_1 \vdash t : \tau \ A \quad \Gamma; \Delta_1, \Delta_2 \vdash \textbf{bind } x = t \text{ in } \nu : \sigma \ B}{\Gamma; \Delta_1, \Delta_2 \vdash \textbf{bind } x = t \text{ in } \nu : \sigma \ B} \quad \text{BIND}
\]

The persistent, or unrestricted, fragment of the surface language is just the simply-typed lambda calculus. Its typing rules are shown in Figure 2.

The remaining two types are \(!\ A\) and \(\tau\), whose typing rules are presented in Figure 3. A linear term that doesn’t use any linear variables can be suspended into a persistent thunk of type \(!\ A\). This thunked event can then be instantiated at any time using \texttt{force}. The \(\vdash\) rules say that we can treat persistent terms of type \(\tau\) as if they were linear with type \(\tau\).

This set of typing rules are sometimes called \textit{adjoint}, which stems from the categorical understanding of \(!\) as a comonad in linear logic. In the linear/non-linear configuration we decompose \(!\) into two parts, \(!\) and \(\vdash\), corresponding to the decomposition of a monad into two adjoint functors. Notice that, like \(!\), \(\vdash\) forms a monad in the persistent fragment. In Section 6 we show that these two monads can be seen as parallel structures in a three-tiered system with linear, temporal, and persistent types.

\textbf{β-equality.} The operational semantics of the surface language is outside the scope of this paper, where we instead focus on the Kripke semantics defined in Section 7. However, \(\beta\)-equality is still a sound way of understanding the behavior of surface language programs, including select functions. In particular, the following equivalences hold in the semantics:

\[
\text{select } (\texttt{return } v \mid t_2) \text{ of } (x_1, x_2) \rightarrow (t'_1 \mid t'_2) =_\beta t'_1\{v/x_1, t_2/x_2\} \\
\text{select } (t_1 \mid \texttt{return } v) \text{ of } (x_1, x_2) \rightarrow (t'_1 \mid t'_2) =_\beta t'_2\{t_1/x_1, v/x_2\}
\]

\textbf{Fix.} The language supports a restricted form of fixpoints that only applies to computations of the form \(\diamond A\). For any function \(\diamond A \rightarrow B\), there is an operator

\[
\text{sfix} : !(\diamond A \rightarrow \diamond B) \rightarrow (\diamond A \rightarrow \diamond B) \rightarrow (\diamond A \rightarrow \diamond B).
\]

The operator is special in that it is non-blocking—it allows other events not in the (possibly non-terminating) loop to execute in parallel.

The \texttt{letrec} construct from the previous section is encoded using \texttt{sfix}:

\[
\text{letrec } f = t_1 \text{ in } t_2 := \\
\text{let } f = \text{ sfix } (\text{ suspend } (\text{ fun } f \rightarrow t_1)) \text{ in } t_2
\]

\textbf{GUI primitives.} We extend the language with three event-specific types:

\[
A ::= \cdots | W \\
\tau ::= \cdots | WData | EData
\]

These can be summarized as follows:

- \(W\) is the type of widgets;
- \(WData\) is the type of data that describes widgets, such as size, positioning, or text to display; and
- \(EData\) is the type of data that describes events, such as the kind of event (a button click or a key press) and potentially extra data (which key was pressed).

In this paper we leave \(WData\) and \(EData\) abstract because they are not the focus of the semantics. The important point is that they describe data in the unrestricted fragment, whereas \(W\) is linear.

The \texttt{onEvent} primitive takes in a widget and produces a computation that will trigger once an event occurs. Its type is

\[
\text{onEvent} : W \rightarrow (W \otimes \diamond \ EData).
\]

The other primitives don’t interact with events, but modify the data associated with widgets.

\[
\begin{align*}
\text{newWidget} : \forall W &\rightarrow W \\
\text{setWData} : W &\rightarrow WData \rightarrow W \\
\text{getWData} : W &\rightarrow W \otimes WData
\end{align*}
\]

Variants on \texttt{onEvent}, like the function \texttt{onClick} in Section 3, can be defined by recursively waiting for events until one of them is a click event.

\[
\text{onClick} : W \rightarrow \diamond W \\
\text{onClick} = \\
\text{letrec } \texttt{onClick'} \ w = \\
\text{let } (v, e) = \texttt{onEvent } w \text{ in } \\
\text{bind } |d| = e \text{ in } \\
\text{if isClick } d \text{ then return } w \\
\text{else force } \texttt{onClick'} \ w
\]

\textbf{5. Target Language}

The surface language computes with monadic events at the type \(\diamond A\); this section will describe in broad strokes how these monadic events might be implemented. The semantics for which we prove safety is described in Section 7. Here, we simply aim to give an intuition about how terms in the surface language behave. We give sketches for implementations of the key constructs of the surface language: the GUI primitives, recursion in the form of \texttt{sfix}, and the monadic operators, \texttt{return} and \texttt{select}.

The key observation is to implement the \(\diamond A\) modality as the continuation monad \(\square \neg A\). The result is code in continuation-passing style (CPS): functions that take in continuations of type \(\square \neg A\) but do not return a meaningful value. Instead, the code simulates returning a value of type \(A\) by passing that value to its continuation.

From the perspective of LTL, we treat these continuations as having type \(\square \neg \square \neg A\), but operationally \(\square \) is ignored, at least for the purposes of this section.

\textbf{5.1 Operational Behavior}

The target of the semantics is an untyped, impure lambda calculus with a mutable heap and an explicit event queue. A store \(\sigma\) is made up of these two parts: a heap and a queue. A heap is a partial function from locations to data, and a queue is a (total) function from locations to multisets of event handlers.

\[
\begin{align*}
(\text{Heaps}) &\quad h \in \text{Loc} \rightarrow \text{Val} \\
(\text{Queues}) &\quad q \in \text{Loc} \rightarrow \mathcal{M}(\text{Val}) \\
(\text{Stores}) &\quad \sigma ::= (h, q)
\end{align*}
\]

Expressions include primitives for manipulating the store.

\[
\begin{align*}
e ::= x | \lambda x.e | e e | \text{in}_1 e | \text{in}_2 e | \text{case } e \text{ of } (\text{in}_1 x \rightarrow e | \text{in}_2 y \rightarrow e) \\
&\quad | (e, e) | \pi_1 e | \pi_2 e \\
&\quad | l | \text{new}(e) | e ::= e | !e \\
&\quad | E | \text{attach}(e, e) \\
&\quad | v ::= \lambda x.e | \text{in}_1 v | \text{in}_2 v | (v, v) | l | E
\end{align*}
\]

The expression \texttt{new}(e) evaluates to a fresh location and initializes that location in the heap with the value of \(e\). Heap locations can be assigned with \(e ::= e'\) and dereferenced with \(le\). The operation \texttt{attach}(e, e') registers \(e'\) as an event handler in the queue at location \(e\). Events \(E\) are passed to these event handlers when triggered by the environment.
The intended behavior of the select rule is to wait for the events \( t_1 \) and \( t_2 \) in parallel. The implementation therefore defines two complementary continuations, giving one to \( t_1 \) and the other to \( t_2 \).

As an example, suppose \( \Gamma; \Delta \vdash t : A \). For an expression \( e \) that interprets \( t \), define

\[
\text{return } e := \lambda k. k e
\]

So \text{return} takes in a continuation \( k \) and immediately passes its value (given by \( e \)) to the continuation.

We will build up the \text{select} rule in stages, starting with the unary version, \text{bind}. But bind is a special case of the more general \text{select} operator, and its implementation should reflect that fact.

The expression first takes in a continuation \( k \). It then evaluates \( e \) by passing it a continuation. The one it chooses is the continuation that uses the result of \( e \) to compute \( e' \) in the original continuation \( k \).

Notice that the time at which \( e' \) is evaluated depends on how \( e \) uses its continuation. For example, if \( e \) registers its continuation in the heap using \text{onEvent}, then \( e' \) will not evaluate until an event triggers the callback.

Next we consider the binary select rule—the \( n \)-ary version can be generalized from this case.

\[
\Gamma; \Delta_1 \vdash t : A \quad \Gamma; \Delta_2 \vdash t' : B \\
\Gamma; \Delta_1, \Delta_2 \vdash \text{bind } x = t \text{ in } t' : B
\]

For a given \text{bind} \( x = e \text{ in } e' := \lambda k. e (\lambda x. e' k) \).

The expression first takes in a continuation \( k \). It then evaluates \( e \) by passing it a continuation. The one it chooses is the continuation that uses the result of \( e \) to compute \( e' \) in the original continuation \( k \).

Notice that the time at which \( e' \) is evaluated depends on how \( e \) uses its continuation. For example, if \( e \) registers its continuation in the heap using \text{onEvent}, then \( e' \) will not evaluate until an event triggers the callback.

Next we consider the binary select rule—the \( n \)-ary version can be generalized from this case.

\[
\Gamma; \Delta_1 \vdash t_1 : A \quad \Gamma; \Delta_2 \vdash t_2 : B \\
\Gamma; \Delta_1, \Delta_2 \vdash \text{select } (t_1 | t_2) : (t_1 | t_2) : C
\]

The intended behavior of the select rule is to wait for the events \( t_1 \) and \( t_2 \) in parallel. The implementation therefore defines two complementary continuations, giving one to \( t_1 \) and the other to \( t_2 \).

As an example, suppose \( t_1 \) is triggered before \( t_2 \). In this case the continuation being fed to \( t_1 \) should evaluate \( t_1' \), and the continuation being fed to \( t_2 \) should do no extra evaluation. In order to distinguish which of the two events came first, it is necessary to synchronize with a mutable reference variable, which we will call \( c \). Over the course of the program, this \text{channel}, which is just a reference cell following a particular protocol, may be in any of the following
linear temporal type theory

5.3 Soundness

Now that we have an understanding of how we expect the implementation to behave, we can consider the properties that characterize a well-behaved program.

Expressions should be type-safe in the traditional sense. That is, if \( t \) is well-typed in the surface language, then its interpretation should not go wrong in the target language. But this definition must be strengthened to take the event queue into account. Consider the expression \( \langle l, x.() x \rangle \). The expression does evaluate to a value (unit), but the updated store is not well-behaved. When an event triggers the callback on \( l \), the program will crash. Thus the expression generated a safe value, but an unsafe store.

So what does it mean for a store to be safe? Intuitively it should mean that any callback in the queue can be triggered and will result in another safe store. As in typical constructions of type safety, this key property can be summed up as:

**Preservation:** If \( \sigma \) is a safe store and \( \sigma \Rightarrow \sigma' \) then \( \sigma' \) is also safe.

The surface language should therefore satisfy the following soundness property, characterized by the fact that the resulting store is safe:

**Soundness:** If \( \Gamma \vdash t : A \) in the surface language and \( \langle ; [t] \rangle \Downarrow \langle \sigma; v \rangle \), then \( \sigma \) is safe.

In order to prove this property, we will need to define a logical relation that classifies valid machine configurations (pairs of a store and a value) for any particular type. Unfortunately, the structure of surface language types is not rich enough to define the necessary logical relation. Consider the bind rule as an example:

\[
\Gamma; \Delta_1 \vdash t : \alpha \rightarrow A \quad \Gamma; \Delta, x : A \vdash t' : B
\]

\[\Gamma; \Delta_1 \vdash \text{bind } x = t \text{ in } v : \alpha \rightarrow B\]

The linear variables in \( \Delta_1 \) and \( \Delta \) may refer to events, which means they rely on the store being well-behaved. But the continuation \( t' \) will not be executed until some arbitrary time has passed—time in which the store may have advanced via the event loop. If \( \sigma \) is valid for \( \Delta \) and \( \sigma \Rightarrow \sigma' \), then how do we know that \( \sigma' \) is valid for \( \Delta' \)?

This property of sets of machine configurations, which we will call *temporal logic*, is exactly characterized by the \( \square \) modality from temporal logic. For a set \( A \) of machine configurations, we expect \( \square A \) to characterize those stores \( \sigma \) such that whenever \( \sigma \Rightarrow \sigma' \), we have \( \sigma' \) valid for \( A \). This is indeed the intended meaning of \( \square \) in \( \alpha \equiv \square \alpha \rightarrow A \), but \( \square \) itself never occurs as the principle type in the surface language.

The logic defined in the next section solves this problem by explicitly separating temporal types from purely linear ones, which characterize stores at a particular moment in time. We embed every surface language type as a temporal proposition, which ensures...
that the heaps they reference satisfy temporality. At the same time, we perform the double negation/CPS translation to unify the □ modality in the encoding of ◇ with this notion of temporality.

6. Adjoint Tensor Logic

This section presents an intermediate logic known as adjoint tensor logic as the first step in the interpretation of the surface language. The logic serves three purposes.

First, tensor logic allows for an interpretation of the monad ◇ in terms of negation and □. The negation operator provides a constructive logical formulation for reasoning about continuations. In the setting of temporal logic, it strengthens the connection between classical temporal logic and continuation passing style. We define a double negation translation from the surface language to derivations of the CPS-style logic.

Second, the logic provides structure for the semantics presented in Section 7. We argue that the □ operator is more primitive than ◇ for the purposes of classifying machine configurations, because it describes states that are valid at any point in the future. Linearity provides the structure to prove that disjoint parts of the heap do not interfere.

Third, the cut elimination theorem we prove about the logic provides assurance that the return and select operators from the surface language make up a sound API for the monadic event primitive. This section presents an intermediate logic known as adjoint tensor logic as the first step in the interpretation of the surface language. The union of adjoint and tensor logics is one of the contributions of the paper.

6.1 Propositions and Judgments

The intermediate logic consists of three different structural fragments, as opposed to the two of the surface language. The persistent fragment, denoted \( P \), corresponds to persistent types \( \tau \) in the surface language. These hypotheses can be used arbitrarily many times by applying weakening and contraction. The temporal fragment, denoted \( T \), represents propositions that are true at any point in time, which are nevertheless stateful, so they must be treated linearly. The linear fragment, denoted \( L \), consists of propositions that are linear and only true at one particular moment in time.

- \( A^L \) :\( = \{ 1 \mid A_1^L \otimes A_2^L \mid 0 \mid -A^L \mid F_L A^T \}
- A^T \) :\( = \{ U_T A^P \mid F_T A^P \}
- A^P \) :\( = \{ Unit \mid A_1^P \times A_2^P \mid Void \mid A_1^P + A_2^P \mid A_1^T \rightarrow A_2^P \mid U^P A^T \}

Note that the linear and temporal fragments do not contain linear implication, which is replaced by the negation operator \( \neg A^L \). Linear negation was introduced by Melliès and Tabareau [10] as tensor logic. Negation can be thought of as the type of continuations, which means that the computational interpretation of the logic satisfies CPS.

The adjoint operators, \( ! A \) and \( \lceil \tau \rceil \), have been replaced by \( U^P A^P \) and \( F_T A^P \). The same operators relate the linear and temporal fragments. These are two instances of the same adjoint relationship, formalized by Reed [17] as adjoint logic. While \( U^P \) and \( F_T \) correspond to the \( ! \) connective from linear logic, the operators \( U^L \) and \( F_L \) correspond to the \( \square \) connective in temporal logic.

The union of adjoint and tensor logics is one of the contributions of this paper. In fact, the linear/temporal/persistent structure is an instance of a more general signature for adjoint tensor logic.

The adjoint operators are based on the idea that one level is "stronger" than another in a particular way. For example, any temporal proposition can be treated as if it were linear, so we write \( L \leq T \). Similarly, any persistent proposition (which is a pure value that does not depend on the store) can be treated as if it were only temporal: \( T \leq P \). These relationships can be generalized to an arbitrary signature of fragments \( r \), with a DAG on fragments denoted \( r_1 \leq r_2 \). We write \( \leq \) for the reflexive transitive closure of the graph. Every fragment corresponds to a collection of propositions \( A^r \), and for every edge \( r \leq s \) in the graph, the propositions include \( A^r := \cdots \mid F_s A^s \mid A^s := \cdots \mid U^s A^r \)

Judgments in the logic have one of two forms:

\[ \Gamma \vdash A^r \]
where \( \Gamma \geq r \) for arbitrary fragments \( r \), and

\[ \Gamma \vdash s \]
where \( \Gamma \geq s \) for fragments \( s \) with negation. Judgments of this form are said to have no conclusion. We will sometimes write \( \Gamma \vdash (A)^s \) to mean that the judgment has either 0 or 1 conclusion.

The logic is structurally linear, as can be seen in the hypothesis and cut rules:

\[ \Gamma \vdash A^r \]

HYP

\[ \Gamma \vdash A^s \]

\[ \Delta, A^r \vdash (B)^r \]

CUT

In the hypothesis rule the context must be empty—all resources must already be consumed. In the cut rule, the contexts \( \Gamma \) and \( \Delta \) are concatenated together and their resources are not duplicated. There are explicit weakening and contraction rules for persistent propositions:

\[ \Gamma \vdash A^r \]

WEAK

\[ \Gamma, A^P, AP^r \vdash (B)^r \]

CONTRACT

Negation has the following inference rules whenever it is defined:

\[ \Gamma \vdash A^r \]

\[ \Gamma, \neg A^r \vdash s \]

\[ \neg L \]

\[ \Gamma, A^r \vdash s \]

\[ \neg R \]

The rules for linear logic operators are given in Figure 8, and the rules for persistent operators in Figure 9. These are straightforward from sequent calculus formulations of linear/non-linear logic [1].

Next are the adjoint operators: \( U^r \) and \( F_s^r \). The inference rules for these operators are given in Figure 10. The U-R rule states that if \( A^r \) is provable using hypotheses that are at least as strong as \( s \), then the \( s \)-proposition \( U^r A^r \) is provable. This is clearly a generalization of the \( ! \)-typing rule from the surface language, and similarly F-R is a generalization of \( \neg \neg \neg \). The intuition should be that stronger fragments lay above the weaker fragments in the ordering, so the \( U \) operator moves \( U \) the chain.

\[ \text{The name "adjoint" hints at a categorical relationship between the logical fragments. Indeed, the labels } F \text{ and } U \text{ derive from the Free and Underlying functors that make up free constructions in category theory [9]. A formal treatment of the categorical semantics of adjoint tensor logic is reserved} \]
for future work, but we expect to build on the categorical semantics for its subsystems [1, 10, 14].

As in the surface language, SELECT is subsumed by the more general \(n\)-ary rule:

\[
\forall i = 1..n : \Gamma, \neg A_1, \neg F_L U^T A_2 \vdash \top \Gamma, \neg F_L U^T A_i, \ldots, \neg F_L U^T A_n \vdash L
\]

6.2 Cut Elimination

Viewed through the lens of the Curry-Howard correspondence, cut elimination is an extremely strong computational claim. The property says that every term, open or closed, can be reduced to normal form. What this means for the surface language is that the concurrency primitives, return and select, are logically well-founded. Other operators might be built on top of these two primitives, but the core language they form is sufficient.

In addition, cut elimination ensures that the event type \(\circ A\) is fully higher-order, so it is safe to compute with events of events or events of functions.

The cut elimination theorem is as follows:

**Theorem 1** (Cut Elimination). If \(\Gamma \vdash A^*\) and \(\Delta, A^* \vdash (B)^r\) are cut-free derivations in adjoint tensor logic, then there is a cut-free derivation of \(\Gamma, \Delta \vdash (B)^r\).

We prove the theorem by induction on the cut term \(A^*\) and on the size of the subderivations. When the second subderivation is a SELECT rule, the proof is not straightforward from the induction hypotheses. We need the help of the following auxiliary lemma:

**Lemma 2.** Let \(D_1\) be a cut-free derivation of \(\Gamma \vdash \neg F_L U^T A\) and \(D_2\) be a cut-free derivation of

\[
D_2 = \Delta \vdash \neg F_L U^T A, \neg B \vdash L
\]

where \(\Delta \geq T\). Then there exists a cut-free derivation of \(\Gamma, \Delta, \neg F_L U^T B \vdash L\).

The problem with this case is the fact that SELECT deals with multiple connectives at once: \(\neg, F, U\). Even if the last rule in \(D_1\) introduces the outermost constructor (the negation) the inductive hypothesis from Theorem 1 does not immediately apply to the subderivation. We strengthen the inductive hypothesis for Lemma 2 in the following way: Given \(D_2\),

1. If \(\Gamma \vdash \neg F_L U^T A\) then \(\Gamma, \Delta, \neg F_L U^T B \vdash L\).
2. If \(\Gamma, F_L U^T A \vdash (C)^L\) then \(\Gamma, \Delta, \neg F_L U^T B, (\neg C)^L \vdash L\).
3. If \(\Gamma, U^T A \vdash (C)^L\) then \(\Gamma, \Delta, \neg F_L U^T B, (\neg C)^L \vdash L\).
4. If \(\Gamma, U^T A \vdash (C)^L\) then \(\Gamma, \Delta, \neg F_L U^T B, (\neg F_L C)^T \vdash L\).
5. If \(\Gamma, A \vdash (C)^L\) then \(\Gamma, \Delta, \neg F_L U^T B, (\neg C)^L \vdash L\).

The full proof of cut elimination can be found in Appendix A.

6.3 Double Negation Translation

In this section we describe how to transform surface language terms into derivations in the logic. Linear types \(A\) from the surface language will be interpreted as temporal propositions \(A^T\), since the surface language assumes that all terms are temporally persistent. Unrestricted types \(\tau\) will be interpreted as persistent propositions \(A^\tau\), taking advantage of the shared type structure. The translation of linear types will integrate the double negation translation, while the persistent types are translated directly. This ensures that the unrestricted fragment of the surface language is semantically just the simply typed lambda calculus.

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The linear translation is broken up into two parts. We define 

\[ [A] ::= U^T \neg [A] \]  

where 

\[
egin{align*}
[1]^- & ::= \neg 1 \\
[A_1 \otimes A_2]^- & ::= \neg (F_L [A_1] \otimes F_L [A_2]) \\
[0]^- & ::= 0 \\
[A_1 \otimes A_2]^- & ::= \neg (F_L [A_1] \otimes F_L [A_2]) \\
[A_1 \rightarrow A_2]^- & ::= F_L [A_1] \otimes \neg F_L [A_2] \\
[\circ A]^- & ::= F_L U^T \rightarrow F_L [A] \\
[\tau]^- & ::= \neg F_L F_T [\tau]
\end{align*}
\]

The following lemma ensures that the double negation translation is flexible enough to encode linear implication:

**Lemma 3.** \( \Gamma \vdash [A] \) if and only if \( \Gamma \geq T \) and \( \Gamma, [A]^- \vdash L \).

The unrestricted types translation simply commutes around the operators:

\[
\begin{align*}
[\text{Unit}] & ::= \text{Unit} \\
[\tau_1 \times \tau_2] & ::= [\tau_1] \times [\tau_2] \\
[\text{Void}] & ::= \text{Void} \\
[\tau_1 + \tau_2] & ::= [\tau_1] + [\tau_2] \\
[\tau_1 \rightarrow \tau_2] & ::= [\tau_1] \rightarrow [\tau_2] \\
[! A] & ::= U^P [A]
\end{align*}
\]

Next, well-typed terms in the surface language are translated into doubly negated derivations \( [\Gamma] \vdash [A] \) in the sequent calculus. Most of the definitions are straightforward from their types, but the following lemmas provide translations for the \( \circ \) typing rules.

**Lemma 4.** The \( \circ \) introduction rule is admissible:

\[
\begin{align*}
\Gamma & \vdash [A] \\
\Gamma & \vdash [\circ A] \\
\Gamma, \circ A & \vdash L
\end{align*}
\]

**Proof.** Recall that \( [\circ A]^- = F_L U^T \neg F_L [A] \). By Lemma 3, the following derivation is sufficient:

\[
\begin{array}{c}
\Gamma \vdash [A] \\
\Gamma \vdash F_L [A] \\
\Gamma, \neg F_L [A] \vdash L \\
\Gamma, U^T \neg F_L [A] \vdash L \\
\Gamma, [\circ A]^- \vdash L
\end{array}
\]

To interpret the select rule, we make the following observations:

**Proposition 5.** \( \Gamma \vdash [\circ A] \) if and only if \( \Gamma, U^T \neg F_L [A] \vdash L \) where \( \Gamma \geq T \).

**Proposition 6.** \( \Gamma, [\circ A] \vdash L \) if and only if \( \Gamma, \neg F_L U^T \neg F_L [A] \vdash L \).

Now we can interpret the surface language select rule.

**Lemma 7.** The following derivation is admissible:

\[
\forall i : \Gamma_i \vdash [\circ A_i] \quad \Gamma, [A_i], [\circ A_i]^{\neq i} \vdash [\circ B] \\
\Gamma_1, \ldots, \Gamma_n, \Gamma \vdash [\circ B]
\]

**Proof.** In order to prove the lemma, it suffices to give a derivation of

\[
\Gamma, U \rightarrow F[B], [\circ A_1], \ldots, [\circ A_n] \vdash L,
\]

which can be turned into the stated conclusion by cutting against the derivations of \( [\circ A_i] \) and applying Proposition 5.

From the statement of the lemma, for each \( 1 \leq i \leq n \) we have the following derivation:

\[
\Gamma, U^T \neg F_L [B], [A_i], [\circ A_i]^{\neq i} \vdash L
\]

By Proposition 6, this is equivalent to

\[
\Gamma, U^T \neg F_L [B], F_L [A_i], \neg F_L U^T \neg F_L [A_i]^{\neq i} \vdash L
\]

for each \( i \). Because \( \Gamma, U^T \neg F_L [B] \geq T \), we can then apply the \( \circ \)-left rule to obtain a derivation of

\[
\Gamma, U^T \neg F_L [B], \neg F_L U^T \neg F_L [A_1], \ldots, \neg F_L U^T \neg F_L [A_n] \vdash L.
\]

Applying Proposition 6 again yields the desired conclusion:

\[
\Gamma, U \neg F_L [B], [\circ A_1], \ldots, [\circ A_n] \vdash L.
\]

\( \square \)

### 6.4 GUI primitives

The GUI specific primitives in the surface language do not have meaning in the intermediate logic—we will give their semantics directly in terms of machine configurations. However, to prove that the implementation is safe with respect to the rest of the semantics, we must provide a typing specification. We extend the signature of propositions as follows:

\[
A^T ::= \cdots | W \\
A^P ::= \cdots | WData | EData
\]

We then add interpretation for the GUI-specific types. Since \( W \) is a temporal proposition, it must satisfy Lemma 3.

\[
[W] ::= U^T \neg [W]^- \\
[W]^- ::= U^T \neg F_L W \\
[EData] ::= EData
\]

### 7. Semantics

With the logical structure of adjoint tensor logic in place, we can move on to defining a Kripke logical relation for the target language semantics.

First, however, we must account for synchronization in the implementation of the select operator. Recall Figure 6, which describes a protocol governing the use of a shared channel \( c \) in the synchronization of select. To prove the implementation abides by the protocol, we augment the target language with a notion of resources, denoted \( A \) and \( B \) in the diagram, that govern which expressions have the permission to modify a shared channel, and how.

Semantic types are collections of expressions along with the permissions that they own in a particular heap. We say an expression owns the resources that allow it to modify the heap. When an expression embedded at a channel in the heap owns some resources, we say that the channel itself is the owner.

Permissions and resources actually form the basis of all heap ownership in the system. We prove that non-shared references treat ownership like a linear resource, taking advantage of the linear logical structure to prove it.

Thus permissions actually track two distinct protocols. The first, required by the select rule, is based on the state transition diagram of Figure 6, and regulates the use of shared state. The second protocol governs the rest of the heap, which is not shared but treated linearly. To simplify the presentation we distinguish shared state as channels, with locations given by a set of channel locations \( \text{Chan} \). This set is disjoint from the locations for linear-use references, which is denoted \( \text{Loc} \). This induces two different kinds of heaps:

- Linear heaps, as in Section 5, are partial functions \( h \) from locations in \( \text{Loc} \) to values.
- Shared heaps are partial functions \( c \) from locations in \( \text{Chan} \) to values.
All linear references will have type $\text{WData}$, while channels may have arbitrary types.

We extend the operational semantics to act on updated stores, which consist of the event queue, the heap, and the shared heap. The syntax of expressions for modifying channels remains the same, except that there are two distinct constructors for references versus channels:

$$\sigma; e \Downarrow ((q, h, c); v) \quad \text{REF}$$

$$\sigma; \text{ref}(e) \Downarrow ((q, [h | l : v]); e;l) \quad \text{CHAN}$$

The sketch of the semantics is as follows: propositions in adjoint tensor logic will be interpreted as semantic types. Derivations references and one to govern the ownership of channels.

A compilation strategy from the surface to the target language. The translation of surface language terms, the fundamental theorem defines a compilation strategy from the surface to the target language.

### 7.1 Resources and Permissions

There are two kinds of resources: one to govern the ownership of references and one to govern the ownership of channels.

$$\text{RefResources} := \{\text{Full}\}$$

$$\text{ChanResources} := \mathbb{N}$$

References are governed by a single resource: an expression either has permission to modify a location, or it does not. For channels, an expression may have permission to take one action but not another. The channel permissions are countably infinite to account for the arbitrary $n$-ary versions of select. In the binary case, as in Section 5, we only use two of these resources, so we write $A$ for 1 and $B$ for 2.

A permission is a partial map from locations to resources. Given some sets of domains, $L \subseteq \text{Loc}$ and $C \subseteq \text{Chan}$, we define $\text{Perm}(L, C)$ to be the type of permissions over $L$ and $C$:

$$\text{Perm}(L, C) := (L \rightarrow \mathcal{P}(\text{RefResources})) \times (C \rightarrow \mathcal{P}(\text{ChanResources}))$$

Next we define the protocols. Protocols govern how the permissions owned by an expression affect how an expression can modify the heap. For a specific location $l$, the relation $v' \sqsubset_X v$ means that the owner of the resources in $X$ allows to mutate the value stored at $l$ from $v$ to $v'$.

If an expression owns the resource Full for a reference, it may freely modify the reference’s value. That is, for any values $v$ and $v'$ we have

$$v' \sqsubset_X v \quad \text{when Full} \in X$$

For channels, the only possible values are those given by the locations in Figure 6. The necessary permissions to move from one state to the other are specified by the labels on the transitions.

### 7.2 Types and Worlds

In this section we start to define the step-indexed Kripke logical relation that will be used to prove soundness of the logic. The full development can be found in Appendix C, but we present a simplified overview of the main ideas. In particular we leave out the details of step-indexing for the sake of clarity, even though they are necessary for well-founded definitions.

Stores now consist of three parts: a queue, a linear heap, and a shared heap. We define the type of a store to consist of the domains of its two heaps as well as a typing assignment for its channels.

$$\text{StoreType} := \{\langle L, C, \tau \rangle \mid \begin{array}{c}
L \subseteq \text{Loc} \land C \subseteq \text{Chan} \\
\tau \in C \rightarrow \text{Type}
\end{array}\}$$

$$\text{Store}(L, C, \tau) := \{ (q, h, c) \mid \begin{array}{c}
\text{dom}(h) = L \land \text{dom}(c) = C \\
\forall l \in L, q(l) = 0
\end{array}\}$$

We say a store $\sigma$ is well-typed if there exists some store type $\Sigma$ such that $\sigma \in \text{Store}(\Sigma)$. Well-typed stores play the role of worlds in the Kripke semantics, meaning that the logical relation is indexed by them.

$$\text{World} := \{ (\sigma, \Sigma) \mid \sigma \in \text{StoreType} \land \sigma \in \text{Store}(\Sigma) \}$$

Semantic types, referenced in the definition of $\text{StoreType}$, map worlds to values with permissions saying how they can modify those worlds.

$$\text{Type} := \text{World} \rightarrow \mathcal{P}((\text{Perm} \times \text{Val}))$$

In Section 5 we considered what it meant for a store to be safe. We now define $\text{Safe}$ to be the semantic type that maps worlds to the unit value with the empty set of permissions.

$$\text{Safe}(w) = \{ (e, ()) \}$$

Expressions in Safe don’t have permission to modify the heap at all, so intuitively they cannot do any harm.

A permission map ascribes a permission to every event handler in the event queue and to every channel in the shared heap. Let $[n]$ denote the set $\{0, \ldots, n-1\}$ of natural numbers. Given the sets $L$ and $C$ of locations and a size function $s : L \rightarrow \mathbb{N}$, a permission map is a function

$$M : (C + \Sigma | l : s(l)) \rightarrow \text{Perm}(L, C)$$

The intuition is that $s(l)$ numbers the callbacks registered at each location $l$, and $M$ associates a permission with each callback and channel. We write $M$ to mean the pointwise union of all the permissions in the image of $M$.

Let $w$ be the world $(\sigma, \Sigma)$ where $\sigma = (q, h, c)$ and $\Sigma = (L, C, \tau)$. A permission map $M$ is well-typed with respect to the world $w$ if the following properties hold:

1. $\forall l \in L, h(l) \in \text{WData}$
2. $\forall l \in L, \forall k_i \in q(l), (M(l, i), k_i) \in \Box \neg \text{EData}(w)$
3. $\forall l \in C, (M(l), c(l)) \in \tau(l)(w)$

In this case we write $M : w$. 

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This definition relies on two operators on semantic types: $\Box$ and $\neg$. The $\Box$ operator implements temporality by ignoring the event queue:

$$\Box A((q, h, c), \Sigma) := A((\emptyset, h, c), \Sigma)$$

What this means is that an expression in $\Box A(w)$ has the same permissions in $A$ given any updated world $w'$ derived from executing callbacks in $w$.

The negation operator describes continuations.

$$\neg A(w) := \{ (R, u) \mid \forall T \neq R, \forall w' \leq T, w, \text{whenever } (S, v) \in A(w'), (R \cdot S, wv) \in E(StaExp)(w') \}$$

Finally, the type $E(A)$ relates expressions to the values in the semantic type of $A$.

$$E(A)(w) := \{ (R, u) \mid \forall w' \leq R w, \forall v, \forall M, \text{if } M : w \text{ and } (w; v) \downarrow (w'; v), \text{then } (R, v) \in A(w') \}$$

As a reminder, these definitions are not completely accurate, but are intended to convey the flavor of the precise definitions given in Appendix C. The two biggest omissions are leaving out step indexing (to ensure well-foundedness) and leaving out the possibility of interference on the shared state from other parties with the rights to modify it.

### 7.3 Logical Relation

Finally, we can interpret propositions from the logic as semantic types. Recall the structure of propositions from Section 6:

$$A^L := 1 \mid A^1 \otimes A^2 \mid 0 \mid A^1 \oplus A^2 \mid \neg A^L \mid F A^T$$

$$A^T := \text{Unit} \mid A^P \times A^P \mid \text{Void} \mid A^1 P + A^2 P$$

$$A^P := A^P \rightarrow A^P \mid U^P A^T$$

We will define three mutually-recursive step-indexed Kripke logical relations, shown in Figure 11, to interpret the three classes of propositions.

Contexts $\Gamma$ are interpreted as pairs $(R; \gamma)$ where $R$ is a permission and $\gamma$ is a substitution of values for variables. We define an indexed family of substitutions for contexts $\Gamma$ by induction:

$$[\Gamma] w = \{ (e, \cdot) \}$$

$$[\Gamma, x : A^P] w = \{ (R; \gamma, v/x) \mid (R; \gamma) \in [\Gamma] w \land v \in [A^P] \}$$

$$[\Gamma, x : A^T] w = \{ (R \cdot R_A; \gamma, v/x) \mid (R; \gamma) \in [\Gamma] w \land (R_A; v) \in [A^T] w \}$$

$$[\Gamma, x : A^L] w = \{ (R \cdot R_A; \gamma, v/x) \mid (R; \gamma) \in [\Gamma] w \land (R_A; v) \in [A^L] w \}$$

Finally we can prove soundness of the semantics. The fundamental property says that every derivation in the logic corresponds to an expression in the correct semantic type. The proof of the theorem, which is provided in Appendix C, constructs the expressions, which leads to a compilation strategy similar to the one set out in Section 5.

**Theorem 8** (Fundamental Property). If $\Gamma \vdash (A')$ then there is an expression $e$ with free variables in $\Gamma$ such that for all worlds $w$ and $(\sigma; \gamma) \in [\Gamma] w$, we have $(\sigma; (\gamma)(e)) \in E[(A')]$.

### 8. Discussion

**Liveness.** An important correctness condition for event-based programs is liveness, which says that the event queue is always responsive enough to handle incoming events. Blocking computations are those computations that never return control to the scheduler. In the target language, these just correspond to non-terminating expressions.

The `sfix` operator defines non-terminating computations that are also non-blocking by attaching each recursive call of the function to the event queue. This technique is commonly used in even non-monadic event programming languages to define non-blocking recursion—see for example the `setTimeout` paradigm in JavaScript [18].

Unfortunately, the step-indexed semantics described in Section 7 is not equipped to prove liveness guarantees, even for the language without `sfix`. We leave a thorough analysis of liveness to future work.

**Message passing.** We have shown that the surface language in Section 4 makes a robust core language for expressing GUI programs. But the monadic style is equally well-suited for other domains of event-driven programming. One promising domain is message passing, in which primitive operations include sending and receiving messages over communication channels (not to be confused with the synchronization channels from Section 7). Channels replace widgets as the type of mutable references in the event queue. The act of receiving along a channel is equivalent to waiting for GUI events by invoking the `onEvent` primitive. The final operator, the `send` function, would correspond to the invocation of callbacks in the event queue.

**Affine events.** The linear restriction on types in the surface language provides a surprisingly clean way to incorporate stateful computations in a pure functional setting. It may be the case that this restriction can be weakened to only `affine` types, allowing variables to be dropped but not duplicated. Computationally this would correspond to flushing event handlers from the event queue.

This paper draws on ideas from type theory, programming language design, propositional logic, and logical relations to study event-based reactive programming from multiple perspectives. We discover that the monadic style used by many real-world languages

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**Figure 11.** Kripke interpretation of propositions.
has logical foundations in temporal and linear logic. We exploit this
foundation to prove the existence of a sound interpretation in an ef-
fectful language. The result of this study is a valuable application
of the Curry-Howard correspondence in a surprising domain.

References

[1] P. Benton. A mixed linear and non-linear logic: Proofs, terms and
models. In L. Pacholski and J. Tiuryn, editors, Computer Science
Logic, volume 933 of Lecture Notes in Computer Science, pages 121–

information flow security with mutable state. Journal of Func-
tional Programming, 15:249–291, 2005. ISSN 1469-7653. doi:
10.1017/S0956796804005441.


1 – 101, 1987. ISSN 0304-3975. doi: http://dx.doi.org/10.1016/0304-
3975(87)90045-4.

[5] P. Haller, A. Prokopec, H. Miller, V. Klang, R. Kuhn, and V. Jo-
vonacci. Futures and promises. Scala Documentation, 2013. URL
http://docs.scala-lang.org/overviews/core/futures.

as types, proofs as functional reactive programs. In Proceedings of
the Sixth Workshop on Programming Languages Meets Program

[7] W. Jeltsch. Towards a common categorical semantics for linear-
time temporal logic and functional reactive programming. Electronic
Notes in Theoretical Computer Science, 286(0):229 – 242, 2012. doi:
http://dx.doi.org/10.1016/j.entcs.2012.08.015. Proceedings of the 28th
Conference on the Mathematical Foundations of Programming Se-
manics (MFPS XXVIII).

[8] N. R. Krishnaswami and N. Benton. Ultrametric semantics of re-
active programs. In Logic in Computer Science (LICS), 2011 26th
Annual IEEE Symposium on, pages 257–266. IEEE, June 2011. doi:
10.1109/LICS.2011.38.


0072. doi: http://dx.doi.org/10.1016/j.apal.2009.07.018. The Third
workshop on Games for Logic and Programming Languages (GaLoP)
Galop 2008.

O’Reilly Media, 2013.


modal lambda calculus for distributed computing. In Logic in Com-

classical linear logic. In S. Alves and I. Cervesato, editors, Proce-
dings Third International Workshop on Linearity, LINEARITY 2014,
Vienna, Austria, 13th July, 2014., volume 176 of EPTCS, pages 9–23,
2015. doi: 10.4204/EPTCS.176.2.

logic. Mathematical structures in computer science, 11(04):511–540,

[16] A. Pnueli. The temporal logic of programs. In Foundations of


9781449396947.